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MAXIMUM RATE OF NORM CONVERGENCE IN THE ERGODIC THEOREM FOR GROUPS $\mathbb{Z}^d$ AND $\mathbb{R}^d$

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Abstract: For d pairwise commuting automorphisms (flows) of a probability space, ergodic averages over parallelepipeds are considered. It is shown that the maximum rate of their convergence in the L_p-norm is $\mathcal{O}(\frac{1}{t_1 t_2 \cdots t_d})$. A spectral criterion is also obtained for the maximum convergence rate in the L_2-norm.

Keywords: rates of convergence in ergodic theorems, spectral measure, coboundaries, bundle of hyperplanes.

1 Introduction

1.1. Let T be a measure-preserving transformation of a probability measure space (Ω, μ). It is well known that ergodic averages

$$A_n^T f(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega), \quad f \in L_p(\Omega, \mu),\ p \in [1, +\infty)$$

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converge in norm as $n \to \infty$ to their limit f^* at a rate no faster than $\mathcal{O}(\frac{1}{n})$. Namely, if $\|A_n^T f - f^*\|_p = o(1/n)$ as $n \to \infty$, then $f = f^*$. For $p > 1$ this result was obtained by Butzer and Westphal [1, Theorem 1], where a more general case of power bounded operators in a reflexive Banach space was considered. Gaposhkin obtained a new proof for the L_2-convergence case in [2, Corollary 5], where the spectral representation of stationary sequences was used. In this case, the fact about the maximum rate immediately followed from the inequality

$$\overline{\lim_{n\to\infty}} n\|A_n^T f\|_2 > 0$$

for a non-zero function f. Sedalishchev in [3, Lemma 1] obtained a simple proof of this inequality for all L_p, $p \in [1, +\infty]$, strengthening it to the following form:

$$\overline{\lim_{n\to\infty}} n\|A_n^T f\|_p \geq \frac{\|f\|_p}{2}.$$

The purpose of this note is to obtain an analogue of the maximum rate of convergence statement for ergodic averages

$$A_{\vec{n}} f(\omega) = A_{n_1}^{T_1} \cdots A_{n_d}^{T_d} f(\omega), \quad \vec{n} = (n_1, ..., n_d) \in \mathbb{N}^d,$$

where T_k, $k = 1, ..., d$, are pairwise commuting automorphisms of the probability measure space (Ω, μ); and also for ergodic averages with continuous time

$$\mathcal{A}_{\vec{t}} f(\omega) = \frac{1}{t_1 t_2 \cdots t_d} \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u}, \quad \vec{t} = (t_1, ..., t_d) \in \mathbb{R}^d,$$

where $\{T_k^t\}_{t\in\mathbb{R}}$, $k = 1, ..., d$, are pairwise commuting flows of the probability measure space (Ω, μ).

We show that for all $p \in [1, +\infty]$ the maximum rate of convergence in the L_p-norm is $\mathcal{O}(\frac{1}{n_1 n_2 \cdots n_d})$ as $n_1, ..., n_d \to +\infty$ for discrete time, and $\mathcal{O}(\frac{1}{t_1 t_2 \cdots t_d})$ as $t_1, ..., t_d \to +\infty$ for continuous time. This is a consequence of Theorem 1. Put

$$\pi(\vec{t}) = t_1 t_2 \cdots t_d, \quad \vec{t} \in \mathbb{R}^d,$$

and we will write $\vec{t} \geq \vec{s}$ if only $t_j \geq s_j$ for each coordinate.

Theorem 1. *Let $f \in L_p(\Omega, \mu)$, $p \in [1, \infty]$; then*

$$\sup_{\vec{m}\geq\vec{n}} \|A_{\vec{m}} f\|_p \geq \frac{\|f\|_p}{2^{2d}\pi(\vec{n})} \quad \textit{for all} \quad \vec{n} \in \mathbb{N}^d,$$

$$\sup_{\vec{s}\geq\vec{t}} \|\mathcal{A}_{\vec{s}} f\|_p \geq \frac{\sup_{\vec{q}\in E_{\vec{t}}} \left\| \int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u} \right\|_p}{2^{2d}\pi(\vec{t})} \quad \textit{for all} \quad \vec{t} \in \mathbb{R}_+^d,$$

where $E_{\vec{t}} = \left\{ \vec{q} = (\frac{t_1}{n_1}, ..., \frac{t_d}{n_d}) : \ \vec{n} \in \mathbb{N}^d, \ \vec{n} \geq \vec{t} \right\} \subset (0, 1]^d.$

1.2. The maximum rate of convergence $\mathcal{O}(\frac{1}{n_1 \cdots n_d})$ or $\mathcal{O}(\frac{1}{t_1 \cdots t_d})$ of ergodic averages is equivalent to the condition

$$\sup_{\vec{n} \in \mathbb{N}^d} \left\| \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} f(T_1^{k_1} \cdots T_d^{k_d} \omega) \right\|_p < \infty \quad \text{for discrete time,}$$

$$\sup_{\vec{t} \in \mathbb{R}_+^d} \left\| \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u} \right\|_p < \infty \quad \text{for continuous time.}$$

For the case of a single measure-preserving transformation, it is well known that the last inequality holds only for L_p-coboundaries (or functions from L_p cohomological to zero), i.e., $f \in (I - T)L_p(\Omega, \mu)$. An excellent exposition of the history of the question can be found in the introduction to Cohen and Lin's paper [4]. In general, it follows from Bradley's paper [5] that f is a d-multiple L_p-coboundary for all d measure-preserving transformations, i.e.,

$$f \in (I - T_1) \cdots (I - T_d) L_p(\Omega, \mu).$$

Cohen and Lin obtained [4], [6, Proposition 3.1] a spectral criterion for two-fold L_2-coboundaries in the case of two commuting transformations. We give this assertion (the equivalence of conditions (1) and (2) below) in the general case $d \geq 1$. Note that for a single measure-preserving transformation the spectral criterion is due to Robinson [7]. In a recent note [8] a similar criterion for uniform convergence on subspaces was obtained.

Theorem 2. *Let $T_1, ..., T_d$ be pairwise commuting automorphisms of the probability measure space (Ω, μ). Then for $f \in L_2(\Omega, \mu)$ the following conditions are equivalent:*

$$f \in (I - T_1) \cdots (I - T_d) L_2(\Omega, \mu); \tag{1}$$

$$\int\limits_{(-\pi,\pi]^d} \frac{d\sigma_f(x_1, \ldots, x_d)}{\sin^2 \frac{x_1}{2} \cdots \sin^2 \frac{x_d}{2}} < \infty; \tag{2}$$

$$\left\| \frac{1}{n^d} \sum_{k_1=0}^{n-1} \cdots \sum_{k_d=0}^{n-1} f(T_1^{k_1} \cdots T_d^{k_d} \omega) \right\|_2 = O\left(\frac{1}{n^d}\right) \quad \textit{as} \quad n \to \infty; \tag{3}$$

$$\left\| \frac{1}{\pi(\vec{n})} \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} f(T_1^{k_1} \cdots T_d^{k_d} \omega) \right\|_2 = O\left(\frac{1}{\pi(\vec{n})}\right) \quad \textit{as } n_1, ..., n_d \to \infty. \tag{4}$$

Here $\sigma_f(x_1, ..., x_d)$ is the spectral measure constructed from the function f and the automorphisms $T_1, ..., T_d$ (see, for example, [9, 10]). In particular, using the spectral measure one can express the norms of ergodic means, namely

$$\|A_{\vec{n}} f\|_2^2 = \frac{1}{n_1^2 \cdots n_d^2} \int\limits_{(-\pi,\pi]^d} \frac{\sin^2 \frac{n_1 x_1}{2} \cdots \sin^2 \frac{n_d x_d}{2}}{\sin^2 \frac{x_1}{2} \cdots \sin^2 \frac{x_d}{2}}\, d\sigma_f(x_1, \ldots, x_d).$$

A similar result holds for continuous time. To formulate it, we define an analogue of L_p-coboundaries. Let R_j be the generator of the group $\{T_j^t\}_{t\in\mathbb{R}}$, $j = 1, ..., d$, i.e., in the strong operator topology $R_j = \lim\limits_{t\to 0+} \frac{1}{t}(I - T_j^t)$. It is easy to verify that on $\operatorname{Dom} R = \cap_{j=1}^d \operatorname{Dom} R_j$ the generators commute with each other, so on this domain the operator $R = R_1 \cdots R_d$ is well defined. We will say that f is a d-multiple L_p-coboundary if it lies in the image of the operator R, i.e., there exists a function $g \in L_p(\Omega, \mu)$ such that $f = Rg$.

Theorem 3. *Let $T_1^{t_1}, ..., T_d^{t_d}$ be pairwise commuting flows of the probability measure space (Ω, μ). Then for $f \in L_2(\Omega, \mu)$ the following conditions are equivalent:*

$$f \in RL_2(\Omega, \mu); \tag{1'}$$

$$\int\limits_{\mathbb{R}^d} \frac{d\sigma_f(x_1, \ldots, x_d)}{x_1^2 \cdots x_d^2} < \infty; \tag{2'}$$

$$\left\| \frac{1}{t^d} \int_{[0,t]^d} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_2 = O\left(\frac{1}{t^d}\right) \quad as \;\; t \to \infty; \tag{3'}$$

$$\left\| \frac{1}{\pi(\vec{t})} \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_2 = O\left(\frac{1}{\pi(\vec{t})}\right) \quad as \; t_1, ..., t_d \to \infty. \tag{4'}$$

Note that in the case $d = 1$ the equivalence of $(1')$ and $(4')$ was proved in [11] (see also [12]), and the equivalence of $(2')$ and $(3')$ was considered in [13].

The problem of transferring known results on convergence rates in ergodic theorems [14] from the action of the group $\mathbb{Z}$ to the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$ (and further to the widest possible class of groups) was set to the first co-author of this paper by A.M. Vershik and A.M. Stepin in the mid-1990s; its solution was started at the same time in [9] and [15]. Theorems 2 and 3, together with the results of [16], complete the solution of this problem for power rates of convergence in the norm (with all possible exponents) for the actions of the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$.

2 Proof of Theorem 1

2.1. Consider a mapping $\mathcal{L} : \mathbb{N}^d \to \mathbb{N}^d$ such that

$$\vec{n} < \mathcal{L}(\vec{n}) \;\text{ for all }\; \vec{n} \in \mathbb{N}^d, \;\text{ and }\; \mathcal{L}(\vec{n}) \le \mathcal{L}(\vec{m}) \text{ for } \;\vec{n} \le \vec{m}.$$

We will say that a net $\{x_{\vec{n}}\}_{\vec{n}\in\mathbb{N}^d}$ of elements of some normed space $(X, \|\cdot\|)$ is $(a, \mathcal{L})$-recurrent for some $a > 0$ if for any $\vec{n} \in \mathbb{N}^d$ there exists an element $s(\vec{n}) \in \mathbb{N}^d$ such that $\vec{n} \le s(\vec{n}) < \mathcal{L}(\vec{n})$ and $\|x_{s(\vec{n})}\| \ge a$. The following theorem holds.

Theorem 4. *Let $\{x_{\vec{n}}\}_{\vec{n}\in\mathbb{N}^d}$ be a $(a, \mathcal{L})$-recurrent net; then for all $\vec{n} \in \mathbb{N}^d$*

$$\sup_{\vec{m}\geq\vec{n}} \left\| \frac{1}{\pi(\vec{m})} \sum_{0\leq\vec{k}<\vec{m}} x_{\vec{k}} \right\| \geq \frac{a}{2^d \pi(\mathcal{L}(\vec{n}))}.$$

Theorem 4 is proved in [17] for complex-valued nets. The proof of the general case for nets with values in a normed space is word for word the same as the original case, since essentially only the triangle inequality for the norm is used.

Theorem 4 immediately implies Theorem 1 for the group time $\mathbb{Z}^d$. Indeed, for the function $f \in L_p(\Omega, \mu)$, consider the net $x_{\vec{n}} = f(T_1^{n_1} \cdots T_d^{n_d}\omega)$ and $\mathcal{L}(\vec{n}) = \vec{n} + (1, ..., 1)$. Taking $a = \|f\|_p$, we obtain that the net is $(a, \mathcal{L})$-recurrent: $s(\vec{n}) = \vec{n}$ and $\|x_{s(\vec{n})}\|_p = a$. Therefore, the inequality from Theorem 4 is valid, which after applying the estimate $\pi(\vec{n} + (1, ..., 1)) \leq 2^d \pi(\vec{n})$ becomes exactly the inequality from Theorem 1.

Now let's consider the continuous case, i.e., the group time $\mathbb{R}^d$, and use the already proven inequality for discrete time. Fix $\vec{t}$ and take an arbitrary vector $\vec{q} \in E_{\vec{t}}$. Put

$$g_{\vec{q}}(\omega) = \int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u}, \quad S_j = T_j^{q_j}, \quad j = 1, ..., d.$$

Let us consider the partition of the interval $[0, \vec{n}]$ into unit cubes, i.e.,

$$[0, \vec{n}] = \bigcup_{0\leq\vec{k}\leq\vec{n}} P_{\vec{k}},\ P_{\vec{k}} = [0,1]^d + \vec{k}, \quad \vec{k} \leq \vec{n}.$$

We denote the coordinate-wise multiplication of vectors in $\mathbb{R}^d$ as

$$\vec{t} \odot \vec{s} = (t_1 s_1, ..., t_d s_d).$$

It is easy to see that

$$\sum_{0\leq\vec{k}\leq\vec{n}} \int_{P_{\vec{k}}} f(S_1^{u_1} \cdots S_d^{u_d}\omega)\, d\vec{u} = \sum_{0\leq\vec{k}\leq\vec{n}} \int_{[0,1]^d} f(S_1^{u_1+k_1} \cdots S_d^{u_d+k_d}\omega)\, d\vec{u} =$$
$$= \frac{1}{\pi(\vec{q})} \sum_{0\leq\vec{k}\leq\vec{n}} \int_{[0,\vec{q}]} f(T_1^{t_1} S_1^{k_1} \cdots T_d^{t_d} S_d^{k_d}\omega)\, d\vec{t} = \frac{\pi(\vec{n})}{\pi(\vec{q})} A_{n_1}^{S_1} \cdots A_{n_d}^{S_d} g_{\vec{q}}(\omega).$$

From here, taking into account that $\vec{t} = \vec{n_0} \odot \vec{q}$ for some $\vec{n_0} \geq \vec{t}$, we obtain

$$\sup_{\vec{s}\geq\vec{t}}\|\mathcal{A}_{\vec{s}}f\|_p \geq \sup_{\vec{s}\geq\vec{t},\vec{s}=\vec{n}\odot\vec{q},\vec{n}\in\mathbb{N}^d}\|\mathcal{A}_{\vec{s}}f\|_p =$$

$$= \sup_{\vec{n}\geq\vec{n_0}}\frac{1}{\pi(\vec{n}\odot\vec{q})}\left\|\int_{[0,\vec{n}\odot\vec{q}]} f(T_1^{u_1}\cdots T_d^{u_d}\omega)\,d\vec{u}\right\|_p =$$

$$= \sup_{\vec{n}\geq\vec{n_0}}\frac{1}{\pi(\vec{n})}\left\|\int_{[0,\vec{n}]} f(T_1^{q_1u_1}\cdots T_d^{q_du_d}\omega)\,d\vec{u}\right\|_p =$$

$$= \sup_{\vec{n}\geq\vec{n_0}}\frac{1}{\pi(\vec{n})}\left\|\sum_{0\leq\vec{k}\leq\vec{n}}\int_{P_{\vec{k}}} f(S_1^{u_1}\cdots S_d^{u_d}\omega)\,d\vec{u}\right\|_p =$$

$$= \sup_{\vec{n}\geq\vec{n_0}}\frac{1}{\pi(\vec{q})}\left\|A_{n_1}^{S_1}\cdots A_{n_d}^{S_d}g_{\vec{q}}(\omega)\right\|_p \geq \frac{\|g_{\vec{q}}\|_p}{2^{2d}\pi(\vec{q})\pi(\vec{n_0})} = \frac{\|g_{\vec{q}}\|_p}{2^{2d}\pi(\vec{t})}.$$

Since $\vec{q}\in E_{\vec{t}}$ was arbitrary, we can take the supremum over all such vectors. Theorem 1 is completely proved.

Remark. V.V. Ryzhikov proposed the following argument to justify the inequality for the discrete case in Theorem 1. The following statement is true: for any $\vec{n}\in\mathbb{N}^d$ there exists $\vec{n'}\geq\vec{n}$, such that $\|\vec{n}-\vec{n'}\|_\infty\leq 1$ and

$$\left\|\sum_{k_1=0}^{n_1'-1}\cdots\sum_{k_d=0}^{n_d'-1} f(T_1^{k_1}\cdots T_d^{k_d}\omega)\right\|_p \geq \frac{\|f\|_p}{2^d}.$$

This inequality strengthens Sedalishchev's inequality from the one-dimensional case. The proof is by induction. We will only show the basis for $d=1$. From the equality

$$\sum_{k=0}^{n} f(T^k\omega) - \sum_{k=0}^{n-1} f(T^k\omega) = f(T^n\omega)$$

it follows that the norms of both sums on the right-hand side cannot be less than $\|f\|_p/2$ at the same time.

Now, having the strengthened Sedalishchev's inequality, we obtain

$$\sup_{\vec{m}\geq\vec{n}}\|A_{\vec{m}}f\|_p \geq \sup_{\|\vec{n'}-\vec{n}\|_\infty\leq 1}\|A_{\vec{n'}}f\|_p \geq \frac{\|f\|_p}{2^d\pi(\vec{n'})} \geq \frac{\|f\|_p}{2^{2d}\pi(\vec{n})}.$$

2.2. Let us show how the statement about the maximum rate of convergence of ergodic means follows from Theorem 1.

Let $\|A_{\vec{n}}(f-f^*)\|_p = o\left(\frac{1}{n_1\cdots n_d}\right)$ for $n_1,...,n_d\to+\infty$. Then

$$\frac{\|f-f^*\|_p}{2^{2d}\pi(\vec{n})} \leq \sup_{\vec{m}\geq\vec{n}}\|A_{\vec{m}}(f-f^*)\|_p = o\left(\frac{1}{\pi(\vec{n})}\right),$$

whence $\|f-f^*\|_p = o(1)$, i.e., $f(\omega)=f^*(\omega)$ a.e.

For continuous time, also assume that $\|\mathcal{A}_{\vec{t}}(f-f^*)\|_p = o\left(\frac{1}{t_1\cdots t_d}\right)$ as $t_1,...,t_d \to +\infty$. Then

$$\frac{\sup_{\vec{q}\in E_{\vec{t}}} \|\int_{[0,\vec{q}]} f(T_1^{u_1}\cdots T_d^{u_d}\omega)\,d\vec{u} - \pi(\vec{q})f^*\|_p}{2^{2d}\pi(\vec{t})} \le \sup_{\vec{s}\ge\vec{t}} \|\mathcal{A}_{\vec{s}}(f-f^*)\|_p = o\left(\frac{1}{\pi(\vec{t})}\right),$$

whence $\sup_{\vec{q}\in E_{\vec{t}}} \|\int_{[0,\vec{q}]} f(T_1^{u_1}\cdots T_d^{u_d}\omega)\,d\vec{u} - \pi(\vec{q})f^*\|_p = o(1)$ as $t_1,...,t_d \to +\infty$.

Let $\vec{t} = n\vec{t_0},\ n\to\infty$, and the vector $\vec{t_0}$ be fixed. Since $E_{\vec{t_0}} \subset E_{n\vec{t_0}} \subset (0,1]^d$, then $\sup_{\vec{q}\in E_{\vec{t_0}}} \|\int_{[0,\vec{q}]} f(T_1^{u_1}\cdots T_d^{u_d}\omega)\,d\vec{u} - \pi(\vec{q})f^*\|_p = o(1)$ as $n\to\infty$, therefore, for any $\vec{q}\in E_{\vec{t_0}}$,

$$\left\|\int_{[0,\vec{q}]} f(T_1^{u_1}\cdots T_d^{u_d}\omega)\,d\vec{u} - \pi(\vec{q})f^*\right\|_p = 0.$$

Then from the local ergodic theorem [18, §7.2] we obtain a.e. equality

$$f^*(\omega) = \lim_{E_{\vec{t_0}}\ni\vec{q}\to 0} \frac{1}{\pi(\vec{q})}\int_{[0,\vec{q}]} f(T_1^{u_1}\cdots T_d^{u_d}\omega)\,d\vec{u} = f(\omega).$$

3 Functions $\mathcal{J}_d(x)$ and $\mathcal{K}_d(x)$

3.1. Let us consider an auxiliary problem, interesting in itself and useful for proving the spectral criterion for the maximum possible rate of convergence, i.e., for Theorems 2 and 3. For $x=(x_1,...,x_d)$ we define the functions

$$\mathcal{J}_d(x) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \prod_{k=1}^d \sin^2(x_k s/2)\,ds,\ \ x\in\mathbb{R}^d;$$

$$\mathcal{K}_d(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N \prod_{k=1}^d \sin^2(x_k n/2),\ \ x\in(-\pi,\pi]^d.$$

Our goal is to find the set of zeros of these functions and to show that their set of values is finite. The reader interested in the proof of Theorems 2 and 3 can skip to the next sections and return to §3 later.

To calculate the limits, we use the following construction. Consider in the space $\mathbb{R}^d$ the bundle $\mathcal{L}(a)$ of all hyperplanes defined by the equalities $x_1\varepsilon_1 + x_2\varepsilon_2 + ... + x_d\varepsilon_d = a$, where $a\in\mathbb{R}$ is fixed, and $\varepsilon_k\in\{-1,0,1\}$ and not all of them are equal to zero simultaneously. It is clear that $\mathcal{L}(a)=\mathcal{L}(-a)$ for each $a\in\mathbb{R}$. The index of a hyperplane L from the bundle $\mathcal{L}(a)$ is the number of all nonzero coefficients in the equation defining it, i.e., the quantity $\mathrm{ind}L = \sum_{k=1}^d |\varepsilon_k| = \|\varepsilon\|_1$. In the case $a=0$, the hyperplanes from the bunch $\mathcal{L}(0)$, defined by the vectors ε and $-\varepsilon$, will be considered different. Many different combinatorial problems are associated with such constructions; see, for example, [19]. We also set

$$\mathcal{L}_d = \mathcal{L}(0)\cup\mathcal{L}(2\pi)\cup\cdots\cup\mathcal{L}\left(2\pi\left\lfloor d/2\right\rfloor\right).$$

Proposition 1. *The following equalities are true*

$$\mathcal{J}_d(x) = \frac{1}{2^d}\left(1 + \sum_{k=1}^{d} \frac{(-1)^k}{2^k} \sum_{\substack{L\in\mathcal{L}(0)\\ \mathrm{ind}L=k}} I_{\{x\in L\}}\right), \quad x \in \mathbb{R}^d;$$

$$\mathcal{K}_d(x) = \frac{1}{2^d}\left(1 + \sum_{k=1}^{d} \frac{(-1)^k}{2^k} \sum_{\substack{L\in\mathcal{L}_d\\ \mathrm{ind}L=k}} I_{\{x\in L\}}\right), \quad x \in (-\pi,\pi]^d.$$

Proof. Let us prove both equalities simultaneously. Let ν be the Lebesgue measure on $\mathbb{R}$ or the counting measure on $\mathbb{Z}$. Then both limits in the definitions of the functions $\mathcal{J}_d$ and $\mathcal{K}_d$ are written in the same way. We have

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{(0,T]} \prod_{k=1}^{d} \sin^2(x_k s/2)\, d\nu(s) = \lim_{T\to\infty} \frac{1}{2^d T} \int\limits_{(0,T]} \prod_{k=1}^{d} (1-\cos(x_k s))\, d\nu(s).$$

Using the formula for the product of cosines, we rewrite the product

$$\begin{aligned}\prod_{k=1}^{d} (1-\cos(x_k s)) = 1 - \sum_{k=1}^{d} \cos(x_k s) + \sum_{1\le i<j\le d} \cos(x_i s)\cos(x_j s) + \ldots \\ + (-1)^k \sum_{1\le i_1<\ldots<i_k\le d} \cos(x_{i_1} s)\cdots\cos(x_{i_k} s) + \ldots \\ + (-1)^d \cos(x_1 s)\cdots\cos(x_d s) = \\ = 1 + \sum_{k=1}^{d} \frac{(-1)^k}{2^{k-1}} \frac{1}{2} \sum_{\|\varepsilon\|_1=k} \cos((x_1\varepsilon_1 + x_2\varepsilon_2 + \ldots + x_d\varepsilon_d)s).\end{aligned}$$

The summation $\sum\limits_{\|\varepsilon\|_1=k}$ is over all vectors $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$, $\varepsilon_k \in \{-1,0,1\}$ with the norm $\|\varepsilon\|_1 = k$. Since the vectors ε and $-\varepsilon$ make the same contribution, each sum has an additional factor $\frac{1}{2}$. It remains to calculate the limits of the form $\lim_{T\to\infty} \frac{1}{T}\int_{(0,T]} \cos(ys)\, d\nu(s)$, where $y = \sum_{k=1}^{d} x_k\varepsilon_k$.

If ν is the Lebesgue measure, then the limit is $I_{\{y=0\}}$. Let's check that for the counting measure on $\mathbb{Z}$ there will be a similar result. Using the known formulas for the sum of cosines, we get

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \cos(ny) = \\ = \lim_{N\to\infty} \frac{1}{N}\left(\frac{\cos(Ny/2)\sin((N+1)y/2)}{\sin(y/2)} - 1\right) = \sum_{k=-\infty}^{+\infty} I_{\{y=2\pi k\}}.\end{aligned}$$

The last sum is in fact finite and equals $\sum_{k=-\lfloor d/2\rfloor}^{\lfloor d/2\rfloor} I_{\{y=2\pi k\}}$, since for $x \in (-\pi,\pi]^d$ the point $y \in (-d\pi, d\pi]$. Putting all the calculations together, we obtain the required equalities. □

It immediately follows from Proposition 1 that the functions $\mathcal{J}_d$ and $\mathcal{K}_d$ take a finite number of values. In addition, for all $x \in (-\pi,\pi]^d$ such that $x \notin \mathcal{L}_d \setminus \mathcal{L}(0)$, the equality $\mathcal{J}_d(x) = \mathcal{K}_d(x)$ holds; in particular, for all $x \in \left(-\frac{\pi}{d}, \frac{\pi}{d}\right]^d$.

3.2. Let us find the zeros of the function $\mathcal{J}_d$, and estimate its minimum positive value.

Proposition 2. *The following statements hold for the function* $\mathcal{J}_d$*:*

(i) $\mathcal{J}_d(x) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{k=1}^{d}\sin^2(x_k s/2)\,ds$ *for all* $x \in \mathbb{Z}^d$.

(ii) $\mathcal{J}_d(x) = 0$ *if and only if* $\prod_{k=1}^{d} x_k = 0$. *Moreover, for the minimal value* $j_d := \min_{\prod_{k=1}^{d} x_k \neq 0} \mathcal{J}_d(x)$ *we have the estimate*

$$\frac{1}{2(2d+1)d^{(d-1)(2d+1)}} \le j_d \le \frac{1}{2^d}.$$

Proof. Considering that for $x \in \mathbb{Z}^d$ the squares of the sines involved in the product will be 2π-periodic, we obtain

$$\begin{aligned}\mathcal{J}_d(x) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\prod_{k=1}^{d}\sin^2(x_k s/2)\,ds = \\ &= \lim_{N\to\infty}\frac{1}{2\pi N}\int_0^{2\pi N}\prod_{k=1}^{d}\sin^2(x_k s/2)\,ds = \\ &= \lim_{N\to\infty}\frac{1}{2\pi N}\sum_{n=0}^{N-1}\int_{2\pi n}^{2\pi(n+1)}\prod_{k=1}^{d}\sin^2(x_k s/2)\,ds = \\ &= \lim_{N\to\infty}\frac{1}{2\pi N}\sum_{n=0}^{N-1}\int_0^{2\pi}\prod_{k=1}^{d}\sin^2(x_k(t+2\pi n)/2)\,dt = \\ &= \lim_{N\to\infty}\frac{1}{2\pi N}\sum_{n=0}^{N-1}\int_0^{2\pi}\prod_{k=1}^{d}\sin^2(x_k t/2)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\prod_{k=1}^{d}\sin^2(x_k t/2)\,dt.\end{aligned}$$

The function $\mathcal{J}_d$ is a homogeneous function of order zero, i.e., for any $\alpha \in \mathbb{R}\setminus\{0\}$ the equality $\mathcal{J}_d(\alpha x) = \mathcal{J}_d(x)$ is true. If the point $x \in \mathbb{R}^d$ does not lie in any hyperplane of the bunch $\mathcal{L}(0)$, then $\mathcal{J}_d(x) = 2^{-d}$. The function $\mathcal{J}_d$ takes the same values on linear subspaces that are intersections of hyperplanes of the bunch $\mathcal{L}(0)$. Since any such intersection contains integer points, the values of the function $\mathcal{J}_d$ are completely determined by its values on $\mathbb{Z}^d$. Moreover, it can be argued that it is sufficient to take integer points from some cube. To find the boundary of this cube, we use the following statement from the theory of Diophantine equations (see [20, Theorem 6.1], [21, Chapter 1, §2.2], and also [22]). Let $m < d$, $a_{i,j} \in \mathbb{Z}$ and $A_i = \sum\limits_{j=1}^{d} |a_{i,j}|, i = 1, ..., m$; then the system of linear homogeneous equations

$$a_{i,1}x_1 + a_{i,2}x_2 + ... + a_{i,d}x_d = 0, \quad i = 1, ..., m$$

has a non-trivial integer solution satisfying the condition

$$\max_{1\le k\le d} |x_k| \le \sqrt[d-m]{A_1 \cdots A_m}.$$

In our case, all $A_i \le d$, and the boundary of the cube is estimated as $\max\limits_{1\le k\le d} |x_k| \le d^{d-1}$.

From (i) and the fact that $\mathcal{J}_d$ is completely determined by the values on $\mathbb{Z}^d$, it is immediately clear that $\mathcal{J}_d$ vanishes only if one of the coordinates $x_k = 0$. Let us now find the lower bound for j_d; the upper bound is obvious. We will use the representation (i), where for the point $x \in \mathbb{Z}^d, x_k \ne 0$ we will assume that $D = \max\limits_{1\le k\le d} |x_k| \le d^{d-1}$. We have

$$\mathcal{J}_d(x) = \frac{1}{2\pi}\int_0^{2\pi} \prod_{k=1}^{d} \sin^2(x_k s/2)\, ds = \\ = \frac{1}{\pi}\int_0^{\pi} \prod_{k=1}^{d} \sin^2(x_k s)\, ds \ge \frac{1}{\pi}\int_0^{\pi/(2D)} \prod_{k=1}^{d} \sin^2(x_k s)\, ds.$$

For $s \in \left[0, \frac{\pi}{2D}\right]$ we have $x_k s \in \left[0, \frac{\pi}{2}\right]$ and hence $\sin^2(x_k s) \ge (\frac{2}{\pi}x_k s)^2$. Thus,

$$\mathcal{J}_d(x) \ge \frac{1}{\pi}\int_0^{\pi/(2D)} \prod_{k=1}^{d} \frac{4x_k^2 s^2}{\pi^2}\, ds = \frac{4^d x_1^2 \cdots x_d^2}{\pi^{2d+1}} \int_0^{\pi/(2D)} s^{2d}\, ds = \\ = \frac{4^d x_1^2 \cdots x_d^2}{\pi^{2d+1}} \frac{\pi^{2d+1}}{(2d+1)(2D)^{2d+1}} = \frac{x_1^2 \cdots x_d^2}{2(2d+1)D^{2d+1}} \ge \frac{1}{2(2d+1)d^{(d-1)(2d+1)}}.$$

□

3.3. Let us proceed to finding the zeros of the function $\mathcal{K}_d(x)$.

Proposition 3. *$\mathcal{K}_d(x) = 0$ if and only if* $\prod\limits_{k=1}^{d} x_k = 0$.

Proof. As can be seen from the definition of the function $\mathcal{K}_d$, if at least one coordinate of the point $x \in (-\pi, \pi]^d$ is equal to zero, then the function vanishes. Let us show that there are no other zeros. Suppose that there is a point $x \in (-\pi, \pi]^d$, such that $\prod\limits_{k=1}^{d} x_k \ne 0$, and at the same time $\mathcal{K}_d(x) = 0$.

We obtain a contradiction using the density theory for subsets of the set of natural numbers. And the transition to this theory is realized using the Koopman—von Neumann lemma (see, for example, [23, lemma 2.41]): for a bounded sequence of non-negative numbers $\{a_n\}_{n\geq 1}$ one have the equivalence

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n=0 \qquad\Leftrightarrow\qquad a_n\to 0 \ \text{ for } \ n\to\infty,\ n\notin J$$

for some set J with zero asymptotic density.

Recall (see, for example, [24]) that the asymptotic density $d(J)$ of some set $J=\{j_1,j_2,...\}$ of natural numbers is the limit (if it exists)

$$d(J)=\lim_{n\to\infty}\frac{\#\{k:j_k\leq n\}}{n}=\lim_{n\to\infty}\frac{n}{j_n}.$$

Let $a_n=\prod\limits_{k=1}^{d}\sin^2(x_kn/2)$. Since

$$0=\mathcal{K}_d(x)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_n,$$

then there exists a set $J\subset\mathbb{N}$ of asymptotic density 1, along which $a_n\to 0$ (since the complement of a set of density 0 always has density 1). We will show that this cannot happen. To do this, we will find out along which increasing sequences m_n of natural numbers it will be $\sin(ym_n)\to 0$ as $m_n\to\infty$ for a given $y\in(-\pi/2,\pi/2]\setminus\{0\}$.

If $y=\pi\frac{a}{b}, a\in\mathbb{Z}, b\in\mathbb{N},\ b\geq 2$, then $m_n=bM_n,\ M_n\in\mathbb{N}$, starting from some number n. The maximally dense set will be the arithmetic progression $m_n=bn,\ n\in\mathbb{N}$; its asymptotic density is $\frac{1}{b}$.

If $y=\pi\alpha, \alpha\in\mathbb{R}\setminus\mathbb{Q}$, then $\{m_n\}_{n\geq 1}$ has zero asymptotic density. Indeed, let us first show that $m_{n+1}-m_n\to+\infty$ as $n\to\infty$. Let this not be so; then there exists a natural number c such that for an infinite number of numbers we have $m_{n+1}-m_n=c$. For such numbers we obtain

$$\begin{aligned}0&=\lim_n\sin(ym_{n+1})=\lim_n\sin(ym_n+yc)=\\&=\lim_n\sin(ym_n)\cos(yc)+\lim_n\cos(ym_n)\sin(yc)=\pm\sin(yc)=\pm\sin(\pi c\alpha).\end{aligned}$$

The sign $\pm$ depends on the limit of $\cos(ym_n)$: is it 1 or -1. Here we conclude that $c=0$; it cannot be. Using Stolz theorem for calculating the limits of sequences, we obtain

$$d(\{m_n\})=\lim_{n\to\infty}\frac{n}{m_n}=\lim_{n\to\infty}\frac{n+1-n}{m_{n+1}-m_n}=0.$$

Let us return to the sequence $a_n=\prod\limits_{k=1}^{d}\sin^2(x_kn/2)$. The set J along which it tends to zero is the union of the sets J_k along each of which $\sin(x_kn/2)$ tends to zero. Among the J_k there are either sets of density zero or arithmetic progressions of the form $\{b_kn\}_{n\geq 1}, b_k\in\mathbb{N}, b_k\geq 2$. A finite union of sets with zero asymptotic density will be a set with zero density. And a finite union of the arithmetic progressions considered cannot have density 1, since there will always be another arithmetic progression in its complement. Indeed, the arithmetic progression $\{1 + b_1 \cdots b_\ell n\}_{n\geq 1}$ has no common terms with any of the progressions $\{b_k n\}_{n\geq 1}$. Since otherwise the Diophantine equation

$$1 + b_1 \cdots b_\ell n = b_k m$$

would have a solution $n, m \in \mathbb{N}$; it cannot be.

Thus, the asymptotic density of the set J cannot be equal to 1. □

4 Proof of Theorem 2

Recall that the equivalence of (1) and (4) follows from [5]. For the remaining items, we prove the following chain: (4) ⇒ (3) ⇒ (2) ⇒ (4).

From (4), we can easily obtain (3) by setting $n_1 = \cdots = n_d = n$.

We will show the implication (3)⇒ (2). Let for some constant $B > 0$ for all natural n we have $\|A_n f\|_2^2 \leq Bn^{-2d}$. Taking into account the representation for the L_2-norm of ergodic means, we rewrite this estimate as

$$B \geq \int\limits_{(-\pi,\pi]^d} \frac{\prod_{i=1}^d \sin^2 \frac{nx_i}{2}}{\prod_{i=1}^d \sin^2 \frac{x_i}{2}}\, d\sigma_f(x_1, \ldots, x_d).$$

Summing this inequality over n from 1 to N, we obtain

$$B \geq \frac{1}{N} \sum_{n=1}^{N} \int\limits_{(\pi,\pi]^d} \frac{\prod_{i=1}^d \sin^2 \frac{nx_i}{2}}{\prod_{i=1}^d \sin^2 \frac{x_i}{2}}\, d\sigma_f(x_1, \ldots, x_d).$$

The Fatou lemma, when passing to the lower limit as $N \to \infty$, yields the following inequality:

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int\limits_{(-\pi,\pi]^d} \frac{\prod_{i=1}^d \sin^2 \frac{nx_i}{2}}{\prod_{i=1}^d \sin^2 \frac{x_i}{2}}\, d\sigma_f(x_1, \ldots, x_d) \geq$$

$$\geq \int\limits_{(-\pi,\pi]^d} \frac{\lim_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N} \prod_{i=1}^d \sin^2 \frac{nx_i}{2}}{\prod_{i=1}^d \sin^2 \frac{x_i}{2}}\, d\sigma_f(x_1, \ldots, x_d).$$

Recalling the definition of the function $\mathcal{K}_d(x)$, we obtain an estimate close to the required one:

$$\int\limits_{(-\pi,\pi]^d} \frac{\mathcal{K}_d(x)}{\prod_{i=1}^d \sin^2 \frac{x_i}{2}}\, d\sigma_f(x_1, \ldots, x_d) \leq B.$$

From condition (3), applying the conditional expectation operator $\mathbb{E}_j$ with respect to the σ-algebra of T_j-invariant sets, we obtain, on the one hand,

$$\left\|\mathbb{E}_j\left(\sum_{k_1=0}^{n-1}\cdots\sum_{k_d=0}^{n-1} f(T_1^{k_1}\cdots T_d^{k_d}x)\right)\right\|_2 \le \left\|\sum_{k_1=0}^{n-1}\cdots\sum_{k_d=0}^{n-1} f(T_1^{k_1}\cdots T_d^{k_d}x)\right\|_2 = \mathcal{O}(1),$$

and on the other hand, because $\mathbb{E}_jT_j = \mathbb{E}_j$,

$$\left\|\mathbb{E}_j\left(\sum_{k_1=0}^{n-1}\cdots\sum_{k_d=0}^{n-1} f(T_1^{k_1}\cdots T_d^{k_d}x)\right)\right\|_2 = n\left\|\sum_{k_1=0}^{n-1}\cdots\sum_{k_d=0}^{n-1} \mathbb{E}_j f(T_1^{k_1}\cdots \widehat{T_j^{k_j}}\cdots T_d^{k_d}x)\right\|_2 .$$

Putting it all together, we find for the ergodic averages generated by $d-1$ automorphisms (all except T_j), the estimate

$$\|A_n^{T_1}\cdots A_n^{T_d}\mathbb{E}_j f\|_2 = o\left(\frac{1}{n^{d-1}}\right).$$

Taking into account Theorem 1, we conclude that $\mathbb{E}_j f = 0$ for each $j = 1, ..., d$. From this we deduce [10, Corollary 1], that the spectral measure $\sigma_f(\mathbb{O}) = 0$ for

$$\mathbb{O} = \left\{x \in (-\pi,\pi]^d :\ \prod_{i=1}^d x_i = 0\right\}.$$

By Proposition 3, the function $\mathcal{K}_d(x)$ also vanishes on $\mathbb{O}$, therefore

$$\int\limits_{(-\pi,\pi]^d} \frac{\mathcal{K}_d(x)}{\prod_{i=1}^d \sin^2\frac{x_i}{2}}\, d\sigma_f(x_1,\dots,x_d) \ge \min_{x\notin\mathbb{O}}\mathcal{K}_d(x) \int\limits_{(-\pi,\pi]^d} \frac{d\sigma_f(x_1,\dots,x_d)}{\prod_{i=1}^d \sin^2\frac{x_i}{2}}.$$

Whence, assuming $\kappa_d = \min\limits_{x\notin\mathbb{O}}\mathcal{K}_d(x)$ and knowing from Propositions 1 and 3 that $\kappa_d > 0$, we obtain

$$\int\limits_{(\pi,\pi]^d} \frac{d\sigma_f(x_1,\dots,x_d)}{\prod_{i=1}^d \sin^2\frac{x_i}{2}} \le \frac{B}{\kappa_d} < \infty.$$

We now show the implication (2)⇒(4). If

$$\int\limits_{(-\pi,\pi]^d} \frac{d\sigma_f(x_1,\dots,x_d)}{\prod_{i=1}^d \sin^2\frac{x_i}{2}} = A < \infty,$$

then for all $\vec{n} \in \mathbb{N}^d$

$$\|A_{\vec{n}}f\|_2^2 = \frac{1}{n_1^2\cdots n_d^2}\int\limits_{(-\pi,\pi]^d} \frac{\sin^2\frac{n_1x_1}{2}\cdots\sin^2\frac{n_dx_d}{2}}{\sin^2\frac{x_1}{2}\cdots\sin^2\frac{x_d}{2}}\, d\sigma_f(x_1,\dots,x_d) \le$$

$$\le \frac{1}{n_1^2\cdots n_d^2}\int\limits_{(-\pi,\pi]^d} \frac{d\sigma_f(x_1,\dots,x_d)}{\sin^2\frac{x_1}{2}\cdots\sin^2\frac{x_d}{2}} \le A\frac{1}{n_1^2\cdots n_d^2}.$$

Theorem 2 is completely proved.

5 Proof of Theorem 3

The equivalence of conditions (2′), (3′), and (4′) is proved word for word, just as in Theorem 2 the equivalence of conditions (2), (3), and (4) is proved. The difference is the use of the function $\mathcal{J}_d$ (instead of the function $\mathcal{K}_d$), whose zeros form the set $\mathbb{O}' = \left\{x \in \mathbb{R}^d : \prod_{i=1}^d x_i = 0\right\}$. We prove the equivalence of (1′) and (4′).

Proposition 4. *Let $T^{t_1}, ..., T^{t_d}$ be pairwise commuting flows of a probability measure space (Ω, μ). For any function $f \in L_p(\Omega, \mu)$, $p \in [1, +\infty)$ the equivalence holds*

$$f \in RL_p(\Omega, \mu) \quad \Leftrightarrow \quad \sup_{\vec{t} \in \mathbb{R}_+^d} \left\| \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_p < \infty.$$

Proof. It is enough to prove the implication $\Leftarrow$. Let us reduce the problem to the already known criterion for discrete time. Let

$$\sup_{\vec{t} \in \mathbb{R}_+^d} \left\| \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_p = C < \infty.$$

Fix an arbitrary vector $\vec{q} \in \mathbb{R}_+^d$, and set, as in the proof of Theorem 1,

$$g_{\vec{q}}(\omega) = \int_{[0,\vec{q}]} f(T_1^{t_1} T_2^{t_2} \cdots T_d^{t_d}\omega)\, d\vec{t}, \quad S_j = T_j^{q_j}, \quad j = 1, ..., d.$$

Then, using the calculations from the proof of Theorem 1, we obtain the estimate

$$\sup_{\vec{t} \in \mathbb{R}^d} \left\| \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_p \geq \sup_{\vec{t} = \vec{n} \odot \vec{q},\ \vec{n} \in \mathbb{N}^d} \left\| \int_{[0,\vec{t}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_p =$$

$$= \sup_{\vec{n} \in \mathbb{N}^d} \left\| \int_{[0,\vec{n} \odot \vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} \right\|_p = \sup_{\vec{n} \in \mathbb{N}^d} \left\| \sum_{0 \leq \vec{k} \leq \vec{n}} g_{\vec{q}}(S_1^{k_1} \cdots S_d^{k_d}\omega) \right\|_p.$$

Thus, for commuting automorphisms $S_j, j = 1, ..., d$ and the function $g_{\vec{q}}$, we obtain

$$\sup_{\vec{n} \in \mathbb{N}^d} \left\| \sum_{0 \leq \vec{k} \leq \vec{n}} g_{\vec{q}}(S_1^{k_1} \cdots S_d^{k_d}\omega) \right\|_p \leq C < \infty.$$

It follows from the work of Bradley [5] that there exists a function $h_{\vec{q}} \in L_p(\Omega, \mu)$ such that $\|h_{\vec{q}}\|_p \leq C$ and $g_{\vec{q}} = \prod_{j=1}^d (I - S_j) h_{\vec{q}}$, i.e.,

$$\int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} = \prod_{i=1}^d (I - T_j^{q_j}) h_{\vec{q}}, \quad \vec{q} \in \mathbb{R}_+^d.$$

Let us write this equality for the vector $\vec{q}/\vec{k} := \left(\frac{q_1}{k_1}, ..., \frac{q_d}{k_d}\right)$, $\vec{k} \in \mathbb{N}^d$:

$$\int_{[0,\vec{q}/\vec{k}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u} = \prod_{j=1}^{d} (I - T_j^{\frac{q_j}{k_j}}) h_{\vec{q}/\vec{k}}.$$

Acting on this equality from the left by the operator

$$\prod_{j=1}^{d} \left(I + T_j^{\frac{q_j}{k_j}} + T_j^{2\frac{q_j}{k_j}} ... + T_j^{(k_j-1)\frac{q_j}{k_j}} \right) = \sum_{0 \le \vec{n} \le \vec{k}} T_1^{n_1 \frac{q_1}{k_1}} \cdots T_d^{n_d \frac{q_d}{k_d}},$$

we obtain the equality

$$\int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u} = \prod_{j=1}^{d} (I - T_j^{q_j}) h_{\vec{q}/\vec{k}}.$$

We will now assume that the vector $\vec{q}$ has binary-rational coordinates. Then there exists a natural number $m_0 = m_0(\vec{q})$ such that for any $m \ge m_0$ we can choose a vector $\vec{k} = \vec{k}(m)$, for which

$$\vec{q}/\vec{k} = (2^{-m}, 2^{-m}, ..., 2^{-m}) := \vec{v}_m.$$

Indeed, let $\vec{q} = \left(\frac{a_1}{2^{b_1}}, ..., \frac{a_d}{2^{b_d}}\right)$, $a_j, b_j \in \mathbb{N}$. Then

$$m_0 = b_1 b_2 \cdots b_d, \quad \vec{k} = (a_1 2^{m-b_1}, a_2 2^{m-b_2}, ..., a_d 2^{m-b_d}).$$

Since the sequence of functions $h_{\vec{v}_m}$ is bounded, then from the reflexivity of the space $L_p(\Omega, \mu)$ for $p \in (1, \infty)$ it follows that that there exists a function $h \in L_p(\Omega, \mu)$ which is a weak limit of some subsequence $h_{\vec{v}_{m_n}}$. Then

$$\prod_{j=1}^{d} (I - T_j^{q_j}) h_{\vec{v}_{m_n}} \xrightarrow{\omega} \prod_{j=1}^{d} (I - T_j^{q_j}) h.$$

But for $m_n \ge m_0(\vec{q})$ there is an equality

$$\prod_{j=1}^{d} (I - T_j^{q_j}) h_{\vec{v}_{m_0}} = \int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u} = \prod_{j=1}^{d} (I - T_j^{q_j}) h_{\vec{v}_{m_n}}.$$

Here we conclude that for any vector $\vec{q}$ with binary-rational coordinates

$$\int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d} \omega)\, d\vec{u} = \prod_{j=1}^{d} (I - T_j^{q_j}) h.$$

In the case of the space $L_1(\Omega, \mu)$, we use the Komlos theorem (see, for example, [25, Theorem 10.10.22]), which states the following. For a sequence of functions $h_{\vec{v}_m}$ bounded in $L_1(\Omega, \mu)$, there exists a subsequence $h_{\vec{v}_{m_n}}$ and a function $h \in L_1(\Omega, \mu)$ such that there is convergence a.e. of Cesaro means for any of its subsequences $h_{\vec{v}_{m_{n_k}}}$:

$$\tilde{h}_k := \frac{h_{\vec{v}_{m_{n_1}}} + h_{\vec{v}_{m_{n_2}}} + ... + h_{\vec{v}_{m_{n_k}}}}{k} \to h \quad \text{as} \quad k \to \infty.$$

Then a.e. $\prod_{j=1}^{d}(I - T_j^{q_j})\tilde{h}_k \to \prod_{j=1}^{d}(I - T_j^{q_j})h$ as $k \to \infty$. We can assume that $m_{n_1} \geq m_0(\vec{q})$. Then the left-hand side of the last limit relation is equal to $\int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u}$. Here we again conclude that for any vector $\vec{q}$ with dyadic rational coordinates

$$\int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} = \prod_{j=1}^{d}(I - T_j^{q_j})h.$$

Thus, the resulting equality is true for all $p \in [1, +\infty)$. Since the binary rational numbers are dense in $\mathbb{R}$, and both sides of the last equality are L_p-continuous, it will be true for all $\vec{q} \in \mathbb{R}_+^d$. Then

$$\frac{1}{\pi(\vec{q})} \int_{[0,\vec{q}]} f(T_1^{u_1} \cdots T_d^{u_d}\omega)\, d\vec{u} = \prod_{j=1}^{d} \frac{I - T_j^{q_j}}{q_j} h.$$

Passing to the limit in L_p for $q_1, ..., q_d \to 0$, taking into account the local ergodic theorem, we obtain $f = Rh$. Which is what was required to be proved. □

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