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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 22, No. 1, pp. 46–53 (2025) УДК 519.17
https://doi.org/10.33048/semi.2025.22.004 MSC 05C20, 05C38

VERTEX-DISJOINT CYCLES OF DIFFERENT LENGTHS IN TOURNAMENTS OF CYCLES

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Communicated by A.V. Pyatkin

Abstract: We show in this paper that every tournament of k cycles $D = (V, A)$ with $k \geq 2$ and minimum out-degree 3, except the digraph D_8^3, contains two disjoint cycles of different lengths.

Keywords: digraph, tournament of cycles, vertex-disjoint cycles, cycles of different lengths.

1 Introduction

We consider here only a *finite simple digraph*, i.e., a digraph that has a finite number of vertices, no loop, and no multiple arc. Unless otherwise indicated, our graph-theoretic terminology will follow [2]. We will also adopt notation and basic definitions that are used in [16].

Let D be a digraph. Then the vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$ (or by V and A for short), respectively. A vertex $v \in V$ is called an *out-neighbor* of a vertex $u \in V$ if $(u, v) \in A$. We denote the set of all out-neighbors of u by $N_D^+(u)$. The *out-degree* of $u \in V$, denoted by $d_D^+(u)$, is $|N_D^+(u)|$. The *minimum out-degree* of D is $\min\{d_D^+(u) \mid u \in V\}$. Similarly, a vertex $w \in V$ is called an *in-neighbor* of a vertex $u \in V$ if $(w, u) \in A$. We denote the set of all in-neighbors of u by $N_D^-(u)$. The *in-degree* of $u \in V$,

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Received May, 9, 2024, Published March, 3, 2025.

denoted by $d_D^-(u)$, is $|N_D^-(u)|$. If $W \subseteq V$, then the subdigraph of D induced by W is denoted by $D[W]$.

Let $D = (V, A)$ be a digraph. Then we write uv for an arc $(u, v) \in A$ for short. By a *cycle* (resp., *path*) in D, we always mean a directed cycle (resp., directed path). By *disjoint cycles* in D, we always mean vertex disjoint cycles. A *chord* of a cycle C of D is an arc $uv \in A \setminus A(C)$ with $u, v \in V(C)$.

Thomassen in [18] has proved that every digraph with minimum out-degree at least 3 contains two vertex-disjoint cycles. Recently, in connection with 2-coloring of hypergraphs, Henning and Yeo have begun to study in [7] the existence of vertex-disjoint cycles of different lengths in digraphs and they have posed there several conjectures. One of these conjectures has been solved by Lichiardopol in [10], which asserts that every digraph with minimum out-degree at least 4 contains two vertex-disjoint cycles of different lengths. By the results obtained by Thomassen [18] and Lichiardopol [10], the investigation of structure for digraphs without vertex-disjoint cycles of different lengths can be restricted to digraphs with minimum out-degree 3. Up to now there has been a lot of research on this topic, especially the results achieved recently (see [4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 16]). Although the results achieved are relatively diverse, the problem has not been completely resolved, there are still many other classes of graphs that need to be researched. Therefore, in this article, we continue to contribute a research result on this topic, contributes to perfecting the research of the problem digraphs without vertex-disjoint cycles of different lengths.

An *oriented graph* is a digraph with no cycle of length 2. A *tournament of k cycles* with $k \geq 2$ is an oriented graph $D = (V, A)$ with a partition $V = V_1 \cup V_2 \cup \ldots \cup V_k$ such that the subdigraphs $D[V_1], D[V_2], \ldots, D[V_k]$ are cycles and for every two vertices $u \in V_i$ and $v \in V_j$ with $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$ exactly one of the arcs uv and vu is in A. We study the existence of vertex-disjoint cycles of different lengths in tournaments of k cycles with $k \geq 2$ and minimum out-degree 3.

In [7], Henning and Yeo have given an example of 3-regular digraphs having no vertex-disjoint cycles of different lengths. We denote this digraph here by D_8^3. This digraph has the vertex set $V(D_8^3) = \{v_0, v_1, \ldots, v_5, v, w\}$ and the arc set $A(D_8^3) = A_1 \cup A_2 \cup A_3$, where $A_1 = \{v_i v_j \mid j - i = 1 \text{ or } 2 \pmod 6\}$, $A_2 = \{vv_i, v_i w \mid i = 0, 2, 4\}$ and $A_3 = \{wv_i, v_i v \mid i = 1, 3, 5\}$. The digraph D_8^3 is illustrated in Figure 1.

We note that the digraph D_8^3 is a tournament of two cycles with minimum out-degree 3. So, it is natural to ask whether there are other tournaments of k cycles with $k \geq 2$ and minimum out-degree 3 without vertex-disjoint cycles of different lengths. In this paper, we will give the answer to this question by proving the following main result.

Theorem 1. *Every tournament of k cycles $D = (V, A)$ with $k \geq 2$ and minimum out-degree 3, except the digraph D_8^3, contains two disjoint cycles of different lengths.*

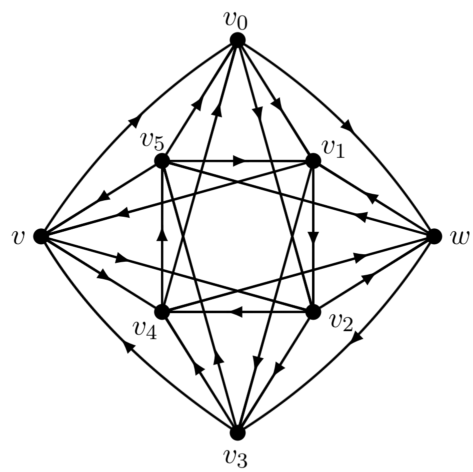

FIG. 1. The digraph D_8^3

Further information can be found for tournaments in the recent surveys [3, 19].

2 Preliminary results

Let $D = (V, A)$ be a tournament of k cycles with $k \geq 2$ and minimum out-degree 3, having no disjoint cycles of different lengths. Further, let $V = V_0 \cup V_1 \cup \ldots \cup V_{k-1}$ be the partition of D and $D[V_j] = A^j$, where $j \in \{0, 1, \ldots, k-1\}$, be the cycle. $A^0, A^1, \ldots, A^{k-1}$ must have same length $t \geq 3$ because D has no disjoint cycles of different lengths. Let

$$A^j = (a_0^j, a_1^j, \ldots, a_{t-1}^j, a_0^j), \text{ where } j \in \{0, 1, \ldots, k-1\}.$$

A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of cycles $C_1, C_2, \ldots, C_k$ in a digraph is called *bad* if there exist two vertex-disjoint cycles of different lengths in $\mathcal{C}$.

Let X and X' be two disjoint subsets of the vertex set V of a digraph $D = (V, A)$. We say that X *dominates* X' or X' *is dominated by* X if $X' \subseteq N_D^+(x)$ for each $x \in X$. We write $X \to X'$ if X dominates X'. If $X \to X'$ and $X = \{v\}$, then we simply say that v dominates X' and simply write $v \to X'$. Similarly, if $X \to X'$ and $X' = \{v'\}$, then we simply say that X dominates v' and simply write $X \to v'$. If $X = \{x\}$ and $X' = \{x'\}$, then we simply say that x dominates x', which means that $xx' \in A$, and simply write $x \to x'$. The set X dominates a subdigraph D' if X dominates $V(D')$. Similarly, a subdigraph D' dominates a subset X' if $V(D')$ dominates X'.

Then we have the following trivial observations.

Observation 2. *Each of A^0, A^1,... and A^{k-1} has no chord.*

Observation 3. *D has no bad collection of cycles.*

Observation 4. *By renaming the vertices of D, we may assume that $d_D^+(a_0^0) = 3$, $a_0^0 \to a_0^1$ and $a_1^1 \to a_0^0$.*

By these observations and assumtion about D, we have immediately the following Lemma 5 and Lemma 6

Lemma 5. $k = 2$.

Proof. For a contradiction, let $k \geq 3$. If $a_0^0 \to a_2^1$ (resp. $a_2^1 \to a_0^0$) then $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_2^1 A^1 a_1^1, a_0^0)$ and A^2 (resp. $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_0^1, a_1^1, a_2^1, a_0^0)$ and A^2) form a bad collection of cycles, a contradiction to Observation 3. Thus, we must have $k = 2$. □

Lemma 6. $t \geq 4$.

Proof. For a contradiction, let $t = 3$. Then

$$|A(D)| \leq \binom{6}{2} = 15 < 18 \leq \sum_{v \in V(D)} d^+(v),$$

a contradiction. Thus, we must have $t \geq 4$. □

From now on, we always assume that

(1) $d_D^+(a_0^0) = 3$, $N_D^+(a_0^0) = \{a_1^0, a_0^1, a_m^1\}$ with $1 < m < t$, $a_1^1 \to a_0^0$,

(2) $a_m^1 \to \{a_p^0, a_q^0\}$ with $0 < p < q < t$ such that if $0 < i < q, i \neq p$ then $a_i^0 \to a_m^1$.

Lemma 7. *a_p^0 and a_q^0 cannot both be out-neighbors of a_0^1.*

Proof. Suppose, on the contrary, that a_p^0 and a_q^0 are both out-neighbors of a_0^1. Since $d_D^+(a_p^0) \geq 3$, there exists a_n^1 in A^1 with $1 < n < t$ such that $a_p^0 \to a_n^1$. Then $a_n^1 \to a_0^0$ because $d_D^+(a_0^0) = 3$.

First, we assume that $1 < n < m$. If $m > n + 1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_p^0, a_n^1 A^1 a_m^1, a_p^0)$ form a bad collection of cycles, a contradiction. Thus, $m = n + 1$. If $a_1^1 \to a_p^0$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_0^0, a_0^1, a_1^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction. Thus, $a_p^0 \to a_1^1$. It follows that $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_q^0, a_1^1, a_0^0, a_0^1, a_q^0)$ form a bad collection of cycles, a contradiction.

Now, we assume that $m < n < t$. Then $q = n$ (if $q \neq n$ then $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$ and $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$ form a bad collection of cycles, a contradiction) and $t - q = m - 1$ (if $t - q \neq m - 1$ then $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$, $(a_0^1, a_1^1, a_p^0, a_n^1 A^1 a_0^1)$ and $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction). If $a_1^1 \to a_p^0$ then $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$, $(a_0^1, a_1^1, a_p^0, a_n^1 A^1 a_0^1)$ and $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction. Thus, $a_p^0 \to a_1^1$. Since $d_D^+(a_1^1) \geq 3$, there exists a_h^0 in A^0 such that $a_1^1 \to a_h^0$. If $0 < h < p$ then $a_h^0 \to a_m^1$. So $(a_0^0, a_m^1, a_q^0 A^0 a_0^0)$, $(a_h^0, a_m^1, a_q^0 A^0 a_h^0)$ and $(a_p^0, a_n^1 A^1 a_0^1, a_p^0)$ form a bad collection of cycles, a contradiction (in the case we are considering, it is easy to see that the digraph H_8^3 is illustrated in Figure 2 is a special case with $u_0 = a_0^0$, $u_1 = a_h^0$, $u_2 = a_p^0$, $u_3 = a_q^0$, $v_0 = a_0^1$, $v_1 = a_1^1$, $v_2 = a_m^1$,

$v_3 = a_n^1$ and $t = 4$. The digraph H_8^3 contains two disjoint cycles of different lengths (u_2, v_3, v_0, u_2) and $(u_1, v_2, u_3, u_0, u_1)$). If $p < h < q$ then again $a_h^0 \to a_m^1$. So $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_p^0 A^0 a_h^0, a_m^1, a_p^0)$ form a bad collection of cycles if $h > p+1$ and $(a_p^0, a_h^0, a_m^1, a_p^0)$, $(a_p^0, a_1^1, a_h^0, a_m^1, a_p^0)$, $(a_0^0, a_0^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles if $h = p+1$, a contradiction. If $q < h < t$ then $(a_m^1, a_h^0 A^0 a_0^0, a_m^1)$, $(a_p^0, a_n^1 A^1 a_0^1, a_p^0)$ form a bad collection of cycles if $a_m^1 \to a_h^0$ and $(a_m^1, a_q^0 A^0 a_h^0, a_m^1)$, $(a_0^1, a_p^0, a_n^1 A^1 a_0^1)$ form a bad collection of cycles if $a_h^0 \to a_m^1$, a contradiction. □

Lemma 8. *a_p^0 and a_q^0 cannot both be in-neighbors of a_0^1.*

Proof. Suppose, on the contrary, that a_p^0 and a_q^0 are both in-neighbors of a_0^1. Since $d_D^+(a_0^1) \geq 3$, there exists a_h^0 and a_l^0 in A^0 with $0 < h < l < t$ such that $a_0^1 \to \{a_h^0, a_l^0\}$.

First, we assume that $0 < h < l < p$. It is clear that $\{a_h^0, a_l^0\} \to a_m^1$. Then $(a_0^1, a_h^0 A^0 a_p^0, a_0^1)$, $(a_0^1, a_l^0 A^0 a_p^0, a_0^1)$ and $(a_m^1, a_q^0 A^0 a_0^0, a_m^1)$ form a bad collection of cycles, a contradiction.

Next, we assume that $0 < h < p < l < q$. It is clear that $\{a_h^0, a_l^0\} \to a_m^1$. If $a_1^1 \to a_h^0$ then $(a_0^1, a_h^0 A^0 a_p^0, a_0^1)$ and $(a_0^1, a_1^1, a_h^0 A^0 a_p^0, a_0^1)$ form a bad collection of cycles, a contradiction. Thus, $a_h^0 \to a_1^1$. So $(a_0^1, a_1^1, a_0^0, a_0^1)$, $(a_0^1, a_h^0, a_1^1, a_0^0, a_0^1)$ and $(a_m^1, a_p^0 A^0 a_l^0, a_m^1)$ form a bad collection of cycles, a contradiction.

Next, we assume that $0 < h < p < q < l < t$. It is clear that $a_h^0 \to a_m^1$. If $a_l^0 \to a_m^1$ (resp., $a_m^1 \to a_l^0$), then $(a_l^0, a_m^1, a_p^0 A^0 a_l^0)$, $(a_l^0, a_m^1, a_q^0 A^0 a_l^0)$ and $(a_0^0, a_0^1, a_1^1, a_0^0)$ (resp., $(a_0^1, a_h^0 A^0 a_p^0, a_0^1)$, $(a_0^1, a_h^0 A^0 a_q^0, a_0^1)$ and $(a_m^1, a_l^0 A^0 a_0^0, a_m^1)$) form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < l < q$. It is clear that $\{a_h^0, a_l^0\} \to a_m^1$. Then $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$, $(a_m^1, a_p^0 A^0 a_l^0, a_m^1)$ and $(a_0^0, a_0^1, a_1^1, a_0^0)$) form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < p < l < t$. It is clear that $a_h^0 \to a_m^1$. If $a_l^0 \to a_1^1$ (resp., $a_1^1 \to a_l^0$), then $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_0^1, a_l^0, a_1^1, a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$ (resp., $(a_0^0, a_0^1, a_l^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_1^1, a_l^0 A^0 a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$) form a bad collection of cycles, a contradiction.

Finally, we assume that $q < h < l < t$. If $a_p^0 \to a_1^1$ (resp., $a_q^0 \to a_1^1$), then $(a_0^0, a_0^1, a_h^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_l^0 A^0 a_0^0)$ and $(a_p^0, a_1^1 A^1 a_m^1, a_p^0)$ (resp., $(a_0^0, a_0^1, a_h^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_l^0 A^0 a_0^0)$ and $(a_q^0, a_1^1 A^1 a_m^1, a_q^0)$) form a bad collection of cycles, a contradiction. Thus, $a_1^1 \to \{a_p^0, a_q^0\}$. So, if $a_m^1 \to a_l^0$ (resp., $a_l^0 \to a_m^1$) then $(a_p^0, a_0^1, a_1^1, a_p^0)$, $(a_p^0 A^0 a_q^0, a_0^1, a_1^1, a_p^0)$ and $(a_0^0, a_m^1, a_l^0 A^0 a_0^0)$ (resp., $(a_l^0, a_m^1, a_p^0 A^0 a_l^0)$, $(a_l^0, a_m^1, a_q^0 A^0 a_l^0)$ and $(a_0^0, a_0^1, a_1^1, a_0^0)$) form a bad collection of cycles, a contradiction. □

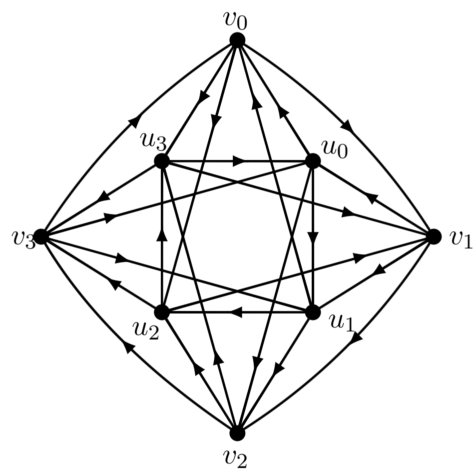

FIG. 2. The digraph H_8^3

3 Proof of Theorem 1

Let $D = (V, A)$ be a tournament of k cycles with $k \geq 2$ and minimum out-degree 3, having no disjoint cycles of different lengths. By Lemma 7 and Lemma 8, we have the following cases.

Case 1. $a_p^0 \to a_0^1$ and $a_0^1 \to a_q^0$.

Since $d_D^+(a_q^0) \geq 3$, there exists a_n^1 in A^1 with $1 < n < t$ such that $a_q^0 \to a_n^1$. It is clear that $a_n^1 \to a_0^0$. We again divide Case 1 into several subcases.

Subcase 1.1. $1 < n < m$.

If $m > n+1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_q^0, a_n^1 A^1 a_m^1, a_q^0)$ form a bad collection of cycles, a contradiction. So $m = n + 1$. It follows that $(a_q^0, a_n^1, a_m^1, a_q^0)$ and $(a_0^0 A^0 a_p^0, a_0^1, a_1^1, a_0^0)$ form a bad collection of cycles, a contradiction.

Subcase 1.2. $m < n < t$.

Since $d_D^+(a_0^1) \geq 3$, there exists a_h^0 in A^0 with $h \notin \{0, p, q\}$ such that $a_0^1 \to a_h^0$.

First, we assume that $0 < h < p$. It is clear that $a_h^0 \to a_m^1$. If $a_n^1 \to a_h^0$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_n^1, a_h^0 A^0 a_q^0, a_n^1)$ form a bad collection of cycles, a contradiction. So, $a_h^0 \to a_n^1$. If $a_1^1 \to a_p^0$ then $(a_h^0, a_n^1, a_0^0 A^0 a_h^0)$, $(a_h^0, a_m^1 A^1 a_n^1, a_0^0 A^0 a_h^0)$ and $(a_0^1, a_1^1, a_p^0, a_0^1)$ form a bad collection of cycles, a contradiction. So, $a_p^0 \to a_1^1$. Thus, $(a_p^0, a_1^1 A^1 a_m^1, a_p^0)$, $(a_p^0, a_0^1, a_1^1 A^1 a_m^1, a_p^0)$ and $(a_h^0, a_n^1, a_0^0 A^0 a_h^0)$ form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < q$. It is clear that $a_h^0 \to a_m^1$. If $h > p + 1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$ form a bad collection of cycles, a contradiction. So $h = p + 1$. It follows that $(a_m^1, a_p^0, a_h^0, a_m^1)$ and $(a_0^0, a_0^1, a_q^0, a_n^1, a_0^0)$ form a bad collection of cycles, a contradiction.

Finally, we assume that $q < h < t$. If $a_p^0 \to a_1^1$ then $(a_0^0, a_0^1, a_h^0 A^0 a_0^0)$, $(a_0^0, a_0^1, a_q^0 A^0 a_0^0)$ and $(a_p^0, a_1^1 A^1 a_m^1, a_p^0)$ form a bad collection of cycles, a contradiction. So, $a_1^1 \to a_p^0$. It follows that $(a_p^0, a_0^1, a_1^1, a_p^0)$ and $(a_0^0, a_m^1, a_q^0, a_n^1, a_0^0)$ form a bad collection of cycles, a contradiction.

Case 2. $a_0^1 \to a_p^0$ and $a_q^0 \to a_0^1$.

Since $d_D^+(a_p^0) \geq 3$, there exists a_n^1 in A^1 with $1 < n < t$ such that $a_p^0 \to a_n^1$. Then $a_n^1 \to a_0^0$ because $d_D^+(a_0^0) = 3$. We again divide Case 1 into several subcases.

Subcase 2.1. $1 < n < m$.

If $m > n+1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_p^0, a_n^1 A^1 a_m^1, a_p^0)$ form a bad collection of cycles, a contradiction. So $m = n + 1$. If $a_1^1 \to a_q^0$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_0^0, a_0^1, a_1^1, a_q^0 A^0 a_0^0)$ form a bad collection of cycles, a contradiction. So, $a_q^0 \to a_1^1$. If $a_q^0 \to a_n^1$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_q^0, a_n^1, a_m^1, a_p^0 A^0 a_q^0)$ form a bad collection of cycles, a contradiction. So, $a_n^1 \to a_q^0$.

Since $d_D^+(a_0^1) \geq 3$, there exists a_h^0 in A^0 with $h \notin \{0, p, q\}$ such that $a_0^1 \to a_h^0$.

First, we assume that $0 < h < p$. It is clear that $a_h^0 \to a_m^1$. If $a_h^0 \to a_1^1$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_h^0, a_1^1, a_0^0, a_0^1, a_h^0)$ form a bad collection of cycles, a contradiction. So, $a_1^1 \to a_h^0$. If $a_n^1 \to a_h^0$ then $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_n^1, a_h^0, a_m^1, a_p^0, a_n^1)$ form a bad collection of cycles, a contradiction. So, $a_h^0 \to a_n^1$. If $a_1^1 \to a_p^0$ then $(a_1^1, a_p^0 A^0 a_q^0, a_1^1)$, $(a_0^1, a_1^1, a_p^0 A^0 a_q^0, a_0^1)$ and $(a_n^1, a_0^0 A^0 a_h^0, a_n^1)$ form a bad collection of cycles, a contradiction. So, $a_p^0 \to a_1^1$. First, we assume that $t = 4$. Let φ be the following mapping from $V(D)$ to $V(D_8^3)$: $a_0^0 \mapsto v_5$, $a_h^0 \mapsto v_1$, $a_p^0 \mapsto v_2$, $a_q^0 \mapsto v_4$, $a_0^1 \mapsto v_0$, $a_1^1 \mapsto w$, $a_n^1 \mapsto v_3$ and $a_m^1 \mapsto v$. Then it is not difficult to verify that φ is an isomorphism between D and D_8^3. Now, we assume that $t > 4$. Then, there exists a_l^1 in A^1 with $l \notin \{0, 1, n, m\}$ such that $a_l^1 \to a_0^0$. If $1 < l < n$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_0^0, a_0^1, a_1^1 A^1 a_l^1, a_0^0)$ form a bad collection of cycles, a contradiction. So, $m < l < t$. If $a_h^0 \to a_l^1$ (resp., $a_l^1 \to a_h^0$), then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_h^0, a_l^1, a_0^0, a_0^1, a_h^0)$ (resp., $(a_0^0, a_0^1, a_1^1, a_0^0)$ and $(a_l^1, a_h^0, a_n^1, a_m^1 A^1 a_l^1)$) form a bad collection of cycles, a contradiction.

Next, we assume that $p < h < q$. It is clear that $a_h^0 \to a_m^1$. Then $(a_0^0, a_0^1, a_1^1, a_0^0)$, $(a_0^0, a_0^1 A^1 a_n^1, a_0^0)$ and $(a_m^1, a_p^0 A^0 a_h^0, a_m^1)$ form a bad collection of cycles, a contradiction.

Finally, we assume that $q < h < t$. If $a_h^0 \to a_1^1$ then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_h^0, a_1^1, a_0^0, a_0^1, a_h^0)$ form a bad collection of cycles, a contradiction. So, $a_1^1 \to a_h^0$. Then $(a_p^0, a_n^1, a_m^1, a_p^0)$ and $(a_1^1, a_h^0 A^0 a_0^0, a_0^1, a_1^1)$ form a bad collection of cycles, a contradiction.

Subcase 2.2. $m < n < t$.

If $a_1^1 \to a_q^0$ then $(a_1^1, a_q^0, a_0^1, a_1^1)$ and $(a_0^0, a_m^1, a_p^0, a_n^1, a_0^0)$ form a bad collection of cycles, a contradiction. So, $a_q^0 \to a_1^1$. Then $(a_q^0, a_1^1 A^1 a_m^1, a_q^0)$, $(a_q^0, a_0^1 A^1 a_m^1, a_q^0)$ and $(a_p^0, a_n^1, a_0^0 A^0 a_p^0)$ form a bad collection of cycles, a contradiction.

The proof of Theorem 1 is complete.

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