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ARE ALMOST ALL n-VERTEX GRAPHS OF GIVEN DIAMETER HAMILTONIAN?

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Abstract: Typical Hamiltonian properties of the class of n-vertex graphs of a fixed diameter k are studied. A new class of typical n-vertex graphs of a given diameter is constructed.

The question of S.V. Avgustinovich on the Hamiltonian property of almost all such n-vertex graphs has been solved. It is proved that almost all n-vertex graphs of fixed diameter $k = 1, 2, 3$ are Hamiltonian, while almost all n-vertex graph of fixed diameter $k \geq 4$ are nonHamiltonian graphs. All found typical Hamiltonian properties of n-vertex graphs of a fixed diameter $k \geq 1$ are also typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) containing the shortest path of length at least k.

Keywords: graph, Hamiltonian graph, Hamiltonian cycle, diameter, typical graphs, almost all graphs.

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Introduction

We study Hamiltonian property for finite labeled ordinary n-vertex graphs of a given diameter. For a connected graph $G = (V, E)$, *the distance* $\rho_G(u, v)$ between its vertices $u, v \in V$ is defined as the length of the shortest path connecting these vertices. In this case, $d(G) = \max_{u,v\in V} \rho_G(u, v)$ is *the diameter* of graph G. A cycle which passes through every vertex of the graph exactly once is called *Hamiltonian*. A graph is *Hamiltonian* (*nonHamiltonian*) if it contains (does not contain) a Hamiltonian cycle.

Hamiltonicity is one of the central concepts of Graph Theory, also arising in various applied problems, when it is required to find out the presence of a Hamiltonian cycle for a graph modeling the problem under consideration. By now, a huge number of papers have been written on this topic. Many ideas that arose here still go back to the classical results of G.A. Dirac and O. Ore, who first opened this "Pandora's box". The main course of research development and the results obtained on the topic of Hamiltonian graphs in various directions can be found in the surveys [9] and [14]. As it turns out, the problem of deciding whether a graph is Hamiltonian is an NP-complete problem, and accordingly one cannot expect a simple classification of graphs that have this property.

The complexity of the problem and the diversity of Hamiltonian graphs encountered also led to the development of an asymptotic or probabilistic approach to the study of Hamiltonicity, in particular, an approach conditioned by the concept of almost all. A number of results were obtained along this path, opening up the subject of research into Hamiltonicity in this direction. Thus, considering all n-vertex graphs, Yu.D. Perepeliza [17] and J.W. Moon [15] proved that almost all graphs are Hamiltonian. There are also a number of papers in which the Hamiltonian property is studied within given classes of graphs. Of particular interest here is the classes in which sufficient conditions for the existence of a Hamiltonian cycle are satisfied for all or almost all graphs, and the verification and construction of such a cycle is implemented polynomially. In particular, problems about Hamiltonicity of regular and Cayley graphs are known. It was found that almost all Cayley graphs [13] and almost all r-regular graphs for every $r \geq 3$ [18] are Hamiltonian.

It is well-known that almost all graphs have diameter 2 [16]. From this result of J.W. Moon and L. Moser, and Yu.D. Perepeliza's Theorem, it is easy to obtain that almost all n-vertex graphs of diameter 2 are Hamiltonian (see, for detail, Section 2). In this regard, the question naturally arises about a Hamiltonian property of almost all n-vertex graphs of a fixed diameter k. This problem was posed by S.V. Avgustinovich.

In the present paper, an answer to this one is obtained. Previously, the author investigated asymptotically the class of n-vertex graphs of a fixed diameter. A number of typical properties of graphs under consideration were found (for more information, see survey article [8]). In the present paper, for every Δ, $0 < \Delta < 1$, a new class $\mathcal{H}_{n,k,\Delta}$ of typical n-vertex graphs of a given diameter k is constructed in Section 2 (Theorem 2). In Section 3, based on the found typical properties and Theorem 2, we establish when almost all such graphs are Hamiltonian. It turned out that almost all n-vertex graphs of given diameter $k = 1, 2, 3$ are Hamiltonian and are nonHamiltonian for every fixed $k \geq 4$ (Theorem 4).

All obtained typical Hamiltonian properties for n-vertex graphs of a fixed diameter $k \geq 1$ remain typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) containing a shortest path of a length at least k (Corollary 2).

1. Preliminary information

The article uses the generally accepted concepts and notation of graph theory [2,12], as well as the standard concepts of combinatorial analysis [10]. We consider only finite ordinary (i.e., without loops and multiple edges) graphs $G = (V, E)$ with set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. As usual, a graph G is *s-connected* if its connectivity is at least s, a set $S \subseteq V$ is *the independence set* of graph G if all vertices in S are pairwise non-adjacent in G, *the number of independence* of a graph is the greatest cardinality of its independent sets. Let $\alpha(G)$ denote the number of independence of graph G, $B_i^G(v) = \{u \in V \mid \rho_G(v, u) \leq i\}$ is *a ball of radius i centered at a vertex* $v \in V$ in the metric space of graph G with the metric ρ_G, $S_i^G(v) = \{u \in V \mid \rho_G(v, u) = i\}$ is *a sphere of radius i centered at a vertex* $v \in V$, K_n — *a complete* n-vertex graph. For a path P with endpoints v_0 and v_n, sequentially passing through vertices $v_0, v_1, \ldots, v_n$, the notation $P(v_0, v_1, \ldots, v_n)$ is used. A shortest path of length $d(G)$ is *the diametral path* of the graph G, and under by *a pair of diametral vertices* we mean an unordered sample of two vertices from the set V, the distance between which is equal to the diameter, a vertex of degree 1 is *pendant*.

We will write $\lfloor x \rfloor$ to denote the largest integer less or equal to a real nonnegative number x and further apply the following well-known binomial identity

$$\binom{n-m}{2} = \binom{n}{2} - nm + \frac{m(m+1)}{2}\,. \tag{1}$$

To denote *the asymptotic equality* of functions $f(n)$ and $g(n)$ as n tends to infinity, we use the notation $f(n) \sim g(n)$, which by definition means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ or, equivalently, $f(n) = g(n)(1 + r(n))$ for all large enough n, where infinitesimal function $r(n)$ is *the approximation error* of $g(n)$.

To estimate the measure of the number of graphs with a certain property, the concept of *almost all* is often used; in this approach, the studied property is considered for graphs with a large number of vertices. Let $\mathcal{J}_n$ be the class of labeled n-vertex graphs with the fixed set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Consider some property $\mathcal{P}$, by which each graph may or may not possess. Through $\mathcal{J}_n^{\mathcal{P}}$ denote the set of all graphs from $\mathcal{J}_n$ that possess the property $\mathcal{P}$. *Almost all graphs possess the property* $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 1$, i.e. $|\mathcal{J}_n^{\mathcal{P}}| \sim |\mathcal{J}_n|$, and *there are almost no graphs with the property* $\mathcal{P}$, if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 0$.

In the study and selection of almost all graphs in the class of graphs under consideration it is often useful to define not characteristic properties themselves for the notion of almost all, but directly select a subclass of typical graphs itself (in [4], a more general concept of a class of typical combinatorial objects for a given class of objects admitting the concept of dimension is formulated, further we will also use this formal concept for graphs when the dimension of a graph is understood as the number of its vertices). Let Ω be an arbitrary class of graphs such that $\Omega_n \neq \varnothing$ for all large enough n, where $\Omega_n = \Omega \cap \mathcal{J}_n$. A subclass $\Omega^* \subseteq \Omega$ is *the class of typical graphs of the class* Ω if $\lim_{n\to\infty} |\Omega_n^*|/|\Omega_n| = 1$. A property of graphs of the class under consideration is *typical* if almost all graphs of this class have this property.

Let $\mathcal{J}_{n,\,d=k},\ \mathcal{J}_{n,\,d\geq k},\ \mathcal{J}^*_{n,\,d\geq k}$ be the following classes of labeled n-vertex graphs: graphs of diameter k; connected graphs of diameter at least k; graphs (not necessarily connected) with the shortest path of length at least k, respectively. In paper [5], it is proved that for $k \geq 3$ all three classes of graphs $\mathcal{J}_{n,\,d=k},\ \mathcal{J}_{n,\,d\geq k},\ \mathcal{J}^*_{n,\,d\geq k}$ have the same asymptotic cardinality, and asymptotically exact value $2^{\binom{n}{2}}\,\xi_{n,k}$ of the number of graphs in these classes is found. Here

$$\xi_{n,k} = q_k\ (n)_{k-1} \left(\frac{3}{2^{k-1}}\right)^{n-k+1}, \quad q_k = \frac{1}{2}\,(k-2)\,2^{-\binom{k-1}{2}},$$

$(n)_k = n(n-1)\cdots(n-k+1)$, $(n)_0 = (0)_0 = 1$ and $(n)_k = 0$ if $n < k$.

In [7], when studying the variety of metric balls in graphs, for every Δ, $0 < \Delta < 1$, it is defined a constant ε_Δ, depending only on Δ and $0 < \varepsilon_\Delta < 1$. Then a class $\mathcal{F}_{n,k,\Delta}$, $k \geq 3$ (the detailed definition of this class is given in Section 2) of typical graphs for the classes $\mathcal{J}_{n,\,d=k},\ \mathcal{J}_{n,\,d\geq k},\ \mathcal{J}^*_{n,\,d\geq k}$ is constructed.

Theorem 1 (asymptotics of $|\mathcal{F}_{n,k,\Delta}|$ [7])**.** *Let* $k \geq 3$*,* $0 < \Delta < 1$*,* $\varepsilon_\Delta < \varepsilon < 1$*, and* k*,* Δ*,* ε *do not depend on* n*. Then there exists a constant* $c > 0$ *independent of* n *such that for every* $n \in \mathbb{N}$ *the following inequalities are valid*

$$2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \leq |\mathcal{F}_{n,k,\Delta}| \leq |\mathcal{J}_{n,\,d=k}|$$
$$\leq |\mathcal{J}_{n,\,d\geq k}| \leq |\mathcal{J}^*_{n,\,d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 + c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big).$$

Note that for $k = 3$ the upper bound in Theorem 1 takes the form $2^{\binom{n}{2}}\xi_{n,3}$ [4]. Moreover, this upper estimate is valid even for a class of graphs containing additionally all disconnected graphs (which do not necessarily have a connected component with shortest path of length 3). Class $\mathcal{F}_{n,3,\Delta}$ is the union of the subclasses $\mathcal{F}_{n,3,\Delta}(x,y)$ over all different $x, y \in V$, and x, y is the unique pair of diametral vertices of every graph from $\mathcal{F}_{n,3,\Delta}(x,y)$. Further we use the following estimate of the number of graphs in class $\mathcal{F}_{n,3,\Delta}(x,y)$ obtained in [7].

Lemma 1 (lower bound [7]). *Let x, y be different vertices in V, Δ is arbitrary constant independent of n, and $0 < \Delta < 1$. Then the following inequality holds as n tends to infinity $|\mathcal{F}_{n,3,\Delta}(x,y) \geq a_n(1 - r(n))$, where $r(n)$ is a positive infinitesimal function and $a_n = 2^{\binom{n}{2}} \frac{8}{9} \left(\frac{3}{4}\right)^n$.*

2. Class of graphs $\mathcal{H}_{n,k,\Delta}$

For every integer $k \geq 3$ and Δ, $0 < \Delta < 1$, the class $\mathcal{F}_{n,k,\Delta}$ of typical graphs of class $\mathcal{J}_{n,\, d=k}$ was constructed by author in [7]. In this section we define a subclass $\mathcal{H}_{n,k,\Delta}$ of class $\mathcal{F}_{n,k,\Delta}$. To define this class, first consider the following properties of n-vertex graphs F of diameter 3 with vertex set V and fixed vertices $x, y \in V$.

a) Non-Pendant condition: vertices x, y are not pendant in F;

b) Existence of a *pole*: $\rho_F(z,x) = \rho_F(z,y) = 2$ for some vertex $z \in V$;

c) Property of diametral vertices: $d(F) = 3$ and graph F has the unique pair of diametral vertices x, y;

d) Nonexistence of a *shuttlecocks*: graph F does not contain shuttlecocks (subgraphs defined in [3]) or, equivalently, does not contain coinciding balls of radius 1 with centers at different vertices;

e) Property of spheres intersections:

$$|S_1^F(u) \cap S_1^F(v)| \geq \left\lfloor \frac{n}{6}\Delta \right\rfloor + 1 \;\; \forall u, v \in V \setminus \{x,y\} \text{ and } u \neq v,$$

$$|S_1^F(u) \cap S_1^F(z)| \geq \left\lfloor \frac{n}{6}\Delta \right\rfloor + 1 \;\; \forall u \in V \setminus \{x,y\} \;\; \forall z \in \{x,y\};$$

f) Property of cardinality of independence sets: $\alpha(F) < \lfloor 2\log_2 n \rfloor$.

In [7], $\mathcal{F}_{n,3,\Delta}(x,y)$ was defined for $x, y \in V$ as the class of all graphs $F \in \mathcal{J}_n$ with the properties a), b), c), d), e). Let $\mathcal{H}_{n,3,\Delta}(x,y)$ be the class of all graphs in $\mathcal{F}_{n,3,\Delta}(x,y)$ possessing property f) additionally. Now, for $k \geq 3$, we define a class $\mathcal{H}_{n,k,\Delta}$ as follows. Let $u = (u_0, u_1, \ldots, u_{k-2})$ be an arbitrary ordered sequence of different vertices from the set V. Fix an arbitrary pair of neighboring elements u_s and u_{s+1}. On the set $V \setminus \{u_0, \ldots, u_{s-1}, u_{s+2}, \ldots, u_{k-2}\}$ of $n-k+3$ vertices, define an arbitrary graph F from the class $\mathcal{H}_{n-k+3,3,\Delta}(u_s, u_{s+1})$. Finally, join by edges the vertices u_i, u_{i+1} for $i \neq s$ and $0 \leq i < k-2$. Denote the so-obtained graph by $G(u,s,F)$. Let $\mathcal{H}_{n,k,\Delta}$ be the class of all graphs $G(u,s,F)$ constructed under condition $0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor$. Note that if, in defining the graphs $G(u,s,F)$, instead of class of graphs $\mathcal{H}_{n-k+3,3,\Delta}(u_s, u_{s+1})$, we use $\mathcal{F}_{n-k+3,3,\Delta}(u_s, u_{s+1})$, then we arrive at the definition of class $\mathcal{F}_{n,k,\Delta}$ [7]. Hence, we have

$$\mathcal{H}_{n,3,\Delta}(x,y) \subseteq \mathcal{F}_{n,3,\Delta}(x,y), \;\; \mathcal{H}_{n,3,\Delta} \subseteq \mathcal{F}_{n,3,\Delta}. \tag{2}$$

Therefore, all properties of graphs $G(u,s,F)$ obtained earlier in [5, 7] will also hold for graphs of class $\mathcal{H}_{n,k,\Delta}$ or can be proven in a similar way. In particular, the properties stated in Lemmas 2 and 3 are valid.

Lemma 2 (properties of graphs $G(u,s,F)$)**.** *Let $k \geq 3$, $0 < \Delta < 1$ and $G = G(u,s,F) \in \mathcal{H}_{n,k,\Delta}$. Then the following properties hold:*

(i) $G \in \mathcal{J}_{n,d=k}$;

(ii) u_s, u_{s+1} *are not pendant vertices in* F*;*

(iii) u_0, u_{k-2} *is the unique pair of diametral vertices of graph* G *and every its diametral path contains vertices* $u_0, u_1, \ldots, u_{k-2}$.

Using Lemma 2, as in [5, 7] one can express the number of graphs of class $\mathcal{H}_{n,k,\Delta}$ through the number of graphs of class $\mathcal{H}_{n,3,\Delta}(x,y)$.

Lemma 3 (number of graphs in $\mathcal{H}_{n,k,\Delta}$)**.** *Let $k \geq 3$, $0 < \Delta < 1$. Then*

$$|\mathcal{H}_{n,k,\Delta}| = \frac{1}{2}(k-2)(n)_{k-1}|\mathcal{H}_{n-k+3,3,\Delta}(x,y)|, \text{ where } x \neq y.$$

Estimate the number of graphs in $\mathcal{H}_{n,3,\Delta}(x,y)$. For this we need the following classes of graphs and estimates of the number of such graphs obtained in Lemma 5 below. Let x, y be different elements of V, $\alpha = \lfloor 2\log_2 n \rfloor$,

$$\mathcal{S}_n(x,y) = \{G \in \mathcal{J}_n \mid B_1^G(x) \cap B_1^G(y) = \varnothing \text{ and } \alpha(F) \geq \lfloor 2\log_2 n \rfloor\},$$

and $\mathcal{S}_n(x,y;x)$, $\mathcal{S}_n(x,y;x,y)$, $\mathcal{S}_n(x,y;\varnothing)$ be the classes of n-vertex graphs $G \in \mathcal{J}_n$ such that $B_1^G(x) \cap B_1^G(y) = \varnothing$, there is an independent α-element set S and the following inclusions hold: $x \in S$, $y \notin S$; $x \in S$, $y \in S$; $x \notin S$, $y \notin S$, respectively. It is obvious that the following inclusions of the sets hold

$$\mathcal{S}_n(x,y) \subseteq \mathcal{S}_n(x,y;\varnothing) \cup \mathcal{S}_n(x,y;x) \cup \mathcal{S}_n(x,y;y) \cup \mathcal{S}_n(x,y;x,y). \tag{3}$$

Lemma 4. *Let $\lambda > 0$, λ does not depend on n and $\alpha = \lfloor 2\log_2 n \rfloor$. Then the following equality is fulfilled as n tends to infinity*

$$\binom{n}{\alpha} 2^{-\binom{\alpha}{2}} = \frac{1}{\lambda^\alpha \sqrt{\alpha}} O(1).$$

Proof. Using Stirling's formula, we obtain

$$\binom{n}{\alpha} = \frac{n^\alpha}{\alpha!} O(1) = \left(\frac{ne}{\alpha}\right)^\alpha \frac{1}{\sqrt{\alpha}} O(1). \tag{4}$$

Using the inequality $\lfloor x \rfloor \geq x - 1$, we obtain

$$2^{-\binom{\alpha}{2}} = 2^{-\alpha(\log_2 n - 1)} O(1) = \left(\frac{2}{n}\right)^\alpha O(1). \tag{5}$$

From (4),(5) we conclude

$$\binom{n}{\alpha} 2^{-\binom{\alpha}{2}} = \left(\frac{2e}{\alpha}\right)^\alpha \frac{1}{\sqrt{\alpha}} O(1) = \left(\frac{1}{\lambda}\right)^\alpha \frac{1}{\sqrt{\alpha}} O(1).$$

□

Lemma 5. *Let x, y be different vertices of V, $q > 1$, q does not depend on n, and $\alpha = \lfloor 2\log_2 n \rfloor$. Then the following equalities are fulfilled as n tends to infinity*

(i) $\mathcal{S}_n(x, y; x) = a_n(\frac{1}{q})^\alpha \frac{1}{\sqrt{\alpha}} O(1)$;

(ii) $\mathcal{S}_n(x, y; x, y) = a_n(\frac{1}{q})^\alpha \frac{1}{\sqrt{\alpha}} O(1)$;

(iii) $\mathcal{S}_n(x, y; \varnothing) = a_n(\frac{1}{q})^\alpha \frac{1}{\sqrt{\alpha}} O(1)$;

Proof. (i) From the definition of class $\mathcal{S}_n(x, y; x)$, it is easy to understand that all graphs of this class are contained among graphs G constructed as follows:

1) choose an $(\alpha - 1)$-element subset $S \subseteq V \setminus \{x, y\}$, there are $\binom{n-2}{\alpha-1}$ possibilities. In graph G, the vertices of set $S \cup \{x\}$ will remain pairwise non-adjacent, resulting in $S \cup \{x\}$ being an α-element independent set;

2) choose an i-element subset $V_x \subseteq V \setminus (S \cup \{x, y\})$, $0 \le i \le n - 1 - \alpha$, and join each vertex from V_x by an edge with x, as a result we have $S_1^G(x) = V_x$;

3) choose a j-element subset of $V_y \subseteq V \setminus (V_x \cup \{x, y\})$, $0 \le j \le n - 2 - i$ and join each vertex from V_y by an edge with y, as a result we obtain $S_1^G(y) = V_y$ and $V_x \cap V_y = \varnothing$;

4) on $(n - 2)$-element set $V \setminus \{x, y\}$ define an arbitrary graph in which there are no $\binom{\alpha-1}{2}$ edges between the vertices of the set S.

Thus, using the Newton's Binomial Theorem, the binomial identity (1), Lemma 4 and the inequality $\alpha \le \lfloor \frac{n}{2} \rfloor$, valid for all large enough n, we obtain as $n \to \infty$

$$
\begin{aligned}
|\mathcal{S}_n(x, y; x)| &= \binom{n-2}{\alpha-1} \sum_{i=0}^{n-1-\alpha} \binom{n-1-\alpha}{i} \sum_{j=0}^{n-2-i} \binom{n-2-i}{j} 2^{\binom{n-2}{2} - \binom{\alpha-1}{2}} O(1) \\
&= 2^{\binom{n-2}{2}} \binom{n}{\alpha} 2^{-\binom{\alpha}{2}+\alpha} \sum_{i=0}^{n-1-\alpha} \binom{n-1-\alpha}{i} 2^{n-2-i} O(1) \\
&= 2^{\binom{n}{2}} \left(\frac{1}{4}\right)^n 4^\alpha \binom{n}{\alpha} 2^{-\binom{\alpha}{2}} \sum_{i=0}^{n-1-\alpha} \binom{n-1-\alpha}{i} 2^{n-1-\alpha-i} O(1) \\
&= 2^{\binom{n}{2}} \left(\frac{3}{4}\right)^n \left(\frac{4}{3}\right)^\alpha \binom{n}{\alpha} 2^{-\binom{\alpha}{2}} O(1) \\
&= a_n \left(\frac{1}{q}\right)^\alpha \frac{1}{\sqrt{\alpha}} O(1).
\end{aligned}
$$

(ii) Similarly to the proof of the statement (i), we construct graphs forming a superclass of the class $\mathcal{S}_n(x,y;x,y)$ and obtain the following estimates

$$\begin{aligned}|\mathcal{S}_n(x,y;x,y)| &= \binom{n-2}{\alpha-2}\sum_{i=0}^{n-\alpha}\binom{n-\alpha}{i}\sum_{j=0}^{n-\alpha-i}\binom{n-\alpha-i}{j}2^{\binom{n-2}{2}-\binom{\alpha-2}{2}}O(1)\\ &= 2^{\binom{n-2}{2}}3^{n-\alpha}\binom{n}{\alpha}2^{-\binom{\alpha-2}{2}}O(1)\\ &= 2^{\binom{n}{2}}\left(\frac{3}{4}\right)^n\left(\frac{4}{3}\right)^\alpha\binom{n}{\alpha}2^{-\binom{\alpha}{2}}O(1)\\ &= a_n\left(\frac{1}{q}\right)^\alpha\frac{1}{\sqrt{\alpha}}O(1).\end{aligned}$$

(iii) The estimate of the number of graphs of class $\mathcal{S}_n(x,y;\varnothing)$ is proved similarly:

$$\begin{aligned}|\mathcal{S}_n(x,y;\varnothing)| &= \binom{n-2}{\alpha}\sum_{i=0}^{n-2}\binom{n-2}{i}\sum_{j=0}^{n-2-i}\binom{n-2-i}{j}2^{\binom{n-2}{2}-\binom{\alpha}{2}}O(1)\\ &= 2^{\binom{n}{2}}\left(\frac{3}{4}\right)^n\binom{n}{\alpha}2^{-\binom{\alpha}{2}}O(1)=a_n\left(\frac{1}{q}\right)^\alpha\frac{1}{\sqrt{\alpha}}O(1).\end{aligned}$$

□

Lemma 6. *Let x,y be different vertices in V, $0<\Delta<1$ and Δ is arbitrary constant independent of n. Then $|\mathcal{H}_{n,3,\Delta}(x,y)\geq a_n(1-r(n))$ as n tends to infinity, where $r(n)$ is a positive infinitesimal function.*

Proof. Directly from the class definitions we obtain

$$\mathcal{F}_{n,3,\Delta}(x,y)\setminus\mathcal{S}_n(x,y)\subseteq\mathcal{H}_{n,3,\Delta}(x,y).$$

Hence, $|\mathcal{H}_{n,3,\Delta}(x,y)|\geq|\mathcal{F}_{n,3,\Delta}(x,y)|-|\mathcal{S}_n(x,y)|$. It remains to apply Lemmas 1, 5 and relation (3). □

Lemma 7 (lower bound)**.** *Let $k\geq 3$ and $0<\Delta<1$ are constants independent of n. Then the following inequality holds as n tends to infinity*

$$|\mathcal{H}_{n,k,\Delta}|\geq 2^{\binom{n}{2}}\xi_{n,k}(1-r(n)),$$

where $r(n)$ is a positive infinitesimal function.

Proof. Using Lemmas 3 and 6, the definitions of numbers a_n and $\xi_{n,k}$, and the binomial identity (1), we conclude

$$\begin{aligned}|\mathcal{H}_{n,k,\Delta}| &\geq \frac{1}{2}(k-2)(n)_{k-1}2^{\binom{n-k+3}{2}}\frac{8}{9}\left(\frac{3}{4}\right)^{n-k+3}(1-r(n))\\ &= 2^{\binom{n}{2}}q_k(n)_{k-1}2^{-(n-k+1)(k-3)}\left(\frac{3}{4}\right)^{n-k+1}(1-r(n))\\ &= 2^{\binom{n}{2}}\xi_{n,k}(1-r(n)).\end{aligned}$$

□

The following theorem follows directly from Lemma 7, relation (2) and Theorem 1.

Theorem 2 (asymptotics of $|\mathcal{H}_{n,k,\Delta}|$). *Let $k \geq 3$, $0 < \Delta < 1$ and k, Δ do not depend on n. Then the following inequalities hold as n tends to infinity*

$$2^{\binom{n}{2}}\xi_{n,k}(1 - r_1(n)) \leq |\mathcal{H}_{n,k,\Delta}| \leq |\mathcal{F}_{n,k,\Delta}| \leq |\mathcal{J}_{n,\,d=k}| \leq 2^{\binom{n}{2}}\xi_{n,k}(1 + r_2(n)).$$

Here $r_1(n)$, $r_2(n)$ are positive infinitesimal functions.

Corollary 1. *Let $k \geq 3$ and $0 < \Delta < 1$ be independent of n. Then $\mathcal{H}_{n,k,\Delta}$ is the class of typical graphs of the class of n-vertex graphs of diameter k and the following asymptotic equalities hold as $n \to \infty$*

$$|\mathcal{H}_{n,k,\Delta}| \sim |\mathcal{F}_{n,k,\Delta}| \sim |\mathcal{J}_{n,\,d=k}| \sim 2^{\binom{n}{2}}\xi_{n,k}.$$

3. Hamiltonian property of almost all graphs of diameter k

Note that K_n is the unique n-vertex graph of diameter $k = 1$. For $n \geq 3$ its Hamiltonian cycle is constructed by graph vertex traversal and K_2 is nonhamiltonian. Therefore, almost all graphs of diameter $k = 1$ are Hamiltonian. A similar fact for graphs of diameter 2 also trivially follows from well-known theorems.

Lemma 8. *Almost all graphs of diameter 2 are Hamiltonian.*

Proof. Through $\mathcal{K}^H$ denote the set of all Hamiltonian graphs from class $\mathcal{K}$. It is well known that almost all graphs are Hamiltonian [17] and almost all graphs have diameter 2 [16]. Thus,

$$|\mathcal{J}_n^H| \sim |\mathcal{J}_n| \sim |\mathcal{J}_{n,\,d=2}|.$$

Hence, as $n \to \infty$ we infer

$$\frac{|\mathcal{J}^H_{n,\,d=2}|}{|\mathcal{J}_{n,\,d=2}|} = 1 - \frac{|\mathcal{J}_{n,\,d=2}| - |\mathcal{J}^H_{n,\,d=2}|}{|\mathcal{J}_{n,\,d=2}|} \geq\ 1 - \frac{|\mathcal{J}_n| - |\mathcal{J}^H_n|}{|\mathcal{J}_n|}\ \frac{|\mathcal{J}_n|}{|\mathcal{J}_{n,\,d=2}|} \longrightarrow 1.$$

□

Now for $k = 3$, let us investigate Hamiltonian property for graphs in the classes of typical graphs $\mathcal{F}_{n,k,\Delta}$ and $\mathcal{H}_{n,k,\Delta}$. Obviously, 2-connectivity is a necessary condition for a graph to be Hamiltonian. Note that not every graph of class $\mathcal{F}_{n,3,\Delta}$ is 2-connected. Indeed, using graph properties a), b) and c), it is easy to see that $\mathcal{F}_{n,3,\Delta} = \varnothing$ if $n < 7$. Let us consider graph F_n^1 shown in Fig. 1. Given the equality $|S_1^{F_n^1}(x) \cap S_1^{F_n^1}(x_1)| = 1$, it is not difficult

to understand that $F_n^1 \in \mathcal{F}_{n,3,\Delta}$ if and only if $7 \leq n < 6/\Delta$. At the same time, for all admissible values of n and Δ, $F_n^1 \in \mathcal{F}_{n,3,\Delta}$ and graph F_n^1 is not 2-connected, since there are no two vertex-disjoint paths connecting its diametral vertices x and y.

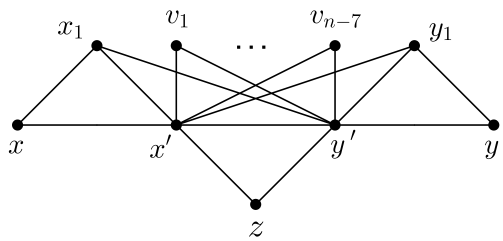

FIG. 1. Graph F_n^1

Lemma 9. *For $n \geq 6/\Delta$ and $0 < \Delta < 1$, all graphs in class $\mathcal{F}_{n,3,\Delta}$ are 2-connected.*

Proof. Let $G \in \mathcal{F}_{n,3,\Delta}$ and $n \geq 6/\Delta$. Then $|S_1^G(u) \cap S_1^G(v)| \geq 2$ if $\{u, v\} \neq \{x, y\}$ due to the property of spheres intersections. By Whitney's Theorem (see, for example, [12]), it sufficient to connect every two vertices u, v by two vertex-disjoint paths.

Let $P(x, x_1, y_1, y)$ is a diametral path of graph G. If u, v are not a pair of diametral vertices then there are exist different vertices $w_1, w_2 \in S_1^G(u) \cap S_1^G(v)$. Hence, $P(u, w_i, v)$, $i = 1, 2$, are two vertex-disjoint paths. Therefore, we further assume that $\{u, v\} = \{x, y\}$. Vertex x is not pendant. Hence, there is exist a vertex $x_2 \in V \setminus \{x_1\}$ adjacent to x. Note that $x_2 \notin \{x, x_1, y_1, y\}$, otherwise $\rho_G(x, y) \leq 2$. Further, there is exist a vertex $y_2 \in (S_1^G(x_2) \cap S_1^G(y)) \setminus \{y_1\}$. Similarly, we have $y_2 \notin \{x, x_2, x_1, y_1, y\}$. Thus, $P(x, x_1, y_1, y)$, $P(x, x_2, y_2, y)$ are required two vertex-disjoint paths. □

Note that not every 2-connected graph in class $\mathcal{F}_{n,3,\Delta}$ is Hamiltonian. Indeed, using the properties of graphs of class $\mathcal{F}_{n,3,\Delta}$, it is easy to prove that $\mathcal{F}_{n,3,\Delta} = \varnothing$ if $6/\Delta \leq n < 9$ (see also [6], page 350). Let us consider graph F_n^2 shown in Fig. 2 for $n \geq 9$. Obviously, F_n^2 is a 2-connected graph. Moreover,

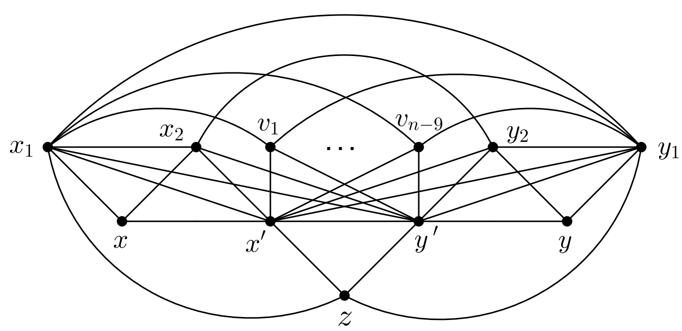

FIG. 2. Graph F_n^2

$|S_1^{F_n^2}(x) \cap S_1^{F_n^2}(x_2)| = 2$. Now, it is not difficult to prove that $F_n^2 \in \mathcal{F}_{n,3,\Delta}$ if and only if $9 \leq n < 12/\Delta$. Note that graph F_n^2 is Hamiltonian for $n = 9, 10, 11$ (as example, the Hamiltonian cycle of graph F_{11}^2 is shown in Fig. 3) and it is nonHamiltonian if $n = 12$. Besides, in all these cases $F_n^2 \in \mathcal{F}_{n,3,\Delta}$ for any Δ, $0 < \Delta < 1$. In addition, $F_n^2 \in \mathcal{F}_{n,3,\Delta}$ and F_n^2 is nonHamiltonian, for example, if $n = 13, 14, \ldots, 18$ and $\Delta < 12/n$. The verification of the Hamiltonian property for the above graphs was performed on a computer, you can also use the online service [11] or similar.

Now we will show that all graphs of class $\mathcal{F}_{n,3,\Delta}$ have sufficiently high connectivity.

Lemma 10. *Let Δ does not depend on n and $0 < \Delta < 1$. Then all graphs in class $\mathcal{F}_{n,3,\Delta}$ are $\varkappa$-connected, where $\varkappa = \lfloor n\,\Delta/18 \rfloor$.*

Proof. Without loss of generality, we assume $\varkappa \geq 1$. Let $G \in \mathcal{F}_{n,3,\Delta}$ and x, y is the unique pair of diametral vertices of graph G. By Whitney's Theorem, it is sufficient to connect every two vertices u, v by $\varkappa$ vertex-disjoint paths.

Let $\rho_G(u, v) < 3$. Then $|S_1^G(u) \cap S_1^G(v)| \geq \frac{n}{6}\Delta > \varkappa$ by the property of sphere intersections. So further we assume $\{u, v\} = \{x, y\}$. Using the property of sphere intersections we obtain the following relations

$$\begin{gathered} B_1^G(x) \cap B_1^G(y) = \varnothing, \\ |S_1^G(x)| \geq \tfrac{n}{6}\Delta,\ |S_1^G(y)| \geq \tfrac{n}{6}\Delta, \\ |S_1^G(x') \cap S_1^G(y')| \geq \tfrac{n}{6}\Delta,\ \text{ if } x' \in S_1^G(x) \text{ and } y' \in S_1^G(y). \end{gathered} \tag{6}$$

We will construct step by step a sequence of pairwise disjoint 3-element sets $V_i = \{x_i, y_i, z_i\}$ such that $P_i(x, x_i, y_i, y)$ is a simple path of graph G.

Step 1. Consider arbitrary vertices $x_1 \in S_1^G(x)$ and $y_1 \in S_1^G(y)$. By virtue of (6) there is a vertex $z_1 \in S_1^G(x_1) \cap S_1^G(y_1)$. Using (6) and property c) for graph G, it is also easy to see that graph G contains a simple path $P_1(x, x_1, z_1, y_1, y)$. We define $V_1 = \{x_1, y_1, z_1\}$.

Step $i + 1$. Let the sets $V_1, \ldots, V_i$ be constructed and $i < \varkappa$. Then

$$|\bigcup_{s=1}^{i} V_s| = 3i < 3\varkappa \leq \frac{n}{6}\Delta. \tag{7}$$

By virtue of (6) and (7) the following vertices exist:

$$x_{i+1} \in S_1^G(x) \setminus \cup_{s=1}^{i} V_s,$$

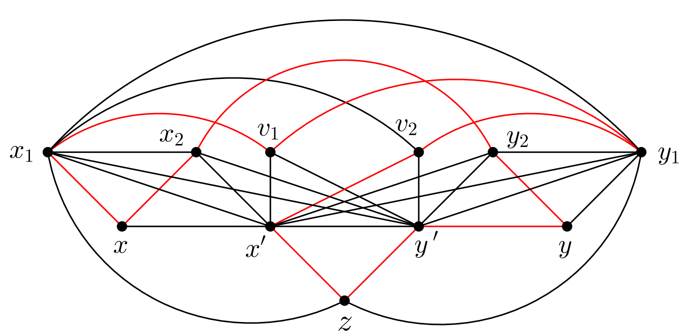

FIG. 3. Hamiltonian cycle of graph $F_{11}^2 \in \mathcal{F}_{11,3,\Delta}$

$$y_{i+1} \in S_1^G(y) \setminus \cup_{s=1}^{i} V_s,$$
$$z_{i+1} \in S_1^G(x_{i+1}) \cap S_1^G(y_{i+1}) \setminus \cup_{s=1}^{i} V_s.$$

We define $V_{i+1} = \{x_{i+1}, y_{i+1}, z_{i+1}\}$. Using property c) for graph G, we obtain $P_{i+1}(x, x_{i+1}, z_{i+1}, y_{i+1}, y)$ is a simple path of graph G.

Thus, at step $\varkappa$, the vertex-disjoint simple paths $P_i, i = 1, \ldots, \varkappa$ with endpoints x, y will be constructed. □

Let us turn to graphs of class $\mathcal{H}_{n,k,\Delta}$. We apply the following sufficient condition of V.Chvátal and P.Erdös for a graph to be Hamiltonian.

Theorem 3 (V.Chvátal and P.Erdös [1]). *Let G be a graph with at least three vertices. If for some s, graph G is s-connected and $\alpha(G) \leq s$, then G has a Hamiltonian cycle.*

Lemma 11. *Let $k \geq 3$. Then all graphs of class $\mathcal{H}_{n,k=3,\Delta}$ are Hamiltonian for all large enough n and every graph in $\mathcal{H}_{n,k,\Delta}$ is nonHamiltonian if $k \geq 4$.*

Proof. There is an integer $N > 0$ such that for all $n \geq N$ the following inequality holds $\lfloor 2\log_2 n \rfloor - 1 \leq \lfloor n\,\Delta/18 \rfloor$. Let $n \geq N$, $G \in \mathcal{H}_{n,3,\Delta}$ and $s = \lfloor 2\log_2 n \rfloor - 1$. By the property of cardinality of independence sets, $\alpha(G) \leq s$. In addition, G is s-connected due to (2) and Lemma 10. Therefore, graph G is Hamiltonian by Theorem 3.

It remains to note that for $k \geq 4$ every graph in class $\mathcal{H}_{n,k,\Delta}$ contains a pendant vertex due to the definition of graph $G(u, s, F)$. □

Lemma 8, Corollary 1 and Lemma 11 imply the following theorem.

Theorem 4. *Almost all n-vertex graphs of fixed diameter $k = 1, 2, 3$ are Hamiltonian, while almost all n-vertex graph of fixed diameter $k \geq 4$ are nonHamiltonian graphs.*

By Theorem 1, for $k \geq 3$ all three classes of graphs $\mathcal{J}_{n,\, d=k}$, $\mathcal{J}_{n,\, d\geq k}$, $\mathcal{J}^*_{n,\, d\geq k}$ have the same asymptotic cardinality. Therefore, we obtain the following corollary.

Corollary 2. *For every fixed $k = 1, 2, 3$, almost all n-vertex graphs of each of the following classes $\mathcal{J}_{n,\, d\geq k}$, $\mathcal{J}^*_{n,\, d\geq k}$ are Hamiltonian, while almost all n-vertex graphs of these classes are nonHamiltonian for every fixed $k \geq 4$.*

4. Conclusion

Note that the existence of a Hamiltonian cycle in Theorem 3 is based on Dirac's generalization of Theorem of Menger on s-connected graphs. This requires considering a large variety of paths to construct s vertex-disjoint paths starting at a given vertex x and terminating in a given cycle C if $x \notin V(C)$. This makes this method of constructing a Hamiltonian cycle algorithmically complex. In this connection, a fairly effective method for constructing a Hamiltonian cycle for almost all n-vertex graphs of diameter 3 is of further interest.

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