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BOUNDARY OPTIMAL CONTROL OF HEAT-CONDUCTING GAS FLOW UNDER RADIATION EXCHANGE CONDITIONS

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Dedicated to 85th birthday of academician Vladimir G. Romanov

Abstract: The problem of controlling one-dimensional viscous gas flow through an interval with a fixed boundary is considered. The flow regime takes into account complex convective conductive radiative heat exchange in the medium. The heat transfer coefficient and the reflection coefficient at the boundaries are chosen to be controls. The existence of optimal control is proved. The necessary conditions for the optimality system are derived. A numerical solution to the optimal control problem is calculated using the Physics Informed Neural Network (PINN) method. The method involves approximating an unknown function with a neural network by minimizing a quadratic functional that includes terms for the residuals of equations, boundary and initial conditions, and additional information. The method avoids the need for linearization and solving optimality systems. The functions of velocity, density, temperature, and radiation intensity are sought for boundary control with two coefficients on the left boundary and two coefficients on the right boundary. All unknowns are approximated by neural networks. The temperatures at observation points match the specified temperature using optimal control of the boundary coefficients.

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The case with observation points inside the region and on the boundaries is considered.

Keywords: Inverse problem, heat and mass transfer, radiation heat exchange, Navier-Stokes equations for a compressible medium, Physics Informed Neural Network.

1 Introduction

In this paper, we study a system of equations for the one-dimensional flow of a viscous compressible gas taking into account radiative, conductive and convective heat transfer. For the case of one spatial variable, the model of a viscous heat-conducting gas under radiative exchange conditions in a limited region $\Omega_0 \subset \mathbb{R}$ is modeled in normalized form by the following system, where the $P1$ (diffusion) approximation is used for the radiation transfer equation [1–3]:

$$\rho(u_t + uu_x) = \nu u_{xx} - R(\rho\,\theta)_x,$$
$$\rho_t + u\rho_x + \rho u_x = 0,$$
$$\rho(\theta_t + u\theta_x) = a\theta_{xx} + (\nu u_x - R\rho\,\theta)u_x - b\,k_\alpha(|\theta|\theta^3 - \varphi),$$
$$-\alpha\varphi_{xx} + k_\alpha(\varphi - |\theta|\theta^3) = 0.$$

The flow of the gas through the interval $\Omega_0 = \{x\colon 0 < x < L_0\}$ with permeable fixed boundaries is considered. At the initial moment of time, the characteristics of the medium are known:

$$u|_{t=0} = u_0(x), \quad \rho|_{t=0} = \rho_0(x) > 0, \quad \theta|_{t=0} = \theta_0(x), \quad x \in \bar{\Omega}_0.$$

At $t > 0$ the flow region is bounded by two boundaries. Gas flows through the left boundary $u|_{x=0} > 0$. Then the conditions for the velocity, temperature, radiation intensity, and density of the medium are set on the left boundary:

$$u|_{x=0} = u_1(t), \quad \rho|_{x=0} = \rho_1(t),$$
$$-a\frac{\partial\theta}{\partial x}\Big|_{x=0} + \beta(\theta|_{x=0} - \theta_1) = 0, \quad -\alpha\varphi_x\big|_{x=0} + \gamma(\varphi|_{x=0} - \theta_1^4) = 0.$$

The gas flows out through the right boundary. Therefore, only the velocity, temperature and radiation intensity of the medium are specified on the right boundary:

$$u|_{x=L_0} = u_2(t),$$
$$a\frac{\partial\theta}{\partial x}\Big|_{x=L_0} + \beta(\theta|_{x=L_0} - \theta_2) = 0, \quad \alpha\frac{\partial\varphi}{\partial x}\Big|_{x=L_0} + \gamma(\varphi|_{x=L_0} - \theta_2^4) = 0.$$

The coefficient γ describes the reflective properties of the boundary, β is the heat transfer coefficient.

The equations describing the processes of convective-conductive transfer of thermal radiation of an incompressible medium are considered in works [4–7] and are well studied. The problems of complex heat exchange in scattering media with reflecting boundaries are presented in works [8–17].

The behavior of solutions of the Navier–Stokes equations «in general» over time for a compressible medium has been exhaustively studied only in the one-dimensional case. The analysis of various boundary value problems associated with the flow of viscous gas is considered in [18, 19].

At the same time, the questions of correctness of initial-boundary value problems for the model of viscous heat-conducting gas, taking into account radiative heat exchange inside the region, as well as the analysis of the stability of stationary solutions, are open. The correctness of the model of viscous heat-conducting gas under conditions of radiative exchange in a limited region is studied in [20].

Problems of optimal control of viscous fluid flow in the one-dimensional case, where the characteristics of the medium were chosen as control, were considered in [21–24].

The questions of correctness in Sobolev spaces of inverse problems on determination of the coefficient in Robin type boundary condition for the convection-diffusion equation with observation point overdetermination conditions are studied in [25]. Based on the reduction of the problem to the Volterra integral equation of the second kind, a theorem of existence and uniqueness of the inverse problem is obtained.

In this paper, we study the correctness of the inverse problem of determining the properties of a medium at the boundary of a domain under conditions of complex radiative heat and mass transfer so that the gas temperature at fixed points of the boundary or inside the domain would take specified values at all moments of time $t \in [0, T]$. A numerical solution of the inverse problem is presented, obtained using a Physics Informed Neural Network method.

2 Problem formulation

When studying problems of gas dynamics, it is convenient to use Lagrangian coordinates. According to the transition formulas [26] the interval $(0, L_0)$ with fixed boundaries in Euler coordinates in the new coordinates will go over to a domain with time-varying boundaries that preserve the length of the interval at each moment of time. We denote by

$$L = \int\limits_0^{L_0} \rho_0(x)\, dx, \quad L \neq 0,$$

where ρ_0 – is the gas density at time $t = 0$. The images of the boundaries $x = 0$ and $x = L_0$ in the new variables will be the functions

$$a(t) = -\int\limits_0^t u_l(\tau)\rho_l(\tau)\, d\tau, \quad b(t) = L - \int\limits_0^t u_r(\tau)\rho_r(\tau)\, d\tau, \tag{1}$$

where $\{u_\ell,\, \rho_\ell\}$, $\ell = l, r$ – are velocity and density of the gas at the boundary points $x = 0$ and $x = L_0$, respectively. The density value at the right boundary $\rho_r(t) = \rho(x,t)|_{x=L_0}$ is determined from the equality

$$\rho_l u_l = \rho_r u_r = m(t) > 0, \quad t \geq 0, \tag{2}$$

which is a consequence of the continuity equation for a compressible medium.

The domain of change of the environment at $t > 0$ in the new coordinates is denoted by

$$Q = \{(x;t)\colon 0 < t < T;\ x \in \Omega_t\}, \quad \Omega_t = \{x\colon a(t) < x < b(t)\}, \tag{3}$$

and at $t = 0$ – respectively through the interval $\Omega_0 = \{x\colon 0 < x < L\}$.

In mass Lagrangian variables, the problem of the flow of a viscous heat-conducting gas under conditions of radiative exchange in a limited region Q has the following form [1]:

$$\begin{gathered} u_t = \nu(\rho u_x)_x - R(\rho\,\theta)_x, \quad \rho_t + \rho^2 u_x = 0, \\ \theta_t = a(\rho\theta_x)_x + (\nu\rho u_x - R\rho\,\theta)u_x - bk_\alpha(\theta^4 - \varphi)\rho^{-1}, \\ -\alpha(\rho\varphi_x)_x + k_\alpha(\varphi - \theta^4)\rho^{-1} = 0. \end{gathered} \tag{4}$$

Here u, ρ, θ – are velocity, density, and normalized temperature of a perfect gas, respectively, the pressure is determined from the Clapeyron equation $p = R\rho\,\theta$, the function φ is interpreted as the normalized radiation intensity. Through ν, R we denote positive physical constants characterizing the medium, ν – is the dynamic viscosity coefficient, R – is the gas constant. The constants b, a, α describe the radiation-thermal properties of the medium, k_α – is the absorption coefficient.

At the initial moment of time, the characteristics of the environment are known

$$u|_{t=0} = u_0(x), \quad \theta|_{t=0} = \theta_0(x), \quad \rho|_{t=0} = \rho_0(x) > 0, \quad x \in \Omega_0. \tag{5}$$

At $t > 0$ the region is bounded by two boundaries. Gas flows in through the left boundary:

$$u|_{x=a(t)} = u_a(t), \quad \rho|_{x=a(t)} = \rho_a(t), \quad 0 \leq t \leq T. \tag{6}$$

Gas flows out through the right boundary:

$$u|_{x=b(t)} = u_b(t), \quad 0 \leq t \leq T. \tag{7}$$

The conditions for temperature and radiation intensity at the boundary take into account the influence of external factors and are described using the Newton-Richmann law:

$$\begin{gathered} -a\rho\theta_x + \beta(\theta - \theta_b) = 0, \quad -\alpha\rho\varphi_x + \gamma(\varphi - \theta^4), \quad x = a(t),\ t \in (0,T) \\ a\rho\theta_x + \beta(\theta - \theta_b) = 0, \quad \alpha\rho\varphi_x + \gamma(\varphi - \theta_b^4) = 0, \quad x = b(t),\ t \in (0,T). \end{gathered} \tag{8}$$

We will consider the reflection process function γ, which takes the values

$$\gamma = \gamma_l(t) \quad \text{at} \quad x = a(t),$$
$$\gamma = \gamma_r(t) \quad \text{at} \quad x = b(t)$$

and the function of the heat transfer process in the form:

$$\beta = \beta_a(t) = \beta_{0a}(t) + \beta_l(t)\Lambda_a(t) \quad \text{at} \quad x = a(t),$$

$$\beta = \beta_b(t) = \beta_{0b}(t) + \beta_r(t)\Lambda_b(t) \quad \text{at} \quad x = b(t). \tag{9}$$

Given the values ρ_0, u_a, ρ_a, u_b, $\gamma_{l,r}$, $\beta_{a,b}$ the problem (4)–(8) is called a direct problem.

Let us formulate an optimal control problem. Let u_0, ρ_0, θ_0, u_a, ρ_a, u_b and the overdetermination conditions be given:

$$\theta(x,t)|_{x=a(t)} = d_a(t), \quad \theta(x,t)|_{x=b(t)} = d_b(t). \tag{10}$$

It is required to find the state u, ρ, θ, φ, both the solution of the system (4)–(8), (10), and the unknown functions $\gamma_l(t)$, $\gamma_r(t)$, $\beta_l(t)$, $\beta_r(t)$. Moreover, the reflection coefficients at the ends of the interval, the functions $\gamma_l(t)$, $\gamma_r(t)$ are selected from a limited set, and $\beta_l(t)$, $\beta_r(t)$ satisfy (9), where β_{0a}, β_{0b}, Λ_a, Λ_b are considered given, and the functions β_a, β_b are positive.

3 Solvability of the optimal control problem

Below we will use the usual notation $L^p(W_p^l)$ for spaces of functions integrable with degree $p \geq 1$ (together with generalized derivatives up to order $l \geq 0$). By $L^2(0,T;X)$ we denote the space of measurable functions (the space of continuous functions with continuous derivatives in $[0,T]$ up to order l) mapping the interval ($[0,T]$) ($[0,T]$) to the space X such that

$$\|f\|^2_{L^2(0,T;X)} = \int\limits_0^T \|f\|^2_X \, dt < \infty, \quad \|f\|_{C^l([0,T];X)} = \max_{0\leq t\leq T} \|f\|_X < \infty.$$

By $H^s(X)$ we will denote the space $W_2^s(X)$, $s > 0$, respectively, $\tilde{H}^{-s}(X) = (H^s(X))'$ — the space conjugate to $H^s(X)$,

$$H^{2,1} = \{q\colon q \in L^\infty(0,T;H^1(\Omega_t)) \cap L^2(0,T;H^2(\Omega_t)),\ q_t \in L^2(0,T;L^2(\Omega_t)\},$$
$$H^{1,1} = \{q\colon q \in L^\infty(0,T;L^2(\Omega_t)) \cap L^2(0,T;H^1(\Omega_t))\colon q_t \in L^2(0,T;L^2(\Omega_t)\},$$
$$H^{1,0} = \{q\colon q \in L^\infty(0,T;L^2(\Omega_t)) \cap L^2(0,T;H^1(\Omega_t))\},$$
$$H = L^2(0,T) \times L^2(0,T) \times H^1(0,T) \times H^1(0,T). \tag{11}$$

The following properties of embeddings take place:

$$H^{2,1} \subset L^2(0,T;H^1(\Omega_t)) \quad \text{continuous and compact,}$$
$$H^{2,1} \subset C(\overline{Q}) \quad \text{continuous.} \tag{12}$$

Let's consider the spaces

$$W = \{q_1,\, q_2,\, q_3,\, q_4\colon q_1 \in H^{1,0},\ q_1|_{x=a(t)} = 0,\ q_1|_{x=b(t)} = 0;$$
$$q_2 \in L^\infty(0,T;L^2(\Omega_t)),\ q_3 \in H^{1,0};\ q_4 \in H^{1,0}\},$$
$$Y = \{q_1,\, q_2,\, q_3,\, q_4\colon q_1 \in H^{2,1};\ q_2 \in H^{1,1};\ q_3 \in H^{2,1};\ q_4 \in L^2(0,T;H^2(\Omega_t))\}. \tag{13}$$

Definition 1. A strong solution of problem (4) – (9) is a set of functions $\{u,\rho,\theta,\varphi\} \in Y$ that satisfies equations (4) almost everywhere in $(0,T) \times \Omega_t$ and takes boundary and initial values (5)–(8) in the sense of traces of functions from the specified classes.

Let the following conditions be satisfied:

$$u_a, u_b, \rho_a \in H^1(0,T), \quad u_0, \theta_0 \in H^1(\Omega), \quad \rho_0 \in L^\infty(\Omega),$$
$$0 < m_0 \le \rho_0 \le M_0 < \infty, \quad \rho_a > 0, \quad u_a > 0; \tag{14}$$
$$\gamma_{1a},\, \gamma_{2a},\, \gamma_{1b},\, \gamma_{2b} \in H^1(0,T),$$
$$\beta_{0a},\, \beta_{0b},\, \Lambda_a,\, \Lambda_b \in L^\infty(0,T), \quad \theta_b \in H^1(0,T), \tag{15}$$
$$\beta_{0a}(t) \ge 0, \quad \beta_{0b}(t) \ge 0, \quad \beta_a \ge \beta_{min} > 0, \quad \beta_b \ge \beta_{min} > 0,$$
$$\Lambda_a(t) \ge \Lambda_{min} > 0, \quad \Lambda_b(t) \ge \Lambda_{min} > 0,$$
$$\gamma_{2a}(t) \ge \gamma_{1a}(t) \ge \gamma_{min} > 0, \quad \gamma_{2b}(t) \ge \gamma_{1b}(t) \ge \gamma_{min} > 0. \tag{16}$$

Here the constants β_{min}, $\Lambda_{\min}$, γ_{min}, m_0, M_0 are given.

Let us define the set of admissible controls $U_{ad} = U_\beta \times U_\gamma$, where

$$U_\beta = \big\{\mathbf{y}_\beta = \{\beta_l,\, \beta_r\} \colon \beta_l \in L^2(0,T), \quad \beta_r \in L^2(0,T)$$
$$\beta_l \ge (\beta_{min} - \min_{t\in(0,T)} \beta_{0a})/\Lambda_a(t), \quad \beta_r \ge (\beta_{min} - \min_{t\in(0,T)} \beta_{0b})/\Lambda_b(t)\ \big\},$$
$$U_\gamma = \big\{\mathbf{y}_\gamma = \{\gamma_l, \quad \gamma_r\} \colon \gamma_l \in H^1(0,T), \quad \gamma_r \in H^1(0,T),$$
$$\gamma_{1a} \le \gamma_l \le \gamma_{2a}, \quad \gamma_{1b} \le \gamma_r \le \gamma_{2b}\big\}. \tag{17}$$

Note that U_β is a closed set in $L^2(0,T) \times L^2(0,T)$, U_γ is a closed convex set in $H^1(0,T) \times H^1(0,T)$.

The correctness of the direct problem (4)–(8) in the case of Dirichlet boundary conditions for temperature was studied in [20]. Note that this result can be extended to the case of Robin-type boundary conditions for the energy equation without loss of generality.

Theorem 1. *Let conditions (14)– (16) be satisfied. Then there exists a unique strong solution to problem (4) – (9), where the functions θ, ρ, u, $\varphi(a(t),t)$, $\varphi(b(t),t)$ are bounded, θ,$\varphi(a(t),t)$, $\varphi(b(t),t)$ are non-negative, and ρ is positive and the following estimates take place:*

$$0 < m_1 \le \rho \le M_1 < \infty, \quad 0 \le \theta \le M_1 < \infty \quad a.e.\ in\ Q,$$
$$0 \le \varphi(a) \le M_1^4 < \infty, \quad 0 \le \varphi(b) \le M_1^4 < \infty \quad for\ a.e.\ t \in (0,T),$$
$$\|u\|_{L^\infty(0,T;H^1(\Omega_t))} + \|\theta\|_{L^\infty(0,T;H^1(\Omega_t))} \le C,$$
$$\|u\|_{H^{2,1}(Q)} + \|\theta\|_{H^{2,1}(Q)} + \|\rho\|_{H^{1,1}(Q)} + \|\varphi\|_{L^2(0,T;H^2(\Omega))} \le C,$$

where constants C, m_1, M_1 do not depend on time. The proof of Theorem 1 is based on the use of a priori estimates, the constants in which depend only on the problem data and T. The obtained estimates allow us to extend the local solution, which is established using the principle of contracted mappings, to the entire time interval. The operator equation equivalent to the problem is constructed by linearizing equations (4) and conditions (5)–(8), just as was done in [5,6]. On a small time interval, the resulting operator is contracting, therefore, Banach's theorem can be applied. The necessary a priori estimates are obtained in a similar way as in [20].

Let's consider the following quality functionality:

$$J[\mathbf{a}] = \frac{1}{2}\int_0^T |\theta(a(t),t) - d_a(t)|^2\, dt + \frac{1}{2}\int_0^T |\theta(b(t),t) - d_b(t)|^2\, dt. \tag{18}$$

Let $\mathbf{y} = \{\mathbf{y}_\beta,\, \mathbf{y}_\gamma\} \in U_{ad}$ be the admissible optimal control.

The optimal control problem is formulated as a problem of minimizing the functional

$$J_\kappa[\mathbf{y}] = J[\mathbf{a}] + \frac{\kappa_l}{2}\int_0^T |\beta_l(t)|^2\, dt + \frac{\kappa_r}{2}\int_0^T |\beta_r(t)|^2\, dt. \tag{19}$$

Here is the state of the system $\mathbf{a}(\mathbf{y}) = \{u(\mathbf{y}),\, \rho(\mathbf{y}),\, \theta(\mathbf{y}),\, \varphi(\mathbf{y})\} \in Y$ is defined as a strong solution to problem (4)–(9). It is required to find $\mathbf{y}_0 \in U_{ad}$ such, as

$$J_\kappa[\mathbf{y}_0] = \underset{\mathbf{y}\in U_{ad}}{inf}\ \{J_\kappa[\mathbf{y}]\}. \tag{20}$$

Theorem 2. *Let $\kappa_l > 0$, $\kappa_r > 0$. Then there exists at least one solution to the problem (20).*

Proof. Since $\kappa_l > 0$, $\kappa_r > 0$, any minimizing sequence $\{\mathbf{y}_k\}_{k=1}^\infty$ is bounded in H, where the space H is defined in (11). By the statement of Theorem 1, for each k there exists a strong solution $\mathbf{a}_k \in Y$ of problem (4) – (9), for which the following estimates [20] hold:

$$\begin{gathered} 0 < m_1 \le \rho_k \le M_1 < \infty, \quad 0 \le \theta_k \le M_1 < \infty \quad \text{a.e. in } Q, \\ 0 \le \varphi_k(a) \le M_1^4 < \infty, \quad 0 \le \varphi_k(b) \le M_1^4 < \infty \quad \text{for a.e. } \ t \in (0,T), \\ \|u_k\|_{L^\infty(0,T;H^1(\Omega_t))} + \|\theta_k\|_{L^\infty(0,T;H^1(\Omega_t))} \le C, \\ \|u_k\|_{H^{2,1}(Q)} + \|\theta_k\|_{H^{2,1}(Q)} + \|\rho_k\|_{H^{1,1}(Q)} + \|\varphi_k\|_{L^2(0,T;H^2(\Omega))} \le C, \end{gathered} \tag{21}$$

where constants C, m_1, M_1 do not depend on k.

To justify the transition to the limit with respect to k in the nonlinear terms of the equations of system (4) written for the sequence $\{\mathbf{a}_k\}_{k=1}^\infty$, it is necessary to obtain additional a priori estimates that guarantee the compactness of the sequences ρ_k, φ_k in $L^2(Q)$. The compactness of the sequences u_k, θ_k in $L^2(Q)$ follows from the compact embedding $H^{1,2}(Q) \subset L^2(Q)$.

Further, we will denote $g(x(t),t) = g(t)$. Consider the second equation in (4) for the sequence ρ_k in the following form:

$$(\rho_k(t) - \rho_k(\tau))_t + \rho_k^2(t) u_{kx}(t) = 0, \quad t, \tau \in (0,T).$$

We multiply this equality by $(\rho_k(t) - \rho_k(\tau))$ and, taking into account the formula for differentiation with respect to the parameter, we integrate with respect to $x(t)$ from $a(t)$ to $b(t)$,

$$\frac{1}{2}\frac{d}{dt}\int_{a(t)}^{b(t)} |\rho_k(t) - \rho_k(\tau)|^2\, dx + m(t)(\rho_{2k}(t) - \rho_{2k}(\tau)) - m(t)(\rho_{1k}(t) - \rho_{1k}(\tau)) =$$

$$= -\int\limits_{a(t)}^{b(t)} \rho_k^2(t) u_{kx}(t)(\rho_k(t) - \rho_k(\tau))\, dx. \tag{22}$$

We integrate (22) with respect to t on the interval $[\tau;\, \tau + h]$. Taking into account the boundedness of the function $\|u_x(t)\|_{L^2(\Omega_t)}$ in $L^\infty(0,T)$, and applying Gronwall's lemma to the last inequality, we obtain the estimate

$$\|\rho_k(\tau + h) - \rho_k(\tau)\|^2_{L^2(\Omega_{\tau+h})} \le Ch, \quad \tau < T - h. \tag{23}$$

From which it follows

$$\int\limits_0^{T-h} \|\rho_k(\tau + h) - \rho_k(\tau)\|^2_{L^2(\Omega_{\tau+h})}\, d\tau \le CTh.$$

Let us proceed to obtaining an estimate of equicontinuity for the sequence φ_k. We denote by $\psi_k = \varphi_k(x(\tau + h), \tau + h) - \varphi_k(x(\tau), \tau)$. We consider the last equation in (4) for $t = \tau + h$ and $t = \tau$ and subtract one from the other. Using integration by parts and taking into account the boundary conditions (8), we find

$$\alpha \int\limits_{a(\tau+h)}^{b(\tau+h)} \rho_k(\tau + h)|\psi_{kx}|^2\, dx + \alpha \int\limits_{a(\tau+h)}^{b(\tau+h)} \varphi_{kx}(\tau)(\rho_k(\tau + h) - \rho_k(\tau))\psi_{kx}\, dx +$$

$$k_\alpha \int\limits_{a(\tau+h)}^{b(\tau+h)} \rho_k^{-1}(\tau + h)|\psi_k|^2\, dx - k_\alpha \int\limits_{a(\tau+h)}^{b(\tau+h)} \rho_k^{-1}(\tau + h)(\theta_k^4(\tau + h) - \theta_k^4(\tau))\psi_k\, dx +$$

$$+k_\alpha \int\limits_{a(\tau+h)}^{b(\tau+h)} (\varphi_k(\tau) - \theta_k^4(\tau))(\rho_k^{-1}(\tau + h) - \rho_k^{-1}(\tau))\psi_k\, dx +$$

$$+\alpha\gamma_k(\tau+h)\psi_k^2(a) + \alpha\gamma_k(\tau+h)\psi_k^2(b) + I(a, \tau+h, \tau) + I(b, \tau+h, \tau) = 0, \tag{24}$$

where

$$I(\ell, \tau+h, \tau) = \alpha[\varphi_k(\ell(\tau+h), \tau+h) - \theta_b^4(\ell(\tau+h), \tau+h)][\gamma_k(\tau+h) - \gamma_k(\tau)]\psi_k(\ell) -$$
$$-\alpha\gamma_k(\tau + h)[\theta_b^4(\ell(\tau + h), \tau + h) - \theta_b^4(\ell(\tau), \tau)]\psi_k(\ell).$$

Considering the positivity and boundedness of the sequence ρ_k, we estimate the integral terms in (24) as follows

$$\alpha m_1\|\psi_{kx}\|^2_{L^2(\Omega_{\tau+h})} + k_\alpha M_1^{-1}\|\psi_k\|_{L^2(\Omega_{\tau+h})} + \alpha\gamma_{min}|\psi_k(a)|^2 + \alpha\gamma_{min}|\psi_k(b)|^2 \le$$
$$\le \alpha \max_{x\in\Omega_{\tau+h}} |\varphi_{kx}|\|\psi_{kx}\|_{L^2(\Omega_{\tau+h})}\|\rho_k(\tau + h) - \rho_k(\tau)\|_{L^2(\Omega_{\tau+h})} +$$
$$+k_\alpha m_1^{-1}\|\psi_k\|_{L^2(\Omega_{\tau+h})}\|\theta_k^4(\tau + h) - \theta_k^4(\tau)\|_{L^2(\Omega_{\tau+h})} +$$
$$+k_\alpha m_1^{-2} \max_{x\in\Omega_{\tau+h}} |\varphi_k(\tau) - \theta_k^4(\tau)|\|\psi_{kx}\|_{L^2(\Omega_{\tau+h})}\|\rho_k(\tau + h) - \rho_k(\tau)\|_{L^2(\Omega_{\tau+h})} +$$
$$+I(a, \tau + h, \tau) + I(b, \tau + h, \tau), \quad \tau \in (0, T - h). \tag{25}$$

Let's get additional estimates of the sequence φ_k. Let's represent φ_k as:

$$\varphi_k(x(t),t) = \int\limits_{a(t)}^{x(t)} \varphi_{k\xi}\, d\xi + \varphi_k(a(t),t),$$

$$\alpha\rho_k(x(t),t)\varphi_{kx}(x(t),t) = \int\limits_{a(t)}^{x(t)} (\rho_k(x(t),t)\varphi_{k\xi})_\xi\, d\xi +$$
$$+\gamma_k(t)\rho_k(a(t),t)(\varphi_k(a(t),t) - \theta_b^4(a(t),t)).$$

From here we get

$$\max_{x\in\Omega_t} |\varphi_k(t)| \le L\|\varphi_{kx}\|_{L^2(\Omega_t)} + M_1^4,$$

$$\max_{x\in\Omega_t} |\varphi_{kx}(t)| \le C\|\varphi_{kxx}\|_{L^2(\Omega_t)} + 2M_1^5|\gamma_k(t)|, \quad \text{for a.e. } t \in (0,T). \tag{26}$$

Let us estimate each term of the right-hand side of (25) in $L^2(0,T-h)$ separately. Taking into account (21), (23), (26) we estimate the first term

$$\alpha \int\limits_0^{T-h} \max_{x\in\Omega_{\tau+h}} |\varphi_{kx}|\|\psi_{kx}\|_{L^2(\Omega_{\tau+h})}\|\rho_k(\tau+h) - \rho_k(\tau)\|_{L^2(\Omega_{\tau+h})}\, d\tau \le$$
$$\le \varepsilon \int\limits_0^{T-h} \|\psi_{kx}\|^2_{L^2(\Omega_{\tau+h})}\, d\tau + C_\varepsilon h \int\limits_0^{T-h} \|\varphi_{kxx}\|^2_{L^2(\Omega_{\tau+h})}\, d\tau \le$$
$$\le \varepsilon \int\limits_0^{T-h} \|\psi_{kx}\|^2_{L^2(\Omega_{\tau+h})}\, d\tau + C_\varepsilon h. \tag{27}$$

To estimate the second term, we use the inequality

$$\|\theta_k^4(\tau+h) - \theta_k^4(\tau)\|_{L^2(\Omega_{\tau+h})} \le 4M_1^3\|\theta_k(\tau+h) - \theta_k(\tau)\|_{L^2(\Omega_{\tau+h})}.$$

Then

$$k_\alpha m_1^{-1} \int\limits_0^{T-h} \|\psi_k\|_{L^2(\Omega_{\tau+h})}\|\theta_k^4(\tau+h) - \theta_k^4(\tau)\|_{L^2(\Omega_{\tau+h})}\, d\tau \le$$
$$\le \varepsilon \int\limits_0^{T-h} \|\psi_{kx}\|^2_{L^2(\Omega_{\tau+h})}\, d\tau + C_\varepsilon \int\limits_0^{T-h} \|\theta_k(\tau+h) - \theta_k(\tau)\|^2_{L^2(\Omega_{\tau+h})}\, d\tau. \tag{28}$$

When evaluating the third term on the right-hand side of (25), we take into account (23), (26)

$$k_\alpha m_1^{-2} \int\limits_0^{T-h} \max_{x\in\Omega_{\tau+h}} |\varphi_k(\tau) - \theta_k^4(\tau)|\|\psi_{kx}\|_{L^2(\Omega_{\tau+h})}\|\rho_k(\tau+h) - \rho_k(\tau)\|_{L^2(\Omega_{\tau+h})}\, d\tau \le$$

$$\leq \varepsilon \int\limits_0^{T-h} \|\psi_{kx}\|^2_{L^2(\Omega_{\tau+h})}\, d\tau + C_\varepsilon h. \tag{29}$$

Let us estimate $I(a, \tau + h, \tau)$.

$$\int\limits_0^{T-h} I(a, \tau{+}h, \tau)\, d\tau = \alpha \int\limits_0^{T-h} [\varphi_k(a, \tau{+}h){-}\theta_b^4(a, \tau{+}h)][\gamma_k(\tau{+}h){-}\gamma_k(\tau)]\psi_k(a)\, d\tau -$$

$$-\alpha \int\limits_0^{T-h} \gamma_k(\tau + h)[\theta_b^4(a, \tau + h) - \theta_b^4(a, \tau)]\psi_k(a) \leq$$

$$\leq 2M_1^4 \alpha \int\limits_0^{T-h} |\psi_k(a)| \int\limits_\tau^{\tau+h} |\gamma_t|\, dt\, d\tau + \alpha\gamma_{min} \int\limits_0^{T-h} |\psi_k(a)| \int\limits_\tau^{\tau+h} 4\theta_b^3 |\theta_{bt}|\, dt\, d\tau \leq$$

$$\leq \delta \|\psi_k(a)\|_{L^2(0,T-h)} + Ch\|\gamma_k\|^2_{H^1(0,T)} + Ch\|\theta_k\|^2_{H^1(0,T;C(\overline{\Omega}_t))}. \tag{30}$$

For $I(b, \tau + h, \tau)$ the estimate is similar to (30). We integrate (25) with respect to τ on the interval $[0, T - h]$. We choose $\varepsilon = \alpha m_1/6$, $\delta = \alpha\gamma_{min}/4$ and substitute the estimates (27)–(30) into the right-hand side of (25), after simple transformations we obtain the estimate

$$\int\limits_0^{T-h} \|\psi_{kx}\|^2_{L^2(\Omega_{\tau+h})}\, d\tau +$$

$$+ \int\limits_0^{T-h} \|\psi_k\|_{L^2(\Omega_{\tau+h})}\, d\tau + \|\psi_k(a)\|_{L^2(\tau,\tau+h)} + \|\psi_k(b)\|_{L^2(\tau,\tau+h)} \leq$$

$$\leq Ch + C \int\limits_0^{T-h} \|\theta_k(\tau + h) - \theta_k(\tau)\|^2_{L^2(\Omega_{\tau+h})}\, d\tau =$$

$$= Ch + C \int\limits_0^{T-h} \int\limits_{a(\tau+h)}^{b(\tau+h)} \Big| \int\limits_\tau^{\tau+h} \theta_{kt}\, dt \Big|^2 dx\, d\tau \leq$$

$$\leq Ch + Ch \int\limits_0^{T-h} \int\limits_{a(\tau+h)}^{b(\tau+h)} \int\limits_\tau^{\tau+h} |\theta_{kt}|^2\, dt\, dx\, d\tau \leq Ch\|\theta_{kt}\|^2_{L^2(Q)} \leq Ch. \tag{31}$$

From (31) we conclude that there exists a subsequence (we will denote it by the same name) such that

$$\varphi_k \to \varphi \quad \text{strong in } L^2(Q),$$

and besides

$$\varphi_k(a(t)) \to \varphi(a(t)) \quad \text{strong in } L^2(0,T), \quad \varphi_k(b(t)) \to \varphi(b(t)) \quad \text{strong in } L^2(0,T).$$

Estimates (21) guarantee the choice of a subsequence (we also denote it by) such that

$$u_k \to u \quad \text{weak in } L^2(0,T;H^2(\Omega_t)), \quad \text{strong in } L^2(Q);$$

$$\rho_k \to \rho \quad \text{weak in } L^2(0,T;H^1(\Omega_t));$$

$$\theta_k \to \theta \quad \text{weak in } L^2(0,T;H^2(\Omega_t)), \quad \text{strong in } L^2(Q);$$

$$\varphi_k \to \varphi \quad \text{weak in } L^2(0,T;H^2(\Omega_t));$$

$$u_{kt} \to u_t \quad \text{weak in } L^2(Q),$$

$$\rho_{kt} \to \rho_t \quad \text{weak in } L^2(Q),$$

$$\theta_{kt} \to \theta_t \quad \text{weak in } L^2(Q).$$

Estimates (21), (31) are sufficient to justify that $\mathbf{a} = \{u,\, \rho,\, \theta,\, \varphi\}$ is a strong solution of problem (4)–(9). Due to the weak lower semicontinuity of the functional $J[\mathbf{a}]$ and the property of weak lower semicontinuity in $\mathbf{y}$ for the remaining part of the functional $J_\kappa[\mathbf{y}]$, we conclude that $\mathbf{y}$ is a solution of problem (20). Consequently, any limit (in the sense of weak convergence) point of the minimizing sequence is a solution of problem (20).

Let us prove auxiliary lemmas.

Lemma 1. *Let Q be the domain defined in (1)–(3). For any $g \in L^2(0,T;H^1(\Omega_t))$ such that $g_t \in L^2(0,T;\tilde{H}^{-1}(\Omega_t))$ the equality*

$$\langle g_t, \xi \rangle_{\tilde{H}^{-1}(\Omega_t)\times H^1(\Omega_t)} = \frac{d}{dt}(g,\xi) + m(t)(g_x,\xi) + m(t)(g,\xi_x) \quad \text{a.e. on } (0,T) \tag{32}$$

$\forall \xi \in H^1(\Omega_t)$, where $m(t) \in C[0,T]$ is defined in (2).

Proof. Let g be the function defined in the lemma, $\xi \in H^1(\Omega_t)$ be an arbitrary function. Using the formula for differentiation of an integral with integration limits depending on a parameter, which is valid for the domain with conditions (1), (2), we find

$$\frac{d}{dt}\int\limits_{a(t)}^{b(t)} g\xi\, dx =$$

$$= b'(t) g|_{x(t)=b(t)} \xi|_{x(t)=b(t)} -$$

$$- a'(t) g|_{x(t)=a(t)} \xi|_{x(t)=a(t)} + \langle g_t, \xi \rangle_{\tilde{H}^{-1}(\Omega_t)\times H^1(\Omega_t)}. \tag{33}$$

By the hypothesis of Lemma $g, \xi \in H^1(\Omega_t)$ almost everywhere on $(0,T)$, therefore,

$$\int\limits_{a(t)}^{b(t)} (g_x\xi + g\xi_x)\, dx = \int\limits_{a(t)}^{b(t)} (g\xi)_x\, dx = (g\xi)|_{x(t)=b(t)} - (g\xi)|_{x(t)=a(t)} < \infty.$$

Considering that $a'(t) = b'(t) = -m(t)$, we obtain the proof of the lemma.

Remark 1. *Let the conditions of Lemma 1 be satisfied and it is known that $g(a(t),t) = g(b(t),t) = 0$, then*

$$\langle g_t, g\rangle_{\tilde{H}^{-1}(\Omega_t)\times H^1(\Omega_t)} = \frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2(\Omega_t)} \quad a.e.\ on\ \ (0,T). \tag{34}$$

Lemma 2. *For an arbitrary function $g \in C([0,T]; H^p(\Omega_t))$, $p = 1, 2$ the inequality holds*

$$\|g(t)\|_{C^{p-1}(\bar{\Omega}_t)} \le C(\|g(t)\|_{H^p(\Omega_t)} + \|g(t)\|_{H^{p-1}(\Omega_t)}). \tag{35}$$

And there exists $x^\star \in C(\Omega_t)$ such that

$$g|_{x(t)=x^\star(t)} = \frac{1-a(t)}{L}\int\limits_{\Omega_t} g\,dx. \tag{36}$$

Proof. Let $g \in C([0,T]; H^1(\Omega_t))$. For an arbitrary $x(t) \in C(\Omega_t)$, we introduce an auxiliary function

$$g_1(x,t) = \int\limits_{a(t)}^{x(t)} g(s,t)\,ds. \tag{37}$$

Note that

$$g_1(a(t),t) = 0, \quad g_1(b(t),t) = \int\limits_{\Omega_t} g\,dx, \quad t \in [0,T].$$

For an arbitrary $v \in C(0,T; C(\bar{\Omega}_t))$ satisfying the conditions $v(a(t),t) = 0$, $v(b(t),t) = 0$ there exists $x^\star(t) \in C(\Omega_t)$ such that $v_x|_{x(t)=x^\star(t)} = 0$. Let

$$v(x,t) = g_1(x,t) - g_1(b(t),t)(x(t) - a(t))/L,$$

or

$$g_1(x,t) = v(x,t) + g_1(b(t),t)(x(t) - a(t))/L, \quad x(t) \in \bar{\Omega}_t, \quad t \in [0,T]. \tag{38}$$

From (38) we find

$$g_{1x}|_{x=x^\star(x)} = v_x|_{x=x^\star(x)} + g_1(b(t),t)(1 - a(t))/L = g_1(b(t),t)(1 - a(t))/L.$$

On the other hand, for an arbitrary $x \in C(\bar{\Omega}_t)$, the equality holds

$$g_{1x}(x(t),t) = \int\limits_{x^\star(t)}^{x(t)} g_{1ss}\,ds + g_{1x}|_{x(t)=x^\star(t)} =$$

$$= \int\limits_{x^\star(t)}^{x(t)} g_{1ss}\,ds + g_1(b(t),t)(1 - a(t))/L. \tag{39}$$

Note that $g_{1x} = g$, we rewrite (39), taking into account (37), as follows:

$$g(x(t),t) = \int\limits_{x^\star(t)}^{x(t)} g_s\,ds +$$

$$+\frac{1-a(t)}{L}\int\limits_{\Omega_t} g(x(t),t)\,dx, \quad x(t)\in\bar{\Omega}_t, \quad t\in[0,T]. \tag{40}$$

From (40) follows the estimate (35) for $p = 1$. Considering (40) for $x = x^\star$, we obtain (36). In the case $p = 2$, the estimate (35) is obtained in a similar way.

Remark 2. *For an arbitrary $g \in C([0,T]; H^1(\Omega_t))$ the following inequalities hold:*

$$|g(x(t),t)|^q \le \varepsilon\|g_x\|^2 + C_\varepsilon\|g_x\|^{2(q-2)}\|g\|^2, \quad \varepsilon > 0, \quad q = 2,3. \tag{41}$$

Proof. The inequality (41) for $q = 2$ is obvious, let's consider $q = 3$. The equality is true

$$g^3(x(t),t) = \int\limits_{x^\star(t)}^{x(t)} (g^3)_s\,ds + g^3(x^\star(t),t),$$

where $x^\star(t) \in C(\Omega_t)$ is the same as in Lemma 2. From the last equality we find

$$|g(x(t),t)|^3 \le 3\int\limits_{\Omega_t}|g|^2|g_x|\,dx + C\int\limits_{\Omega_t}|g|^3\,dx \le$$

$$\le C\|g_x\|\|g\|^2_{L^4(\Omega_t)} + C\|g\|^3_{L^4(\Omega_t)} \le C\|g_x\|^2\|g\| + C\|g_x\|^{3/2}\|g\|^{3/2} \le$$
$$C_1\|g_x\|^2\|g\| \le \varepsilon\|g_x\|^2 + C_\varepsilon\|g_x\|^2\|g\|^2.$$

Lemma 3. *Let $\{u_1, \rho_1, \theta_1, \varphi_1\}$, $\{u_2, \rho_2, \theta_2, \varphi_2\}$ be two strong solutions of problem (4)–(9). Then for the functions*

$$A_1 = \nu u_{1x} - R\theta_1, \quad A_2 = \rho_2(\nu u_{2x} - R\theta_2), \quad B = k_\alpha(\theta_2^4 - \varphi_2)/(\rho_1\rho_2),$$
$$\tilde{\theta} = s'\theta_1 + (1-s')\theta_2, \quad s' \in (0,1) \tag{42}$$

the following inclusions are valid

$$A_1, A_2, B \in H^{1,0}, \quad \tilde{\theta},\, B \in L^\infty(0,T; L^\infty(\Omega_t)).$$

For the functions u_i, θ_i, φ_i, $i = 1,2$ the following estimates are valid:

$$\|u_i\|_{L^\infty(0,T;C(\bar{\Omega}_t))} + \|u_{ix}\|_{L^2(0,T;C(\bar{\Omega}_t))} \le C,$$
$$\|\theta_i\|_{L^\infty(0,T;C(\bar{\Omega}_t))} + \|\theta_{ix}\|_{L^2(0,T;C(\bar{\Omega}_t))} \le C,$$
$$\|\varphi_i\|_{L^\infty(0,T;C(\bar{\Omega}_t))} + \|\varphi_{ix}\|_{L^2(0,T;C(\bar{\Omega}_t))} \le C. \tag{43}$$

If $\theta = \theta_1 - \theta_2$, $\varphi = \varphi_1 - \varphi_2$, then

$$\|\theta\|_{L^2(0,T;C(\bar{\Omega}_t))} \le \|\theta_x\|_{L^2(0,T;L^2(\Omega_t))},$$
$$\|\varphi\|_{L^2(0,T;C(\bar{\Omega}_t))} \le \|\varphi_x\|_{L^2(0,T;L^2(\Omega_t))}. \tag{44}$$

Proof. The boundedness of A_1, A_2, B in $L^\infty(0,T;L^2(\Omega_t))\cap L^2(0,T;H^1(\Omega_t))$ follows from the definition of a strong solution to problem (4)–(9). As a consequence of Lemma 2, estimates (43), (44) hold.

4 Differential properties of control-state mapping

Let us study the differential properties of the mapping

$$\mathbf{y} \to \{u(\mathbf{y}),\, \rho(\mathbf{y}),\, \theta(\mathbf{y}),\, \varphi(\mathbf{y})\}, \quad \mathbf{y}\colon H \to W,$$

where the spaces H and W are defined in (11), (13), respectively.

Let $\mathbf{a} = \mathbf{a_1} - \mathbf{a_2}$, where $\mathbf{a_i} = \mathbf{a}(\mathbf{y_i}; F_0)$, $i = 1,2$ – strong solutions of problem (4)–(9), $F_0 = \{u_0,\, \rho_0,\, \theta_0,\, u_a,\, u_b,\, \rho_a,\, \Lambda_a,\, \Lambda_b,\, \beta_{0a},\, \beta_{0b}\}$, $\mathbf{y} = \mathbf{y_1} - \mathbf{y_2} = \{\beta_l,\, \beta_r,\, \gamma_l,\, \gamma_r\}$, $\mathbf{y_i} = \{\beta_{li},\, \beta_{ri},\, \gamma_{li},\, \gamma_{ri}\}$, $\quad i = 1,2.$

Let's consider integral operators

$$\begin{gathered} L_1\colon H^{-1}(\Omega_t) \to H_0^1(\Omega_t)), \quad L_{2,3}\colon \tilde{H}^{-1}(\Omega_t) \to H^1(\Omega_t), \\ R_1\colon H^{-1}(\Omega_t) \to H_0^1(\Omega_t)), \quad R_{2,3}\colon \tilde{H}^{-1}(\Omega_t) \to H^1(\Omega_t), \\ R_{4,5}\colon L^2(\Omega_t) \to L^2(\Omega_t), \quad R_6\colon L^1(\Omega_t) \to L^\infty(\Omega_t) \end{gathered} \tag{45}$$

and functionals $f_i \in L^2(0,T)$, $i = 1,\dots,4$, valid for any $u, \xi_1 \in H_0^1(\Omega_t)$, $\theta, \varphi, \xi_2, \xi_3 \in H^1(\Omega_t)$, $\tau_i \in L^2(0,T)$, $i = 1,\dots,4$ and corresponding to the formulas

$$\begin{gathered} (L_1 u, \xi_1) = \nu(\rho_2 u_x,\, \xi_{1x}), \\ (L_2\theta, \xi_2) = a(\rho_2\theta_x,\, \xi_{2x}) + 4bk_\alpha(\rho_1^{-1}\tilde{\theta}^3\theta,\, \xi_2) + \\ + \beta_{2a}[\theta\xi_2]|_{x(t)=a(t)} + \beta_{2b}[\theta\xi_2]|_{x(t)=b(t)}, \\ (L_3\varphi, \xi_3) = \alpha(\rho_2\varphi_x,\, \xi_{3x}) + k_\alpha(\rho_1^{-1},\, \varphi\xi_3) \\ + \gamma_{2l}[\varphi\xi_3]|_{x(t)=a(t)} + \gamma_{2r}[\varphi\xi_3]|_{x(t)=b(t)}; \end{gathered} \tag{46}$$

$$\begin{gathered} (R_1\mathbf{a}, \xi_1) = (A_1\rho - R\theta\rho_2,\, \xi_{1x}), \\ (R_2\mathbf{a}, \xi_2) = a(\rho\theta_{1x},\, \xi_{2x}), \\ (R_3\mathbf{a}, \xi_3) = \alpha(\rho\varphi_{1x},\, \xi_{3x}), \\ (R_4\mathbf{a}, \xi_2) = -((\nu u_x - R\theta)\rho_2 u_{1x} + A_2 u_x + bB\rho + bk_\alpha\rho_1^{-1}\varphi,\, \xi_2), \\ (R_5\mathbf{a}, \xi_3) = -(B\rho + 4k_\alpha\rho_1^{-1}\tilde{\theta}^3\theta,\, \xi_3)\, dx, \\ (R_6\mathbf{a}, \xi_2) = -(A_1 u_{1x}\rho,\, \xi_2); \end{gathered} \tag{47}$$

$$(f_1(\beta_l), \tau_1)_{L^2(0,T)} = -\int_0^T \beta_l(t)\Lambda_a(t)(\theta_1(a(t),t) - \theta_b(t))\tau_1(t)\, dt,$$

$$(f_2(\beta_r), \tau_2)_{L^2(0,T)} = -\int_0^T \beta_r(t)\Lambda_b(t)(\theta_1(b(t),t) - \theta_b(t))\tau_2(t)\, dt,$$

$$(f_3(\gamma_l), \tau_3)_{L^2(0,T)} = -\int_0^T \gamma_l(t)(\varphi_1(a(t),t) - \theta_b^4(a(t),t))\tau_3(t)\, dt,$$

$$(f_4(\gamma_r), \tau_4)_{L^2(0,T)} = -\int\limits_0^T \gamma_r(t)(\varphi_1(b(t),t) - \theta_b^4(b(t),t))\tau_4(t)\,dt. \tag{48}$$

Note that the operators $L_{1,2,3}$ defined in (46) are symmetric, positive definite, and such that

$$\begin{aligned}(L_1 u, u) &\geq C_1\|u\|^2_{H^1_0(\Omega_t)},\\ (L_2\theta, \theta) &\geq C_1\|\theta\|^2_{H^1(\Omega_t)},\\ (L_3\varphi, \varphi) &\geq C_1\|\varphi\|^2_{H^1(\Omega_t)}\end{aligned} \tag{49}$$

almost everywhere on $(0,T)$, where the constant $C_1 > 0$. For $R_{1,2,3,4,5}$ defined in (47) the following inequalities hold:

$$\begin{aligned}\|R_1\mathbf{a}\|_{H^{-1}} &= \sup_{\|\xi_1\|_{H^1_0(\Omega_t)}=1}(R_1\mathbf{a}, \xi_{1x}) \leq C\max_{x\in\Omega_t}|A_1|\|\rho\| + C\|\rho_2\|_{L^\infty(Q)}\|\theta\|,\\ \|R_2\mathbf{a}\|_{\tilde{H}^{-1}} &= \sup_{\|\xi_2\|_{H^1(\Omega_t)}=1}(R_2\mathbf{a}, \xi_{2x}) \leq C\max_{x\in\Omega_t}|\theta_{1x}|\|\rho\|,\\ \|R_3\mathbf{a}\|_{\tilde{H}^{-1}} &= \sup_{\|\xi_3\|_{H^1(\Omega_t)}=1}(R_3\mathbf{a}, \xi_{2x}) \leq C\max_{x\in\Omega_t}|\varphi_{1x}|\|\rho\|.\\ \|R_4\mathbf{a}\| &= \sup_{\|\xi_2\|_{L^2(\Omega_t)}=1}(R_4\mathbf{a}, \xi_2) \leq C\max_{x\in\bar{\Omega}_t}|u_{1x}|(\|u_x\| + \|\theta\|) +\\ &\quad + C\max_{x\in\bar{\Omega}_t}|A_2|\|u_x\| + C\|\rho\| + C\|\varphi\|,\\ \|R_5\mathbf{a}\| &= \sup_{\|\xi_3\|_{L^2(\Omega_t)}=1}(R_5\mathbf{a}, \xi_3) \leq C\|\rho\| + C\|\theta\|,\\ \|R_6\mathbf{a}\|_{L^1(\Omega_t)} &= \sup_{\|\xi_2\|_{L^\infty(\Omega_t)}=1}(R_6\mathbf{a}, \xi_2) \leq C\max_{x\in\Omega_t}|A_1|\|u_{1x}\|\|\rho\|.\end{aligned} \tag{50}$$

To find estimates of the functionals (48), we will use estimates (43). Applying the Cauchy inequality in (48), we obtain

$$\begin{aligned}\|f_1\|_{L^2(0,T)} &= \sup_{\|\tau_1\|_{L^2(0,T)}=1}(f_1, \tau_1) \leq C\|\beta_l\|_{L^2(0,T)},\\ \|f_2\|_{L^2(0,T)} &= \sup_{\|\tau_2\|_{L^2(0,T)}=1}(f_2, \tau_2) \leq C\|\beta_r\|_{L^2(0,T)},\\ \|f_3\|_{L^2(0,T)} &= \sup_{\|\tau_3\|_{L^2(0,T)}=1}(f_3, \tau_3) \leq C\|\gamma_l\|_{L^2(0,T)},\\ \|f_4\|_{L^2(0,T)} &= \sup_{\|\tau_4\|_{L^2(0,T)}=1}(f_4, \tau_4) \leq C\|\gamma_r\|_{L^2(0,T)}.\end{aligned} \tag{51}$$

Theorem 3. *The mapping* $\mathbf{y} \to \mathbf{a}(\mathbf{y})$ *is defined and acts continuously from* H *to* W. Proof. We obtain conditions for the difference of strong solutions of the problem (4)–(9) $\{u_1 - u_2,\ \rho_1 - \rho_2,\ \theta_1 - \theta_2,\ \varphi_1 - \varphi_2\}$. For this purpose, the first, third and fourth equations of the system (4), considered for strong solutions $\{u_i,\ \rho_i,\ \theta_i,\ \varphi_i\}$ for each $i = 1, 2$ are multiplied by $\xi_1 \in H^1_0(\Omega_t)$, $\xi_2, \xi_3 \in H^1(\Omega_t)$, respectively, as a scalar in $L^2(\Omega_t)$, taking into account the boundary conditions, and subtract one from the other, and the equation for the density function is rewritten as follows: $(1/\rho_i)_t = u_{ix}$, $i = 1, 2$.

Taking into account (45)–(48), we obtain conditions for the functions $\{u,\, \rho,\, \theta,\, \varphi\} = \{u_1 - u_2,\, \rho_1 - \rho_2,\, \theta_1 - \theta_2,\, \varphi_1 - \varphi_2\}$:

$$
\begin{gathered}
u_t + L_1 u + R_1\mathbf{a} = 0, \quad \text{a.e. on } \ t \in (0, T),\\
\rho = -\rho_1\rho_2 \int\limits_0^t u_x\, ds, \quad \text{a.e. on } \ x \in \Omega_t, \quad t \in (0, T),\\
\theta_t + L_2\theta + R_2\mathbf{a} + R_4\mathbf{a} + R_6\mathbf{a} = f_1 + f_2, \quad \text{a.e. on } \ t \in (0, T),\\
L_3\varphi + R_3\mathbf{a} + R_5\mathbf{a} = f_3 + f_4, \quad \text{a.e. on } \ t \in (0, T),\\
u|_{t=0} = 0, \quad \theta|_{t=0} = 0.
\end{gathered} \tag{52}
$$

We obtain a priori estimates for the functions u, ρ, θ, φ and their derivatives through the data of the problem. From the second equation (52) we find the inequality

$$
\|\rho(t)\|^2 \le C \int\limits_0^t \|u_x(s)\|^2\, ds, \quad t \in (0, T). \tag{53}
$$

We rewrite the first equation (52) as

$$
(u_t, u) + (L_1 u, u) + (R_1\mathbf{a}, u) = 0 \quad \text{a.e. on } (0, T).
$$

Taking into account (34), (49), (50), we obtain the estimate

$$
\begin{gathered}
\frac{1}{2}\frac{d}{dt}\|u\|^2 + C_1\|u\|^2_{H^1(\Omega_t)} \le \|R_1\mathbf{a}\|_{H^{-1}(\Omega_t)}\|u\|_{H^1(\Omega_t)} \le\\
\le C(\max_{x\in\Omega_t}|A_1|\|\rho\| + \|\theta\|)\|u\|_{H^1(\Omega_t)} \le\\
\le \frac{C_1}{2}\|u\|^2_{H^1(\Omega_t)} + C(\max_{x\in\Omega_t}|A_1|^2\|\rho\|^2 + \|\theta\|^2).
\end{gathered}
$$

Moving the first term to the left and taking into account (53), we find

$$
\frac{d}{dt}\|u\|^2 + \|u\|^2_{H^1(\Omega_t)} \le C \max_{x\in\Omega_t}|A_1| \int\limits_0^t \|u(s)\|^2_{H^1(\Omega_t)}\, ds + C\|\theta\|. \tag{54}
$$

Let us denote by

$$
\Lambda_1(t) = \int\limits_0^t \|u(s)\|^2_{H^1(\Omega_t)}\, ds
$$

and integrate over t from 0 to t (54), we get

$$
\|u(t)\|^2 + \Lambda_1(t) \le C \int\limits_0^t \max_{x\in\Omega_t}|A_1(s)|\Lambda_1(s)\, ds + C \int\limits_0^t \|\theta\|\, ds.
$$

From here, applying Gronwall's lemma, we find the estimate

$$\max_{t\in(0,T)} \|u(t)\|^2 + \int_0^t \|u(t)\|^2_{H^1(\Omega_t)}\, dt \le C \int_0^t \|\theta\|\, dt \quad t\in[0,T]. \tag{55}$$

Let us rewrite the fourth equation (52) as

$$(L_3\varphi,\varphi) + (R_3\mathbf{a},\varphi) + (R_5\mathbf{a},\varphi) = f_3\varphi(a(t),t) + f_4\varphi(b(t),t). \tag{56}$$

Taking into account (44), (49), (50), we obtain the estimate

$$\begin{gathered} C_1\|\varphi\|^2_{H^1(\Omega_t)} \le \|R_3\mathbf{a}\|_{\tilde H^{-1}(\Omega_t)}\|\varphi\|_{H^1(\Omega_t)} + \\ + \|R_5\mathbf{a}\|\|\varphi\| + |f_3||\varphi(a(t))| + |f_4||\varphi(b(t))| \le \\ C((1+\max_{x\in\bar\Omega_t}|\varphi_{1x}|^2)\|\rho\| + \|\theta\| + |f_3| + |f_4|)\|\varphi\|_{H^1(\Omega_t)}. \end{gathered}$$

We integrate the last inequality from 0 to an arbitrary t. Taking into account (53), (55), we obtain the estimate

$$\begin{gathered} \int_0^t \|\varphi(s)\|^2_{H^1(\Omega_t)}\, ds \le C\|\rho(t)\|^2 \int_0^t (1+\max_{x\in\bar\Omega_s}|\varphi_{1x}|^2)\, ds + \\ + C\int_0^t \|\theta\|^2\, ds + \|f_3\|^2 + \|f_4\|^2 \le \\ \le C\int_0^t \|\theta\|^2\, ds + C\|\gamma_l\|^2_{L^2(0,T)} + \|\gamma_r\|^2_{L^2(0,T)}. \end{gathered} \tag{57}$$

Let us rewrite the second equation (52), taking into account Lemma 1, in the form

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 + m(t)(\theta,\theta_x) + (L_2\theta,\theta) + (R_2\mathbf{a},\theta) + (R_4\mathbf{a},\theta) + (R_6\mathbf{a},\theta) = \\ = f_1\theta(a(t),t) + f_2\theta(b(t),t). \end{gathered} \tag{58}$$

Taking into account (46), (47), we find

$$\begin{gathered} \frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \frac{C_2}{2}\|\theta\|^2_{H^1(\Omega_t)} \le C(1+\max_{x\in\bar\Omega_t}|u_{1x}|^2 + \max_{x\in\bar\Omega_t}|A_2|^2)\|\theta\|^2 + \\ + C(1+\max_{x\in\bar\Omega_t}|A_1|^2\|u_{1x}\|^2 + \max_{x\in\bar\Omega_t}|\theta_{1x}|^2)\|\rho\|^2 + \\ + C\|u_x\|^2 + C\|\varphi_x\|^2 + C|f_1|^2 + C|f_2|^2. \end{gathered} \tag{59}$$

Let us denote by

$$\Lambda_2(t) = \|\theta(t)\|^2 + \int_0^t \|\theta(s)\|^2_{H^1(\Omega_s)}\, ds,$$

$$d(t) = 1 + \max_{x\in\bar\Omega_t}(|u_{1x}(t)|^2 + |A_2(t)|^2 + |\theta_{1x}(t)|^2 + |A_1(t)|^2).$$

We integrate (59) from 0 to an arbitrary t and take into account that $\|u_{1x}\| \in L^\infty(0,T)$, as a consequence of Lemma 3 $d(t) \in L^2(0,T)$. From (59) we obtain

$$\Lambda_2(t) \le C\int\limits_0^t d(s)\Lambda_2(s)\,ds + C\|\rho\|^2_{L^\infty(0,t)} + C\int\limits_0^t \|u_x\|^2\,ds + C\int\limits_0^t \|\varphi_x\|^2\,ds +$$
$$+C\|\beta_l\|^2_{L^2(0,T)} + C\|\beta_r\|^2_{L^2(0,T)} \le$$
$$\le C\int\limits_0^t (1+d(s))\Lambda_2(s)\,ds + C\|\beta_l\|^2_{L^2(0,T)} +$$
$$+C\|\beta_r\|^2_{L^2(0,T)} + C\|\gamma_l\|^2_{L^2(0,T)} + \|\gamma_r\|^2_{L^2(0,T)}.$$

From here, applying Gronwall's lemma, we obtain the inequality

$$\|\theta(t)\|^2 + \int\limits_0^t \|\theta\|^2_{H^1(\Omega_t)} \le C\|\beta_l\|^2_{L^2(0,T)} + C\|\beta_r\|^2_{L^2(0,T)} +$$
$$+C\|\gamma_l\|^2_{L^2(0,T)} + \|\gamma_r\|^2_{L^2(0,T)}. \tag{60}$$

Substituting (60) into the right-hand sides of (55), (57), and then into (53), we find the estimate

$$\max_{t\in(0,T]}(\|\rho(t)\|^2 + \|u(t)\|^2 + \|\theta(t)\|^2 + \|\varphi(t)\|^2) +$$
$$+\int\limits_0^T (\|u(s)\|^2_{H^1(\Omega_t)} + \|\theta(s)\|^2_{H^1(\Omega_t)} + \|\varphi(s)\|^2_{H^1(\Omega_t)})\,ds \le$$
$$\le C(\|\gamma_l\|^2_{L^2(0,T)} + \|\gamma_r\|^2_{L^2(0,T)} + \|\beta_l\|^2_{L^2(0,T)} + \|\beta_r\|^2_{L^2(0,T)}). \tag{61}$$

Estimate (61) proves the statement of the theorem.

Theorem 4. *For any* $\mathbf{z} = \{z_1,\, z_2,\, z_3,\, z_4\} \in H$ *there exists a Gateaux differential* $D\mathbf{a}(\mathbf{y})(\mathbf{z}) = \{v,\, \pi,\, \eta,\, \psi\}$ *of the mapping* $\mathbf{a}(\mathbf{y})$ *in the direction* $\mathbf{z}$ *satisfying the conditions*

$$< v_t,\, \phi_1 >_{H^{-1}(\Omega_t)\times H^1_0(\Omega_t)} + (\rho v_x,\, \phi_{1x}) + (A\pi - R\eta\rho,\, \phi_{1x}) = 0, \tag{62}$$

$\forall \phi_1 \in H^1_0(\Omega_t)$,

$$\pi = -\rho^2\int\limits_0^t v_x\,ds \quad \text{для п. в. } (x,t)\in\Omega_t\times(0,T), \tag{63}$$

$$< \eta,\, \phi_{2t} >_{\tilde H^{-1}(\Omega_t)\times H^1(\Omega_t)} + m(t)[(\eta_x,\, \phi_2) + (\eta,\, \phi_{2x})] + a(\rho\eta_x,\, \phi_{2x}) + a(\pi\theta_x,\, \phi_{2x}) -$$
$$-((\nu v_x - R\eta)\rho u_x,\, \phi_2) - (A\rho v_x,\, \phi_2) - ((Au_x + bB)\pi,\, \phi_2) +$$
$$+4bk_\alpha(\rho^{-1}\theta^3\eta,\, \phi_2) - bk_\alpha(\rho^{-1}\psi,\, \phi_2) +$$
$$+\beta_a\eta(a)\phi_2(a) + \beta_b\eta(b)\phi_2(b) =$$
$$= -\Lambda_a(\theta(a)-\theta_b)z_1\phi_2(a) - \Lambda_b(\theta(b)-\theta_b)z_2\phi_2(b), \tag{64}$$

$\forall \phi_2 \in H^1(\Omega_t)$,

$$\alpha(\rho\psi_x,\, \phi_{3x}) + \alpha(\pi\varphi_x,\, \phi_{3x}) + k_\alpha(\rho^{-1}\psi,\, \phi_3) - 4k_\alpha(\rho^{-1}\theta^3\eta,\, \phi_3) - (B\pi, \phi_3) +$$
$$+\gamma_l\psi(a)\phi_3(a) + \gamma_r\psi(b)\phi_3(b) =$$
$$= -(\varphi - \theta_b^4)z_3\phi_3(a) - (\varphi - \theta_b^4)z_4\phi_3(b), \tag{65}$$

$\forall \phi_3 \in H^1(\Omega_t)$, *where*

$$A = \nu u_x - R\theta \in H^{1,0}, \quad B = k_\alpha(\theta^4 - \varphi)/\rho^2 \in L^\infty(0,T; L^\infty(\Omega_t)). \tag{66}$$

Proof. Let $\mathbf{z} = \{z_1,\, z_2,\, z_3,\, z_4\} \in H$ and $h \in R$. Consider strong solutions of problem (4)–(9) with initial and boundary data $\{\mathbf{y} + h\mathbf{z};\, F_0\}$ and $\{\mathbf{y};\, F_0\}$. Denote by

$$\mathbf{a}_h = \{u_h,\, \rho_h,\, \theta_h,\, \varphi_h\} = \frac{\mathbf{a}(\mathbf{y} + h\mathbf{z};\, F_0) - \mathbf{a}(\mathbf{y};\, F_0)}{h}.$$

In (45)–(52) we set $\mathbf{a}_1 = \mathbf{a}(\mathbf{y} + h\mathbf{z};\, F_0)$, $\mathbf{a}_2 = \mathbf{a}(\mathbf{y};\, F_0)$. It is easy to see that $\mathbf{a}_h$ satisfies the following system:

$$u_{ht} + L_1 u_h + R_1\mathbf{a}_h = 0, \quad \text{a.e. on } (0,T),$$

$$\rho_h = -\rho(\mathbf{y} + h\mathbf{z};\, F_0)\rho(\mathbf{y};\, F_0)\int_0^t u_{hx}\, ds, \quad \text{a.e. on } \Omega_t,$$

$$\theta_{ht} + L_2\theta_h + R_2\mathbf{a}_h + R_4\mathbf{a}_h + R_6\mathbf{a}_h = f_1(z_1) + f_2(z_2), \quad \text{a.e. on } t \in (0,T),$$
$$L_3\varphi_h + R_3\mathbf{a}_h + R_5\mathbf{a}_h = f_3(z_3) + f_4(z_4), \quad \text{a.e. on } (0,T), \tag{67}$$
$$u_h|_{t=0} = 0, \quad \theta_h|_{t=0} = 0. \tag{68}$$

When $h \neq 0$ the vector function $\mathbf{a}_h$ has the following properties:

$$\max_{t\in(0,T]}(\|\rho_h(t)\|^2 + \|u_h(t)\|^2 + \|\theta_h(t)\|^2 + \|\varphi_h(t)\|^2) +$$
$$+\int_0^T (\|u_h(s)\|^2_{H^1(\Omega_t)} + \|\theta_h(s)\|^2_{H^1(\Omega_t)} + \|\varphi_h(s)\|^2_{H^1(\Omega_t)})\, ds \leq K_0, \tag{69}$$

where K_0 does not depend on h. Theorem 1 implies the following property of solutions to the problem (67), (68)

$$\|u_{ht}\|^2 + \|\theta_{ht}\|^2 \leq C(h).$$

As a consequence of the last estimate, the condition (72) makes sense.

The estimates (69) allow us to choose a sequence of values $h \to 0$ such that

$$u_h \to v \quad \text{weak in } L^2(0,T; H_0^1(\Omega_t)),$$
$$\rho_h \to \pi \quad \text{weak in } L^2(0,T; L^2(\Omega_t)),$$
$$\theta_h \to \eta \quad \text{weak in } L^2(0,T; H^1(\Omega_t)),$$
$$\varphi_h \to \psi \quad \text{weak in } L^2(0,T; H^1(\Omega_t)). \tag{70}$$

The results on convergence (70) are sufficient for the limit transition in the obtained relations (45)–(52), (67).

We denote by $\mathbf{a}_0 = \lim\limits_{h\to 0} \mathbf{a}_h$, $\mathbf{a}_0 = \{v,\, \pi,\, \eta,\, \psi\}$. We pass to the limit with respect to $h \to 0$ in the system of equations (67), and obtain the following equalities

$$v_t + L_1 v + R_1 \mathbf{a}_0 = 0, \quad \text{a.e. on } \ (0,T),$$
$$\pi = -\rho^2 \int\limits_0^t v_x\, ds, \quad \text{a.e. on } \Omega_t,$$
$$\eta_t + L_2\eta + R_2\mathbf{a}_0 + R_4\mathbf{a}_0 + R_6\mathbf{a}_0 = f_1(z_1) + f_2(z_2), \quad \text{a.e. on } \ t \in (0,T),$$
$$L_3\psi + R_3\mathbf{a}_0 + R_5\mathbf{a}_0 = f_3(z_3) + f_4(z_4), \quad \text{a.e. on } \ (0,T), \tag{71}$$

Here the operators $L_{1,2,3}$, $R_{1,2,3,4,5,6}$ are defined in (46)-(48), where u_i, ρ_i, θ_i, ψ_i, $i = 1,2$ should be replaced by u, ρ, θ, ψ respectively. Note that by definition of the operators L_1, R_1 and L_2, $R_{2,4,6}$ from the first and third equations of the system (71) it follows that $v_t \in L^2(0,T;H^{-1}(\Omega_t))$, $\eta_t \in L^2(0,T;\tilde{H}^{-1}(\Omega_t))$. We define the trace $v|_{t=0} \in H^{-1}(\Omega)$ and the trace $\eta|_{t=0} \in \tilde{H}^{-1}(\Omega)$ by the formulas

$$\left\langle v|_{t=0}, \phi_1 \right\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0, \quad \forall \phi_1 \in H_0^1;$$
$$\left\langle \eta|_{t=0}, \phi_2 \right\rangle_{\tilde{H}^{-1}(\Omega)\times H^1(\Omega)} = 0 \quad \forall \phi_2 \in H^1. \tag{72}$$

From (71), (72) follows (62)–(65).

5 Necessary optimality conditions

Consider the following problem. Given $\{u,\, \rho,\, \theta,\, \varphi\} \in Y$, it is required to find $\{\xi,\, \tau,\, \zeta,\, \chi\}$ that in the domain $\Omega_t \times (0,T)$ satisfy the following equations:

$$-\xi_t - \nu(\rho\xi_x)_x + \tau_x + [(\nu\rho u_x + A\rho)\zeta]_x = 0, \tag{73}$$
$$\tau_t/\rho^2 + A\xi_x + a\theta_x\zeta_x + \alpha\varphi_x\chi_x - (Au_x + bB)\zeta - B\chi = 0, \tag{74}$$
$$-\zeta_t - a(\rho\zeta_x)_x + R\rho u_x\zeta - R\rho\xi_x + 4k_\alpha\rho^{-1}\theta^3(b\zeta - \chi) = 0, \tag{75}$$
$$-\alpha(\rho\chi_x)_x + k_\alpha\rho^{-1}(\chi - b\zeta) = 0 \tag{76}$$

and boundary conditions

$$\xi|_{t=T} = 0, \quad \tau|_{t=T} = 0, \quad \zeta|_{t=T} = 0, \quad x \in \Omega_T; \tag{77}$$
$$\xi|_{x=a(t)} = 0, \quad \xi|_{x=b(t)} = 0, \tag{78}$$
$$(-a\rho\zeta_x + \beta_a(t)\zeta - m(t)\zeta)|_{x=a(t)} = \theta(a(t),t) - d_a(t),$$
$$(a\rho\zeta_x + \beta_b(t)\zeta + m(t)\zeta)|_{x=b(t)} = \theta(b(t),t) - d_b(t), \tag{79}$$
$$(-\alpha\rho\chi_x + \gamma_l(t)\chi)|_{x=a(t)} = 0, \quad (\alpha\rho\chi_x + \gamma_r(t)\chi)|_{x=b(t)} = 0, \quad t \in (0,T). \tag{80}$$

Here the functions A, B are defined in (66), and γ_l, γ_r, β_a, β_b are given and satisfy the conditions:

$$0 < \gamma_{1a}(t) \le \gamma_l(t) \le \gamma_{2a}(t) < \infty, \quad 0 < \gamma_{1b}(t) \le \gamma_r(t) \le \gamma_{2b}(t) < \infty, \tag{81}$$
$$\beta_a(t) = \beta_{0a}(t) + \Lambda_a\beta_l(t), \quad \beta_b(t) = \beta_{0b}(t) + \Lambda_b\beta_r(t),$$

where

$$|\beta_l(t)| \le K_1 \max\{K_2;\, |\zeta(a(t),t)|\},$$

$$|\beta_r(t)| \leq K_1 \max\{K_2;\ |\zeta(b(t),t)|\}, \tag{82}$$

where $\gamma_{1a}, \gamma_{2a}, \gamma_{1b}, \gamma_{2b}, \in H^1(0,T)$, constants $K_1 > 0$, $K_2 > 0$ do not depend on t.

Note that the adjoint system is nonlinear. The nonlinearity is expressed by the multiplicative boundary condition for the function adjoint to temperature.

Definition 2. A weak solution of problem (73)–(80) is a function $\{\xi,\ \tau,\ \zeta,\ \chi\} \in W$, where W is defined in (13), such that

$$\xi_t \in L^2((0,T); H^{-1}(\Omega_t)), \quad \langle \xi|_{t=T}, v_1\rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} = 0 \quad \forall v_1 \in H_0^1(\Omega_t),$$

$$\zeta_t \in L^2((0,T); \tilde{H}^{-1}(\Omega_t)), \quad \langle \zeta|_{t=T}, v_2\rangle_{\tilde{H}^{-1}(\Omega_t)\times H^1(\Omega_t)} = 0 \quad \forall v_2 \in H^1(\Omega_t)$$

and almost everywhere on $(0,T)$ the following conditions are satisfied:

$$-\langle \xi_t, v_1\rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} + (\rho\xi_x, v_{1x}) -$$
$$-(\tau, v_{1x}) - ((\nu\rho u_x + A\rho)\zeta, v_{1x}) = 0 \quad \forall v_1 \in H_0^1(\Omega_t), \tag{83}$$

$$\tau = \int_t^T \rho^2 (A\xi_x + a\theta_x\zeta_x + \alpha\varphi_x\chi_x - (Au_x + bB)\zeta - B\chi)\, ds, \tag{84}$$

$$-\langle \zeta_t, v_2\rangle_{\tilde{H}^{-1}(\Omega_t)\times H^1(\Omega_t)} + a(\rho\zeta_x, v_{2x}) +$$
$$+R(\rho u_x \zeta, v_2) - R(\rho\xi_x, v_2) +$$
$$+4\kappa_\alpha(\rho^{-1}\theta^3(b\zeta - \chi), v_2) + I_1(t) = 0 \quad \forall v_2 \in H^1(\Omega_t), \tag{85}$$

$$\alpha(\rho\chi_x, v_{3x}) + \kappa_\alpha(\rho^{-1}(\chi - b\zeta), v_3) + I_2(t) = 0 \quad \forall v_3 \in H^1(\Omega_t), \tag{86}$$

where

$$I_1(t) = \beta_a(t)[\zeta v_2]|_{x=b(t)} + \beta_b(t)[\zeta v_2]|_{x=a(t)} - \tag{87}$$
$$-[(\theta - d_b)v_2]|_{x=b(t)} - [(\theta - d_a)v_2]|_{x=a(t)}, \tag{88}$$
$$I_2(t) = \gamma_r(t)[\chi v_3]_{x=b(t)} + \gamma_l(t)[\chi v_3]_{x=a(t)}. \tag{89}$$

Theorem 5. Let $\{u,\ \rho,\ \theta,\ \varphi\} \in Y$ and the inequality

$$\int_0^T (|\theta(a) - d_a|^2 + |\theta(b) - d_b|^2)\, dt \leq \frac{\tilde{m}}{2c}, \tag{90}$$

where the constant $c > 1$ does not depend on t, and $\tilde{m} = \min\{m_1,\ am_1,\ \alpha m_1\}$. Then there exists a weak solution to problem (73)–(80).

Proof. Note that for the weak solution of problem (73)–(80) the following equalities are valid

$$\langle \xi_t, \xi\rangle_{H^{-1}(\Omega_t)\times H_0^1(\Omega_t)} = -\frac{1}{2}\frac{d}{dt}\|\xi\|^2,$$

$$\langle \zeta_t, \zeta\rangle_{\tilde{H}^{-1}(\Omega_t)\times H^1(\Omega_t)} = -\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 - m(t)(\zeta|_{x=b(t)} - \zeta|_{x=a(t)}). \tag{91}$$

Taking into account (91), from (83)–(86), we find

$$-\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \int\limits_{\Omega_t} \rho|\xi_x|^2\,dx \le \|\xi_x\|\|\tau\| + \max_{x\in\Omega_t}|\nu\rho u_x + A\rho|\|\zeta\|\|\xi_x\|,$$

$$-\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + a\int\limits_{\Omega_t}\rho|\zeta_x|^2\,dx + 4\kappa_\alpha b\int\limits_{\Omega_t}\rho^{-1}\theta^3|\zeta|^2\,dx \le$$

$$\le R\max_{x\in\Omega_t}|\rho u_x|\|\zeta\|\|\zeta_x\|^2 + M_1R\|\xi_x\|\|\zeta\| + 4\kappa_\alpha M_1^3 m_1^{-1}\|\chi\|\|\zeta\| +$$

$$+K_1|\zeta_a|^2 + K_1|\zeta_b|^2 + K_2|\zeta_a|^3 + K_2|\zeta_b|^3 + |\theta(b) - d_d||\zeta_b| + |\theta(a) - d_a||\zeta_a|,$$

$$\alpha\int\limits_{\Omega_t}\rho|\chi_x|^2\,dx + \kappa_\alpha m_1^{-1}\int\limits_{\Omega_t}|\chi|^2\,dx + \gamma_r|\chi_b|^2 + \gamma_l|\chi_a|^2 \le b\kappa_\alpha m_1^{-1}\|\chi\|\|\zeta\|. \quad (92)$$

Here $\zeta_a = \zeta|_{x=a(t)}$, $\zeta_b = \zeta|_{x=b(t)}$, $\gamma_a = \gamma|_{x=a(t)}$, $\gamma_b = \gamma|_{x=b(t)}$.

Applying the Cauchy inequality to the right-hand sides of the inequalities (92), and taking into account that $\rho \ge m_0 > 0$, we obtain the estimate

$$-\frac{d}{dt}\|\xi\|^2 - \frac{d}{dt}\|\zeta\|^2 + m_1\|\xi_x\|^2 + m_1 a\|\zeta_x\|^2 + \alpha m_1\|\chi_x\|^2 \le$$

$$\le C\|\tau\|^2 + C\|\zeta\|^2\|\zeta_x\|^2 + C(1+\max_{x\in\Omega_t}|u_x|^2)\|\zeta\|^2 + |\theta(a) - d_a|^2 + |\theta(b) - d_b|^2, \quad (93)$$

where the constant C does not depend on t. We obtain an estimate for τ:

$$\|\tau\|^2 \le (k/C)\int\limits_t^T(|\xi_x|^2 + |\zeta_x|^2 + |\chi_x|^2)\,ds, \quad (94)$$

where

$$k = C_1\int\limits_0^T \max_{x\in\Omega_t}(|A|^2 + |\theta_x|^2 + |\varphi_x|^2 + |Au_x + bB|^2 + |B|^2)\,ds < \infty.$$

Let us denote $\tilde{m} = \min\{m_1;\ am_1;\ \alpha m_1\}$,

$$r(t) = C(1 + \max_{x\in\Omega_t}|u_x|^2), \quad \delta(t) = |\theta(a) - d_a|^2 + |\theta(b) - d_b|^2,$$

$$d_1(t) = \|\xi(t)\|^2 + \|\zeta(t)\|^2, \quad d_2(t) = \int\limits_t^T(\|\xi_x(s)\|^2 + \|\zeta_x(s)\|^2 + \|\chi_x(s)\|^2)\,ds. \quad (95)$$

Note that

$$d_1(t) > 0, \quad d_2(t) > 0, \quad -d_2'(t) > 0 \quad \text{for a.e. } t \in (0,T);$$

$$d_1(T) = 0, \quad d_2(T) = 0,$$

$$\int\limits_0^T r(t)dt < \infty, \quad \int\limits_0^T \delta(t)dt < \infty. \quad (96)$$

Let's substitute (94), (95) into (93), we get

$$-d_1'(t) - \tilde{m} d_2'(t) \le k d_2(t) + r(t) d_1(t) - c d_1(t) d_2'(t) + \delta(t), \quad t \in (0,T). \quad (97)$$

Here the constants $k \ge 0$, $c > 1$.

Next, we obtain an estimate for the functions d_1, d_2 from inequality (97). First, we get rid of the first and second terms on the right-hand side (97).

Let $r_1 \in C[0,T]$ be an arbitrary function. Let us denote by

$$r_1(t) = e^{-\frac{2k}{\tilde{m}}(T-t) - \int_t^T r(s)\,ds}, \quad r_0 = r_1(0) \le r_1(t) \le 1, \quad t \in [0,T].$$

Multiply (97) by $r_1(t) > 0$ and transform to the form

$$-[(d_1(t) + \frac{\tilde{m}}{2} d_2(t)) r_1(t) + \Lambda(t)]' - \frac{\tilde{m}}{2} d_2'(t) r_1(t) \le c d_1(t)(-d_2'(t)) r_1(t),$$
$$t \in (0,T). \quad (98)$$

Here

$$\Lambda(t) = -\int_t^T r_1(s)\delta(s)\,ds, \quad \Lambda(T) = 0.$$

Now let us estimate the right-hand side (98). Let $r_2 \in C[0,T]$ be an arbitrary function. Let us denote by

$$r_2(t) = e^{N\int_t^T d_2'(s)\,ds}, \quad N = const > 0, \quad t \in (0,T).$$

Multiply (98) by $r_2(t) > 0$ and transform to the form

$$-[((d_1(t) + \frac{\tilde{m}}{2} d_2(t)) r_1(t) + \Lambda(t)) r_2(t)]' + (-d_2'(t))[N - c] d_1(t) r_1(t) r_2(t) +$$
$$+ (-d_2'(t))[\frac{\tilde{m} r_0}{2} + N\Lambda(t)] r_2(t) \le 0, \quad t \in (0,T). \quad (99)$$

Let us set $N = c$ and find the condition for the restriction of the function $\Lambda(t) < 0$ from the inequality

$$\frac{\tilde{m} r_0}{2} + N\Lambda(t) \ge \frac{\tilde{m} r_0}{2}\varepsilon, \quad \varepsilon \in [0,1), \quad t \in (0,T).$$

The last inequality will be satisfied if

$$\int_0^T \delta(t)\,dt \le \frac{\tilde{m}}{2c}(1-\varepsilon), \quad \varepsilon \in [0,1).$$

We integrate (99) with respect to t from an arbitrary t to T, taking into account the conditions $d_1(T) = 0$, $d_2(T) = 0$, $\Lambda(T) = 0$ and (90), we obtain the inequality

$$(d_1(t) + \frac{\tilde{m}}{2} d_2(t)) r_1(t) + \frac{\tilde{m} r_0}{2}\varepsilon r_2^{-1}(t) \int_t^T (-d_2'(s)) r_2(s)\,ds \le -\Lambda(t) \le$$

$$\leq \int\limits_0^T \delta(t)\,dt = \int\limits_0^T (|\theta(a)-d_a|^2 + |\theta(b)-d_b|^2)\,dt.$$

Note that $r_1(t) \geq r_0 > 0$, $-d_2'(t) > 0$, $r_2(t)$ is a monotonically increasing function for all $t \in [0,T]$, therefore, transforming the integral term on the left-hand side of the last inequality, we find

$$d_1(t) + \frac{\tilde{m}}{2} d_2(t) \leq \frac{1}{r_0} \int\limits_0^T (|\theta(a)-d_a|^2 + |\theta(b)-d_b|^2)\,dt. \tag{100}$$

Thus, taking into account (95), from (100) we obtain

$$\max_{t\in[0,T]} (\|\xi(t)\|^2 + \|\zeta(t)\|^2) + \int\limits_0^T (\|\xi_x(s)\|^2 + \|\zeta_x(s)\|^2 + \|\chi_x(s)\|^2)\,dt \leq$$

$$\leq C \int\limits_0^T (|\theta(a)-d_a|^2 + |\theta(b)-d_b|^2)\,dt. \tag{101}$$

From (94) we obtain the estimate

$$\max_{t\in[0,T]} \|\tau\|^2 \leq C \int\limits_0^T (|\theta(a)-d_a|^2 + |\theta(b)-d_b|^2)\,dt. \tag{102}$$

Given u, ρ, θ, φ for the difference of functions $\xi = \xi_1 - \xi_2$, $\tau = \tau_1 - \tau_2$, $\zeta = \zeta_1 - \zeta_2$, $\chi = \chi_1 - \chi_2$, in the same way we obtain the following a priori estimates

$$\max_{t\in[0,T]} (\|\xi(t)\|^2 + \|\zeta(t)\|^2 + \|\tau\|^2) + \int\limits_0^T (\|\xi_x(s)\|^2 + \|\zeta_x(s)\|^2 + \|\chi_x(s)\|^2)\,dt \leq 0.$$

guaranteeing the uniqueness of the solution.

The system (73)–(80) will be called the adjoint system for the problem

Theorem 6. For each solution $\mathbf{y}$ of the extremal problem (19) there exists a unique solution of the adjoint problem (73)–(80) and the following relations are valid:

$$\beta_l(t) = \max\big\{\frac{1}{\kappa_l}\Lambda_a(t)[\theta(a(t),t) - \theta_b(t)]\zeta(a(t),t);\ \beta_{01}(t)\big\},$$

$$\beta_r(t) = \max\big\{\frac{1}{\kappa_r}\Lambda_b(t)[\theta(b(t),t) - \theta_b(t)]\zeta(b(t),t);\ \beta_{02}(t)\big\} \tag{103}$$

almost everywhere on $(0,T)$, where

$$\beta_0(t) = \{\beta_{01}(t),\ \beta_{02}(t)\} =$$

$$= \{(\beta_{min} - \min_{t\in(0,T)} \beta_{0a})/\Lambda_a(t),\ (\beta_{min} - \min_{t\in(0,T)} \beta_{0a})/\Lambda_b(t)\}. \tag{104}$$

And the variational inequality

$$\int_0^T [\varphi(a(t),t) - \theta_b^4(t)]\chi(a(t),t)[w(t) - \gamma_l(t)]\,dt +$$

$$+ \int_0^T [\varphi(b(t),t) - \theta_b^4(t)]\chi(b(t),t)[r(t) - \gamma_r(t)]\,dt \le 0 \quad \forall \{w, r\} \in U_\gamma. \tag{105}$$

Proof. Let us calculate the Gateaux differential of the functional $J_\kappa[\mathbf{y}]$ on the element $\mathbf{y}$ in the direction $\mathbf{z}$:

$$DJ_\kappa[\mathbf{y}](\mathbf{z}) = \int_0^T (\theta(a(t),t) - d_a(t))D\theta(\mathbf{y})|_{x=a(t)}\,dt +$$

$$+ \int_0^T (\theta(a(t),t) - d_a(t))D\theta(\mathbf{y})|_{x=b(t)}\,dt +$$

$$+ \kappa_l \int_0^T \beta_l(t) z_1\,dt + \kappa_r \int_0^T \beta_r(t) z_2\,dt. \tag{106}$$

Here $D\theta(\mathbf{y}) = \eta$. Let $\phi_1 = \xi$, $\phi_2 = \zeta$, $\phi_3 = \chi$ in (62), (64), (65), and $v_1 = v$, $v_2 = \eta$, $v_3 = \psi$ in (83), (85), (86), respectively. Subtract the obtained relations and integrate with respect to t from 0 to T, taking into account the following property, which is valid for any continuous functions $f(t)$ and $g(t)$:

$$\int_0^T f(t) \int_t^T g(s)\,ds\,dt - \int_0^T g(t) \int_0^t f(s)\,ds\,dt = 0$$

and, taking into account (106), we find

$$\int_0^T (\kappa_l \beta_l(t) - \Lambda_a(t)[\theta(a(t),t) - \theta_b(t)]\zeta(a(t),t)) z_1(t)\,dt +$$

$$+ \int_0^T (\kappa_r \beta_r(t) - \Lambda_b(t)[\theta(b(t),t) - \theta_b(t)]\zeta(b(t),t)) z_2(t)\,dt +$$

$$- \int_0^T [\varphi(a(t),t) - \theta_b^4(t)]\chi(a(t),t) z_3(t)\,dt -$$

$$- \int_0^T [\varphi(b(t),t) - \theta_b^4(t)]\chi(b(t),t) z_4(t)\,dt = DJ_\kappa[\mathbf{y}](\mathbf{z}). \tag{107}$$

Note that U_{ad} is convex, so we can set $z_1 = w_1 - \beta_l$, $z_2 = w_2 - \beta_r$, $z_3 = w_3 - \gamma_l$, $z_4 = w_4 - \gamma_r$ for any $\mathbf{w} \in U_{ad}$, where $\mathbf{w} = \{w_1,\, w_2,\, w_3,\, w_4\}$.

Let us introduce the following notations for vector functions:

$$\beta(t) = \{\beta_l(t),\, \beta_r(t)\} \in U_\beta, \quad \gamma(t) = \{\gamma_l(t),\, \gamma_r(t)\} \in U_\gamma, \quad \kappa = \{\kappa_l,\, \kappa_r\},$$

$$P_\zeta(t) = \{\Lambda_a(t)[\theta(a(t),t) - \theta_b(t)]\zeta(a(t),t),\ \Lambda_b(t)[\theta(b(t),t) - \theta_b(t)]\zeta(b(t),t)\},$$

$$P_\chi(t) = \{[\varphi(a(t),t) - \theta_b^4(t)]\chi(a(t),t),\ [\varphi(b(t),t) - \theta_b^4(t)]\chi(b(t),t)\}. \quad (108)$$

Due to the necessary optimality condition $\mathbf{y}$, the inequality $DJ_\kappa[\mathbf{y}](\mathbf{z}) \geq 0$ $\forall \mathbf{z} \in U_{ad}$ holds. Taking into account (108), we rewrite (107) as follows

$$\int_0^T (\kappa\beta - P_\zeta(t))(w_\beta - \beta)\, dt - \int_0^T P_\chi(t)(w_\gamma - \gamma)\, dt \geq 0 \qquad (109)$$

for arbitrary $w_\beta \in U_\beta$ and $w_\gamma \in U_\gamma$, where $w_\beta = \{w_1,\ w_2\}$, $w_\gamma = \{w_3,\ w_4\}$. Here $\kappa\beta$ is a vector function with components $\{\kappa_1\beta_l,\, \kappa_2\beta_r\}$. In (109), the vector function $\gamma \in U_\gamma$ will be determined from the condition

$$\int_0^T P_\chi(t)(w_\gamma - \gamma)\, dt \leq 0 \quad \forall w_\gamma \in U_\gamma. \qquad (110)$$

Then from (109), (110) we obtain

$$\int_0^T (\kappa\beta - P_\zeta(t))(w_\beta - \beta)\, dt \geq 0 \quad \forall w_\beta \in U_\beta. \qquad (111)$$

It follows that either $\beta = \beta_0$ and $\kappa\beta - P_\zeta(t) = 0$, or $\beta = \beta_0$, or $\kappa\beta - P_\zeta(t) = 0$, where β_0 is defined in (104). The first option implies $P_\zeta(t) = \kappa\beta_0$ and is of no interest. Let us consider the cases when the following options are fulfilled:

(1) $\beta = \beta_0, \quad \kappa\beta > P_\zeta(t);$
(2) $\beta > \beta_0, \quad \kappa\beta = P_\zeta(t).$

Then the optimality conditions for β can be written as

$$\beta = \kappa^{-1} \max\{P_\zeta;\, \kappa\beta_0\}. \qquad (112)$$

Next, we obtain the variational maximum principle for the optimal solution $\gamma \in U_\gamma$. Let us denote by

$$P_{\chi 1}(t) = [\varphi(a(t),t) - \theta_b^4(t)]\chi(a(t),t),$$

$$P_{\chi 2}(t) = [\varphi(b(t),t) - \theta_b^4(t)]\chi(b(t),t) \quad t \in (0,T).$$

Then from the variational inequality (111) we find in the standard way

$$P_{\chi 1}(t)(w_3 - \gamma_l) + P_{\chi 2}(t)(w_4 - \gamma_r) \leq 0 \quad \forall w_3 \in [\gamma_{1a};\ \gamma_{2a}],$$

$$\forall w_4 \in [\gamma_{1b};\ \gamma_{2b}] \quad \text{a.e. on } (0,T).$$

Inequality (109) provides an analogue of the «bang-bang» principle. Indeed,

$$\gamma_l = \begin{cases} \gamma_{2a}, & \text{если } P_{\chi 1} > 0; \\ \gamma_{1a}, & \text{если } P_{\chi 1} < 0; \\ \gamma_{2a}, & \text{если } P_{\chi 1} > 0; \\ \gamma_{1a}, & \text{если } P_{\chi 1} < 0; \end{cases} \qquad \gamma_r = \begin{cases} \gamma_{2b}, & \text{если } P_{\chi 2} > 0; \\ \gamma_{1b}, & \text{если } P_{\chi 2} < 0; \\ \gamma_{1b}, & \text{если } P_{\chi 2} < 0; \\ \gamma_{2b}, & \text{если } P_{\chi 2} > 0. \end{cases} \tag{113}$$

Equalities (103), (113) together with the direct (5)–(9) and adjoint problem (73)–(80) are necessary conditions for optimality.

6 PINN method

The method of using neural networks to solve differential equations proposed in [27] consists of training neural networks to minimize a quadratic functional G, which includes terms for the residuals of equations, initial and boundary conditions, and additional information.

To solve optimal control problem the functional has the following form:

$$G = G_r + G_0 + G_1 + G_2 + G_{data}, \tag{114}$$

where G_r is the term for the residual of equations (1)–(5), G_0, G_1, G_2 are the terms for the initial and boundary conditions (1)–(5), respectively, G_{data} is the term with additional information (1)–(5).

The terms G_r, G_0, G_1, G_2, G_{data} have the following form:

$$G_r = \frac{K_r}{N_r} \sum_{i=1}^{N_r} \left[r_1^2(x_i^r, t_i^r) + r_2^2(x_i^r, t_i^r) + r_3^2(x_i^r, t_i^r) + r_4^2(x_i^r, t_i^r) \right],$$

$$G_0 = \frac{K_0}{N_0} \sum_{i=1}^{N_0} \Big[\big(\widehat{u}(x_i^0, t_i^0) - u_0(x_i^0, t_i^0)\big)^2 + \big(\widehat{\rho}(x_i^0, t_i^0) - \rho_0(x_i^0, t_i^0)\big)^2 + \\ + \Big(\widehat{\theta}(x_i^0, t_i^0) - \theta_0(x_i^0, t_i^0)\Big)^2 \Big],$$

$$G_1 = \frac{K_1}{N_1} \sum_{i=1}^{N_1} \Big[\big(\widehat{u}(x_i^1, t_i^1) - u_1(x_i^1, t_i^1)\big)^2 + \big(\widehat{\rho}(x_i^1, t_i^1) - \rho_1(x_i^1, t_i^1)\big)^2 + \\ + \left(-\alpha \frac{\partial \widehat{\varphi}(x_i^1, t_i^1)}{\partial x} + \widehat{\gamma}_l (\widehat{\varphi}(x_i^1, t_i^1) - \theta_b^4) \right)^2 + \\ + \left(-a \frac{\partial \widehat{\theta}(x_i^1, t_i^1)}{\partial x} + \widehat{\beta}_l (\widehat{\theta}(x_i^1, t_i^1) - \theta_b) \right)^2 \Big],$$

$$G_2 = \frac{K_2}{N_2}\sum_{i=2}^{N_2}\Bigg[\left(\widehat{u}(x_2^1,t_i^2) - u_2(x_i^2,t_i^2)\right)^2 + \\ + \left(a\frac{\partial\widehat{\theta}(x_i^2,t_i^2)}{\partial x} + \widehat{\beta}_r(\widehat{\theta}(x_i^2,t_i^2)-\theta_b)\right)^2 + \\ + \left(\alpha\frac{\partial\widehat{\varphi}(x_i^2,t_i^2)}{\partial x} + \widehat{\gamma}_r(\widehat{\varphi}(x_i^2,t_i^2)-\theta_b^4)\right)^2\Bigg],$$

$$G_{data} = J[\mathbf{a}] + \int_0^T \frac{\kappa_l}{2}\left|\widehat{\beta}_l(t)\right|^2 dt + \int_0^T \frac{\kappa_l}{2}\left|\widehat{\beta}_r(t)\right|^2 dt,$$

$$J[\mathbf{a}] = \int_0^T \left(\left|\widehat{\theta}(0,t) - \theta_d(0,t)\right|^2 + \left|\widehat{\theta}(L,t) - \theta_d(L,t)\right|^2\right) dt,$$

where $x_i^r \in (0,1), t_i^r \in (0,T]$, $x_i^0 \in [0,L], t_r^i = 0$, $x_1^r = 0, t_1^r \in [0,T]$, $x_2^r = L, t_2^r \in [0,T]$, N_r, N_0, N_1, N_2 – are total numbers of point sets for the corresponding regions, K_r, K_0, K_1, K_2 – are weighting coefficients.

Here $\widehat{u}, \widehat{\rho}, \widehat{\theta}, \widehat{\varphi}, \widehat{\beta}_l, \widehat{\beta}_r, \widehat{\gamma}_l, \widehat{\gamma}_r$ – are approximations of unknowns by neural networks, which have the form:

$$f(X) = H_{swish}\left(H_{tanh}\left(H_{tanh}\left(XW_1^f\right)W_2^f\right)W_3^f\right)W_4^f, \qquad f = \widehat{u}, \widehat{\rho}, \widehat{\theta},$$

$$f(t) = H_{tanh}\left(H_{tanh}\left(tW_1^f\right)W_2^f\right)W_3^f, \qquad f = \widehat{\varphi}, \widehat{\beta}_l, \widehat{\beta}_r, \widehat{\gamma}_l, \widehat{\gamma}_r,$$

where $W_1^f, W_2^f, W_3^f, W_4^f$ – are weight matrices between layers of a neural network, $H_{tanh}(x) = (e^x - e^{-x})/(e^x + e^{-x}), H_{swish}(x) = x/(1+e^{-x})$.

Thus, the optimal control problem is reduced to the problem of minimizing the functional G:

$$G \to min. \tag{115}$$

7 Numerical experiments

The following characteristics were chosen for all numerical experiments:

$$K_r = 1, \qquad K_0 = K_1 = K_2 = 10, \qquad k_l = k_r = 10^{-2}.$$

When using the PINN method, the problem is solved in dimensionless variables. All graphs will also be presented in dimensionless variables.

In total, $N_r = 10000$ collocation points were generated, while at the boundaries and at the initial time, $N_0 = N_1 = N_2 = 200$ points were generated.

In the first numerical experiment, temperature observations θ_d are considered at the boundary. The physical characteristics of the medium correspond to carbon dioxide and, together with the other parameters of the problem, have the following form:

$$X = 50\,[\mathrm{m}]\,, \qquad T = 30\,[\mathrm{s}]\,, \tag{116}$$

$$c_v = 700 \left[\frac{\mathrm{J}}{\mathrm{kg}\cdot\mathrm{K}}\right], \qquad R = 8.314 \left[\frac{\mathrm{J}}{\mathrm{mole}\cdot\mathrm{K}}\right], \tag{117}$$

$$\mu = 5\cdot 10^{-5}\,[\mathrm{Pa}\cdot\mathrm{s}]\; k = 2.2\cdot 10^{-2}\left[\frac{\mathrm{W}}{\mathrm{m}\cdot\mathrm{K}}\right] \tag{118}$$

$$u_0 = 10\left[\frac{\mathrm{m}}{\mathrm{s}}\right], \qquad u_1 = u_2 = 10 + 2t\left[\frac{\mathrm{m}}{\mathrm{s}}\right], \tag{119}$$

$$\theta_0(x) = 389.7 - 0.866x^2\,[\mathrm{K}],$$

$$\theta_b(x,t) = 389.7 - 0.866x^2 - 0.866t\,[\mathrm{K}], \tag{120}$$

$$\rho_0 = 1.8\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \qquad \rho_1 = 1.8 - 0.2t\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \tag{121}$$

$$\theta_d(x) = 346.4 + 0.433x\,[\mathrm{K}], \tag{122}$$

During the calculation, the functional G reached the value 10^{-4}, which the authors consider to be corresponding to the solution of the problem. The obtained solution is shown in figures 1.a, 1.b, 1.c, 1.d at different moments in time.

The profiles of the predicted boundary coefficients are shown in Figures 2.a, 2.b. The coefficients γ_1 and γ_2 are nonlinear functions, but when calculated in the equations they passed through a mask function, turning them into piecewise constant functions:

$$f_{mask}(\gamma) = \begin{cases} min(\gamma), & \gamma \le \frac{min(\gamma)+max(\gamma)}{2}, \\ max(\gamma), & \gamma > \frac{min(\gamma)+max(\gamma)}{2}. \end{cases}$$

The temperature profile at the observed points is shown in figures 3.a, 3.b.

In the second numerical experiment, temperature observations θ_d are considered inside the domain. The parameters of the problem are:

$$X = 50\,[\mathrm{m}], \qquad T = 30\,[\mathrm{s}], \tag{123}$$

$$c_v = 700 \left[\frac{\mathrm{J}}{\mathrm{kg}\cdot\mathrm{K}}\right], \qquad R = 8.314 \left[\frac{\mathrm{J}}{\mathrm{mole}\cdot\mathrm{K}}\right], \tag{124}$$

$$k = 2.2\cdot 10^{-2}\left[\frac{\mathrm{W}}{\mathrm{m}\cdot\mathrm{K}}\right]\mu = 5\cdot 10^{-5}\,[\Pi\mathrm{a}\cdot\mathrm{c}], \tag{125}$$

$$u_0 = 10\left[\frac{\mathrm{m}}{\mathrm{s}}\right], \qquad u_1 = u_2 = 10 + 2t\left[\frac{\mathrm{m}}{\mathrm{s}}\right], \tag{126}$$

$$\theta_0(x) = 389.7 - 0.866x^2\,[\mathrm{K}],$$

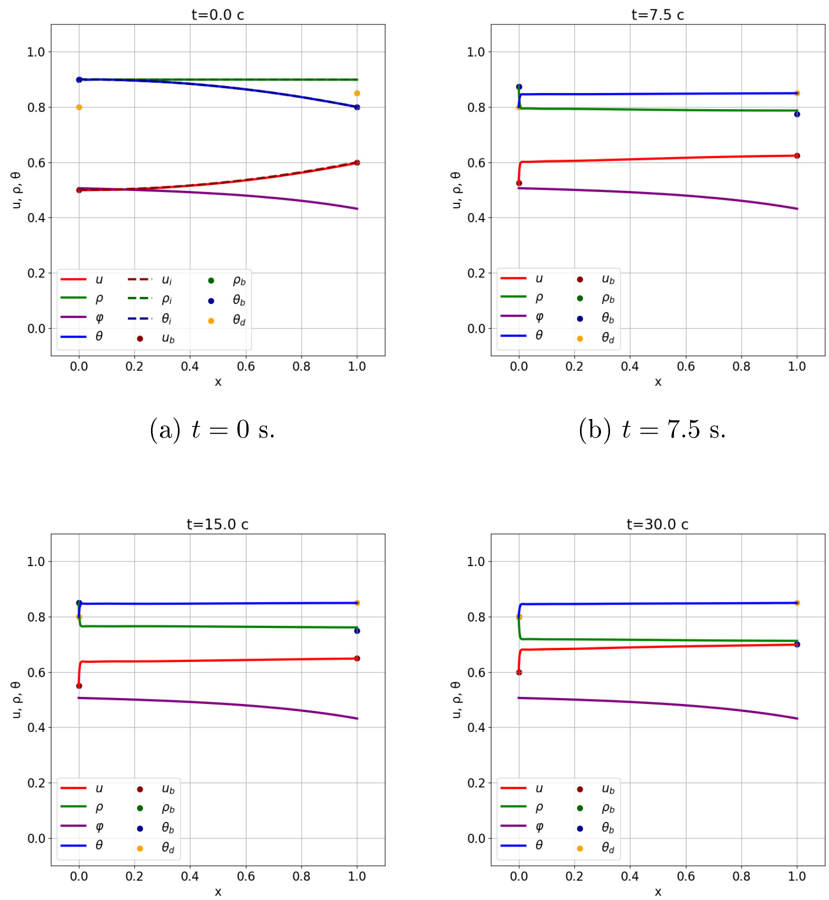
$$\theta_b(x,t) = 389.7 - 0.866x^2 - 0.866t\,[\mathrm{K}], \tag{127}$$

$$\rho_0 = 1.8\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \qquad \rho_1 = 1.8 - 0.2t\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right], \tag{128}$$

$$\theta_d(x) = 346.4 + 0.433x\,[\mathrm{K}], \tag{129}$$

The obtained solution is shown in figures 4.a, 4.b, 4.c, 4.d at different moments in time.

The profiles of the predicted boundary coefficients is presented in figures 5.a, 5.b.



(a) $t = 0$ s. (b) $t = 7.5$ s.

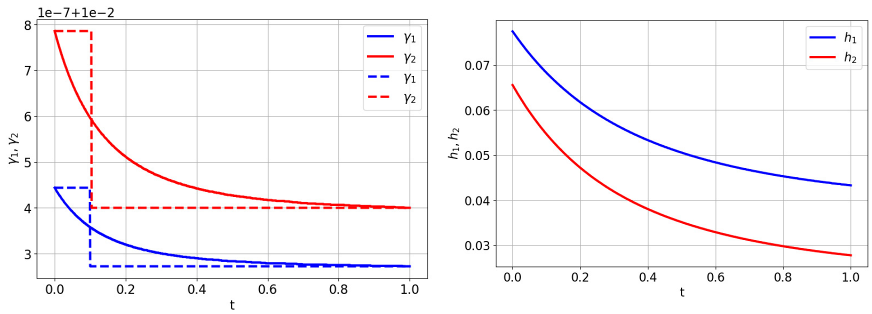
(c) $t = 15$ s. (d) $t = 30$ s.

FIG. 1. Solution of the optimal control problem (115) with parameters (116)-(122) at different moments of time.

The temperature profile at the observed points is shown in figures 6.a, 6.b and 6.c.

8 Conclusion

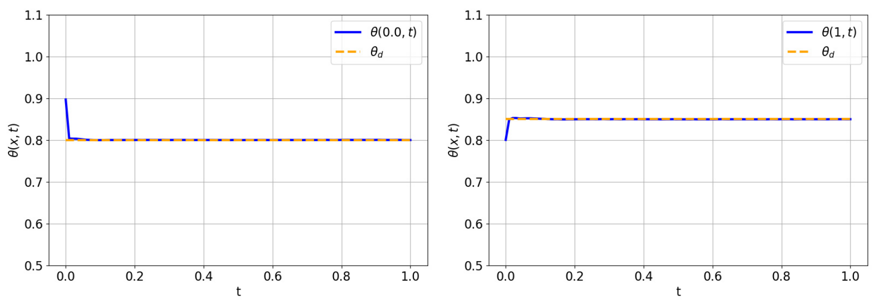
The paper presents a numerical and theoretical study of the problem of controlling one-dimensional viscous gas flow. The theorem of the existence of optimal control is proved, and the necessary conditions for the optimality system are derived. A numerical solution to the problem of optimal control of the viscous heat-conducting gas flow in a one-dimensional region is obtained



(a) γ_1, γ_2..

(b) β_1, β_2.

FIG. 2. Boundary coefficients.



(a) $x = 0.$ s.

(b) $x = 1.$ s.

FIG. 3. Temperature profile at observed points on the boundaries.

using the PINN method, based on the neural networks usage. The possibility of solving the inverse problem of controlling one of several unknown functions describing the state of the system is shown, where the heat transfer coefficient and the reflection coefficient from the boundary of the region are selected as the control. The advantage of using the PINN method to solve the inverse problem of a strongly nonlinear, singularly perturbed system of equations is the absence of the need to linearize the system and solve the optimality system, as well as the possibility of solving the problem on a uniform and not excessively dense grid. It is shown that the use of the method does not require excessive computing resources for operation.

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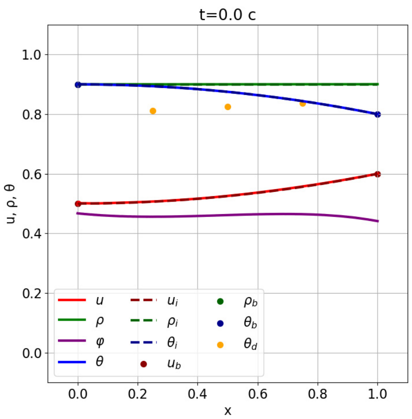
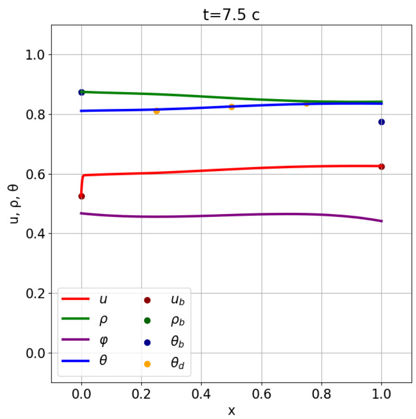
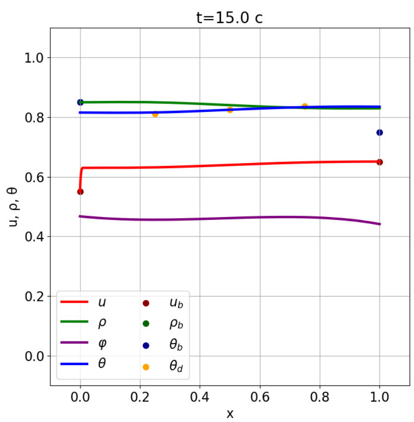


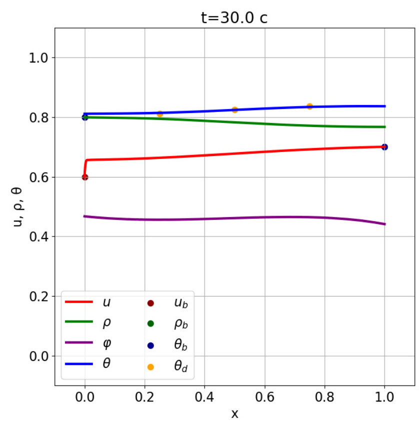
(a) $t = 0$ s.



(b) $t = 7.5$ s.



(c) $t = 15$ s.



(d) $t = 30$ s.

FIG. 4. Solution of the optimal control problem (115) with parameters (123)-(129) at different moments of time.

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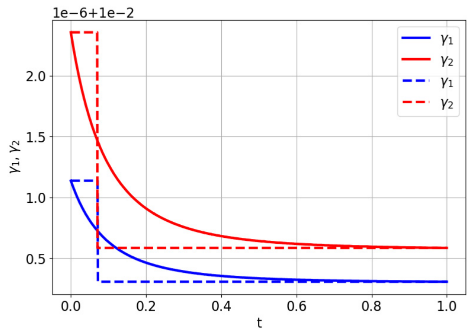
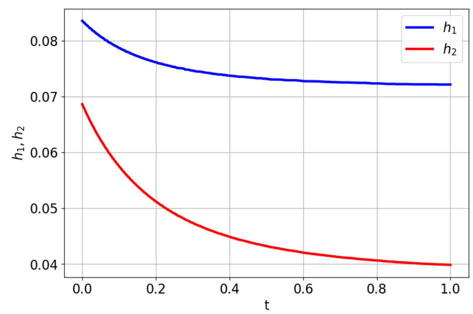
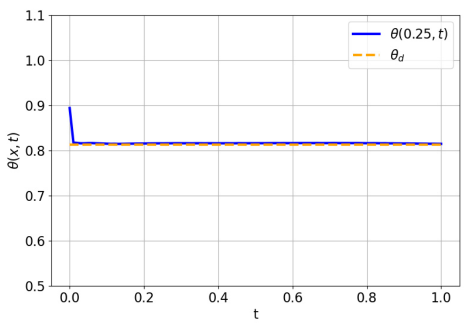


(a) γ_1, γ_2..

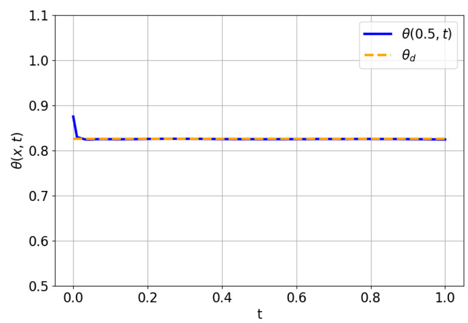


(b) β_1, β_2..

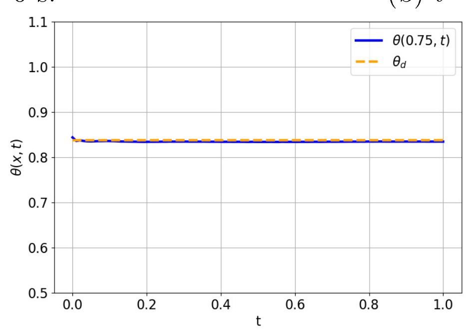
FIG. 5. Boundary coefficients.



(a) $t = 0$ s.



(b) $t = 15$ s.



(c) $t = 30$ s.

FIG. 6. Temperature profile at observed points inside the domain.

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