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TO THE IWASAWA AND HUPPERT THEOREMS ON SUPERSOLVABILITY OF FINITE GROUPS

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Abstract: Let $G \neq 1$ be a finite group and let $\mathbb{P}$ be the set of all primes. A chain $1 = M_0 < M_1 < \ldots < M_{n-1} < M_n = G$ such that M_i is a maximal subgroup of M_{i+1} for every i is called a maximal chain of G. Every chain is associated with a sequence of non-negative integers $j_1, j_2, \ldots, j_n$, where $j_i = |M_i : M_{i-1}|$. A maximal chain is a $\mathbb{P}$-chain if $j_i \in \mathbb{P}$ for every i. We say that a $\mathbb{P}$-chain is a $\mathbb{P}^{<}$-chain ($\mathbb{P}^{>}$-chain) if $j_1 \leq j_2 \leq \ldots \leq j_n$ ($j_1 \geq j_2 \geq \ldots \geq j_n$, respectively). We investigate finite groups in which some maximal chains are $\mathbb{P}$-chains. In particular, we obtain the following criteria for finite groups to be supersolvable: a group G is supersolvable if and only if there are a $\mathbb{P}^{>}$-chain and $\mathbb{P}^{<}$-chain in G; a group G is supersolvable if and only if G has a Sylow tower of supersolvable type and there is a $\mathbb{P}^{<}$-chain in G. The obtained results are used for characterization of generally supersolvable groups.

Keywords: finite group, maximal subgroup, chain of subgroups, subgroup index, supersolvable group.

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1 Introduction

All groups in this paper are finite. We use the standard notation and terminology of [1].

Let $G \neq 1$ be a group. A subgroup chain

$$1 = M_0 \lessdot M_1 \lessdot \ldots \lessdot M_i \lessdot M_{i+1} \lessdot \ldots \lessdot M_{n-1} \lessdot M_n = G \qquad (1)$$

such that M_i is a maximal subgroup of M_{i+1} for every i is called a maximal chain of G, and n is the length of this chain. Every chain is associated with a sequence of non-negative integers $j_1, j_2, \ldots, j_n$, where $j_i = |M_i : M_{i-1}|$ is the index of M_{i-1} in M_i, $i = 1, \ldots, n$. We use $\mathbb{P}$ to denote the set of all primes.

In 1941, Iwasawa proved the following result.

Theorem (Iwasawa)**.** *A group is supersolvable if and only if its maximal chains have the same length,* [2].

Huppert proved the following fundamental theorem 13 years later.

Theorem (Huppert)**.** *A group is supersolvable if and only if all its maximal subgroups are of prime indices,* [3, Theorem 9].

It follows from the Huppert Theorem that all indices of every maximal chain in a supersolvable group are primes. Therefore, the following concepts are quite natural.

Definition 1. *Let G be a group. If $j_i \in \mathbb{P}$ for every i, then a maximal chain (1) is called a $\mathbb{P}$-chain of G. We say that a $\mathbb{P}$-chain is a $\mathbb{P}^<$-chain ($\mathbb{P}^>$-chain) if $j_1 \leq j_2 \leq \ldots \leq j_n$ ($j_1 \geq j_2 \geq \ldots \geq j_n$, respectively). A maximal chain of G is said to be a monotone $\mathbb{P}$-chain if it is a $\mathbb{P}^<$-chain or a $\mathbb{P}^>$-chain.*

In this paper, we investigate groups with $\mathbb{P}$-chains. We enumerate groups in which all maximal chains of every proper subgroup are monotone $\mathbb{P}$-chains, Theorem 1. In Theorem 2, we indicate the properties of the class of all groups with $\mathbb{P}^<$-chain. The following supersolvability criteria follow from this theorem: *a group G is supersolvable if and only if there are a $\mathbb{P}^>$-chain and $\mathbb{P}^<$-chain in G; a group G is supersolvable if and only if G has a Sylow tower of supersolvable type and there is a $\mathbb{P}^<$-chain in G.* The obtained results are used for characterization of generally supersolvable groups, Theorems 3, 4.

2 Used notation and concepts.

If X is a subgroup (proper subgroup, maximal subgroup) of a group Y, then we write $X \leq Y$ ($X < Y$, $X \lessdot Y$, respectively). We use $A \leq_t B$ if $A \leq B$ and $|B : A| = t$. We write $r = \max \pi(G)$ ($r = \min \pi(G)$) to indicate that r is the greatest prime divisor (the lowest prime divisor, respectively) of the order of a group G. Here and later, $\pi(G)$ is the set of all prime divisors of $|G|$. We use $\mathbb{P}$ and $\mathbb{N}$ to denote the sets of all primes and all non-negative integers, respectively; A_n and S_n denotes alternating and symmetric groups of degree n, respectively; C_n is a cyclic group of order n, and C_n^t denotes a direct product of t copies of C_n. We use $A \rtimes B$ to denote the semidirect product of a normal subgroup A and a subgroup B.

Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, $p_i \in \mathbb{P}$, $\alpha_i \in \mathbb{N}$, $i = 1, \dots, n$. We say that G has *a Sylow tower* if G has a normal series

$$1 = G_0 < G_1 < \dots < G_{n-1} < G_n = G \tag{2}$$

such that $|G_i| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$ for every i. In that case, G_{i+1}/G_i is isomorphic to a Sylow p_i-subgroup of G for every i. If $p_1 > p_2 > \dots > p_n$, then we say that G has *a Sylow tower of supersolvable type*, and if $p_1 < p_2 < \dots < p_n$, then G has *a Sylow tower of anti-supersolvable type*. The classes of all groups with Sylow towers of supersolvable type and anti-supersolvable type are denoted by $\mathfrak{D}$ and $\overline{\mathfrak{D}}$, respectively. It is easy to check $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ are subgroup-closed hereditary Fitting formations.

Recall a Schmidt group is a non-nilpotent group with all proper subgroups nilpotent. A group G is a minimal non-supersolvable group if G is not supersolvable but every proper subgroup of G is supersolvable. The properties of Schmidt groups and minimal non-supersolvable groups are well known, see, for example, [4, 5]. A group is primary if it is of prime power order.

3 Groups with monotone $\mathbb{P}$-chains

Since composition factors of a solvable group $G \neq 1$ have prime orders, every composition series of a solvable group is a $\mathbb{P}$-chain.

Example 1. (1) In A_4, there is no subgroup of index 2, hence A_4 has no $\mathbb{P}^{>}$-chains, but in A_4, there is a $\mathbb{P}^{<}$-chain: $1 <_2 C_2 <_2 C_2^2 <_3 A_4$.

(2) In $G = C_3^2 \rtimes C_4$ [6, SmallGroup(36,9)], there is no subgroup of index 3, hence G has no $\mathbb{P}^{<}$-chains, but G has a $\mathbb{P}^{>}$-chain: $1 <_3 C_3 <_3 C_3^2 <_2 C_3^2 \rtimes C_2 <_2 G$.

(3) In $G = C_3^2 \rtimes SL(2,3)$ [6, SmallGroup(216,153)], there is no subgroup of index 2, hence G has no $\mathbb{P}^{>}$-chain. Among the maximal chains of G, only three are $\mathbb{P}$-chains:

$$1 <_2 C_2 <_3 S_3 <_3 C_3 \rtimes S_3 <_2 C_3^2 \rtimes C_4 <_2 PSU(3,2) <_3 G,$$

$$1 <_3 C_3 <_2 S_3 <_3 C_3 \rtimes S_3 <_2 C_3^2 \rtimes C_4 <_2 PSU(3,2) <_3 G,$$

$$1 <_3 C_3 <_3 C_3^2 <_2 C_3 \rtimes S_3 <_2 C_3^2 \rtimes C_4 <_2 PSU(3,2) <_3 G,$$

but each of them is not monotone. In particular, in $G = C_3^2 \rtimes SL(2,3)$, there is a $\mathbb{P}$-chain, but there are no $\mathbb{P}^{>}$-chain and no $\mathbb{P}^{<}$-chain.

(4) In A_6, there is no maximal subgroup of prime index, hence A_6 has no $\mathbb{P}$-chain.

Lemma 1. *Let G be a group. The following statements hold.*

(1) *G contains a $\mathbb{P}^{>}$-chain if and only if G has a Sylow tower of supersolvable type.*

(2) *If G has a Sylow tower of anti-supersolvable type, then there is a $\mathbb{P}^{<}$-chain in G.*

(3) *In any supersolvable group, there is a* $\mathbb{P}^{<}$*-chain.*

Proof. (1) Assume that in G, there is a $\mathbb{P}^{>}$-chain (1). If $n = 1$, then G is a group of prime order and the statement is true. Therefore we can assume that $n > 1$. Use induction on n. By induction, M_{n-1} has a Sylow tower of supersolvable type. Hence a Sylow r-subgroup R of M_{n-1} is normal in M_{n-1} for $r = j_1 = \max \pi(M_{n-1})$. If $r = j_n$, then G is an r-group, and the statement is true. Therefore, $r > j_n$ and R is normal in G in view of the Sylow Theorem. By induction, $G/R \in \mathfrak{D}$. Consequently, $G \in \mathfrak{D}$.

Conversely, let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, $p_1 > p_2 > \ldots > p_n$, with a Sylow tower of supersolvable type. In that case, G has a normal series (2) such that $|G_i| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i}$ for every i. In particular, G_1 is a normal Sylow p_1-subgroup of G for $p_1 = \max \pi(G)$. We have that G_1 contains a chain

$$1 \lessdot G_1^1 \lessdot \ldots \lessdot G_1^{\alpha_1} = G_1, \ |G_1^j : G_1^{j-1}| = p_1, \ j = 1, 2, \ldots, \alpha_1.$$

Since G/G_1 has a Sylow tower of supersolvable type, in G/G_1 there is a maximal $\mathbb{P}^{>}$-chain

$$1 = G_1/G_1 \lessdot M_1/G_1 \lessdot \ldots \lessdot M_m/G_1 = G/G_1$$

by induction. Now, $1 \lessdot G_1^1 \lessdot \ldots \lessdot G_1^{\alpha_1} \lessdot M_1 \lessdot M_2 \lessdot \ldots \lessdot M_m = G$ is a $\mathbb{P}^{>}$-chain of G.

(2) Use induction on $|G|$. Let $G \in \overline{\mathfrak{D}}$ and let $r = \max \pi(G)$. In that case, G has a normal Hall r'-subgroup H. By induction, there is a $\mathbb{P}^{<}$-chain in H. Continuing this chain to G, we obtain a $\mathbb{P}^{<}$-chain.

(3) Use induction on $|G|$. Let G be supersolvable and let $r = \max \pi(G)$. Then G contains a subgroup H of index r. By induction, there is a $\mathbb{P}^{<}$-chain in H. Hence in G, there is a $\mathbb{P}^{<}$-chain. □

Example 2. In S_4, A_5, $PSL(2,7)$, there are $\mathbb{P}^{<}$-chains, but these groups have no Sylow towers. Therefore, Statement (2) of Lemma 1 is not converse.

Theorem 1. *Assume that in all proper subgroups of a group G, every maximal chain is a monotone $\mathbb{P}$-chain. Then $1 \leq |\pi(G)| \leq 3$ and G is a group of one of the following type.*

(1) *G is a supersolvable group of order p^n, pq, p^2q or pqr;*

(2) *G is a p-closed Schmidt group with Frattini subgroup of prime order p;*

(3) *G is a non-supersolvable Schmidt group with identity Frattini subgroup.*

Conversely, in all proper subgroups of the enumerated groups, every maximal chain is a monotone $\mathbb{P}$-chain.

Proof. Assume that every maximal chain of G is a monotone $\mathbb{P}$-chain. In that case, all maximal subgroups of G have prime indices, and G is supersolvable by the Huppert Theorem. If G is not primary, then in G, there is a normal subgroup C_p of prime order p and a subgroup C_q of prime order q, $p \neq q$. Suppose that $H = C_pC_q$ is a proper subgroup of G. If $p > q$, the chain $1 \lessdot_q C_q \lessdot_p H$ is a $\mathbb{P}^{<}$-chain and the chain $1 \lessdot_p C_p \lessdot_q H$ is a $\mathbb{P}^{>}$-chain. Fix a monotone $\mathbb{P}$-chain from H to G: $H \lessdot H_1 \lessdot \ldots \lessdot H_{n-1} \lessdot H_n = G$. In that case, one of the chains

$$1 \lessdot_q C_q \lessdot_p H \lessdot H_1 \lessdot \ldots \lessdot H_{n-1} \lessdot H_n = G,$$

$$1 \lessdot_p C_p \lessdot_q H \lessdot H_1 \lessdot \ldots \lessdot H_{n-1} \lessdot H_n = G$$

are not monotone, a contradiction. Therefore, $G = C_pC_q$. Thus, if every maximal chain of G is monotone $\mathbb{P}$-chain, then either G is primary or G has order pq.

In the sequel, we can assume that G contains a maximal chain that is not a monotone $\mathbb{P}$-chain. By the above, every proper subgroup of G is either primary or of order pq. Let G be a supersolvable. Then $1 \leq |\pi(G)| \leq 3$, and $|G| = pqr$ if $|\pi(G)| = 3$. If $|\pi(G)| = 2$, then $|G| = p^2q$. Let G be non-supersolvable. In that case, G is a minimal non-supersolvable group, $G = P \rtimes Q$, where Q is a supplement to the normal Sylow p-subgroup P of G and $|P/\Phi(P)| = p^n > p$ by [7, Lemma 2.1]. If $Q_1 \lessdot Q$, $|Q_1| \neq 1$, then $|PQ_1| = p^n|Q_1| \neq pq$, a contradiction. Consequently, $|Q| = q \in \mathbb{P}$. If $\Phi(P) = 1$, then $\Phi(G) = 1$ and G is a Schmidt group with identity Frattini subgroup, i. e. G is a group from Statement (3) of the theorem. If $\Phi(P) \neq 1$, then $|\Phi(P)Q| = pq$ and $M = \Phi(P)Q \lessdot G$. Suppose that $M = \Phi(P) \rtimes Q$ is not nilpotent. It follows q divides $p - 1$. In view of $M_G = \Phi(P)$, we have $G/M_G = P/M_G \rtimes M/M_G$ is a Schmidt group with identity Frattini subgroup. Hence $|G : M| = |P/M_G| = p^n$, where n is the multiplicative order of p modulo q, and $n = 1$, a contradiction. Hence M is nilpotent, and G is a p-closed Schmidt group with Frattini subgroup of order p, i. e. G is a group from Statement (2) of the theorem.

It remains to show that in proper subgroups of the enumerated groups, every maximal chain is a monotone $\mathbb{P}$-chain. Obviously, the statement is true for primary groups, groups of order pq, p^2q, or pqr. If G is a p-closed Schmidt group with Frattini subgroup of prime order p, then $G = P \rtimes Q$, $|P| = p^{1+m}$, $|Q| = q$, where m is the multiplicative order of p modulo q, and maximal subgroups of G have order p^{1+m} or pq. It is clear that in all proper subgroups of G, every maximal chain is a monotone $\mathbb{P}$-chain. In a non-supersolvable Schmidt group with identity Frattini subgroup, all proper subgroups are primary. Therefore such a group also satisfies the conditions of the theorem. □

4 On the class of groups with $\mathbb{P}^{<}$-chains.

Let $\mathcal{P}^{<}$ ($\mathcal{P}^{>}$) be the class of all groups in which there is a $\mathbb{P}^{<}$-chain (respectively, $\mathbb{P}^{>}$-chain). By $\mathfrak{U}$ and $\mathfrak{S}$ we denote the formations of all supersolvable and all solvable groups, respectively. In view of Lemma 1, we have $\mathcal{P}^{>} = \mathfrak{D}$, $\overline{\mathfrak{D}} \subseteq \mathcal{P}^{<}$ and $\mathfrak{U} \subseteq \mathcal{P}^{<}$. In particular, $\mathcal{P}^{>}$ is a subgroup-closed hereditary Fitting formation. The class $\mathcal{P}^{<}$ is more complicated.

Theorem 2. *The following statements hold.*

(1) *$\mathcal{P}^<$ is closed under taking normal subgroups, quotients, and direct products.*

(2) *$\mathcal{P}^< \cap \mathfrak{S}$ is a subgroup-closed formation.*

(3) *$\mathcal{P}^< \cap \mathfrak{D} = \mathcal{P}^< \cap \mathcal{P}^> = \mathfrak{U}$.*

(4) *If a simple non-abelian group $G \in \mathcal{P}^<$, then $G \in \{PSL(2,7),\ PSL(2,11),\ PSL(2,2^n)\}$, where 2^n+1 is a Fermat prime.*

Proof. (1) Let $G \in \mathcal{P}^<$ and let N be a normal subgroup of G. Then in G, there is a $\mathbb{P}^<$-chain (1) with prime indices $j_1 \le j_2 \le \ldots \le j_n$. Put $K_i = M_i \cap N$ and consider a chain

$$1 = K_0 \le K_1 \le \ldots \le K_{n-1} \le K_n = N. \tag{3}$$

Since N is normal in G, we get K_{i+1} is normal in M_{i+1} and $M_i \le K_{i+1}M_i \le M_{i+1}$. In view of $|M_{i+1} : M_i| = j_{i+1} \in \mathbb{P}$, either $M_i = K_{i+1}M_i$ or $K_{i+1}M_i = M_{i+1}$. If $M_i = K_{i+1}M_i$, then $K_i = K_{i+1}$. If $K_{i+1}M_i = M_{i+1}$, then $j_{i+1} = |M_{i+1} : M_i| = |K_{i+1}M_i : M_i| = |K_{i+1} : K_i|$. In chain (3), we leave only one of the coincide subgroups in chain (3) and we get a $\mathcal{P}^<$-chain of N, i. e. $N \in \mathcal{P}^<$. Thus, $\mathcal{P}^<$ is closed under taking normal subgroups.

It is clear that there is a chain

$$N/N = M_0N/N \le \ldots \le M_{n-1}N/N \le M_nN/N = G/N. \tag{4}$$

Note that $|M_{i+1}N/N : M_iN/N| = |M_{i+1}N : M_iN| = |M_{i+1} : M_i|/|(M_{i+1} \cap N) : (M_i \cap N)|$. In view of $|M_{i+1} : M_i| = j_{i+1} \in \mathbb{P}$, we have either $|(M_{i+1} \cap N) : (M_i \cap N)| = |M_{i+1} : M_i|$ and $|M_{i+1}N/N : M_iN/N| = 1$ or $M_{i+1} \cap N = M_i \cap N$ and $|M_{i+1}N/N : M_iN/N| = |M_{i+1} : M_i| = j_{i+1}$. In chain (4), we leave only one of the coincide subgroups in chain (4) and we get a $\mathcal{P}^<$-chain of G/N, i. e. $G/N \in \mathcal{P}^<$. Thus, $\mathcal{P}^<$ is closed under taking quotients.

Now, we check that $\mathcal{P}^<$ is closed under taking direct products. If $A, B \in \mathcal{P}^<$, then there are $\mathbb{P}^<$-chains

$$1 = A_0 < A_1 < \ldots < A_{n-1} < A_n = A, \tag{5}$$

$$a_i = |A_i : A_{i-1}| \in \mathbb{P}, \quad a_1 \le a_2 \le \ldots \le a_n,$$

$$1 = B_0 < B_1 < \ldots < B_{m-1} < B_m = B, \tag{6}$$

$$b_i = |B_i : B_{i-1}| \in \mathbb{P}, \quad b_1 \le b_2 \le \ldots \le b_n.$$

Use induction on $|A \times B|$. Let $r = \max\{a_n, b_m\}$. In that case, in chain (5), A_α is a Hall r'-subgroup of A for some α. Similarly, in chain (6), B_β is a Hall r'-subgroup of B for some β. By induction, $A_\alpha \times B_\beta$ contains a $\mathbb{P}^<$-chain. It is clear that $A_\alpha \times B_\beta$ is a Hall r'-subgroup of $A \times B$, and we can construct a chain from $A_\alpha \times B_\beta$ to $A \times B$ with indices equal to r. So, $A \times B \in \mathcal{P}^<$.

(2) Let $G \in \mathcal{P}^<$ and let G be solvable. Then in G, there is a $\mathbb{P}^<$-chain

$$1 \lessdot M_1 \lessdot \ldots \lessdot M_{n-1} = M \lessdot M_n = G.$$

Hence $|G : M| = r = \max \pi(G)$, and by induction, there is a $\mathbb{P}^<$-chain in every subgroup of M. Let $H \lessdot G$. If H and M are conjugate, then H contains a $\mathbb{P}^<$-chain. Assume that H and M are not conjugate. Since G is solvable, we have $G = MH$ and $|G : H| = |H : H \cap M| = r = \max \pi(G)$. In view of

$H \cap M \le M$, we deduce that $H \cap M$ has a $\mathbb{P}^<$-chain, and so, H also has a $\mathbb{P}^<$-chain. Thus, there is a $\mathbb{P}^<$-chain in every maximal subgroup of G. By induction, in every subgroup of G, there is a $\mathbb{P}^<$-chain, therefore the class $\mathcal{P}^< \cap \mathfrak{S}$ is closed under taking subgroups, i. e. it is subgroup-closed. From Statement (1), we deduce that $\mathcal{P}^< \cap \mathfrak{S}$ is a subgroup-closed formation.

(3) By Lemma 1 (2-3), $\mathcal{P}^< \cap \overline{\mathfrak{D}} = \overline{\mathfrak{D}}$ and $\mathfrak{U} \subseteq \mathcal{P}^<$. Since $\mathfrak{U} \subseteq \mathfrak{D}$, we get $\mathfrak{U} \subseteq \mathcal{P}^< \cap \mathfrak{D}$. Suppose that $\mathcal{P}^< \cap \mathfrak{D} \not\subseteq \mathfrak{U}$, and let G be a group of least order such that $G \in \mathcal{P}^< \cap \mathfrak{D} \setminus \mathfrak{U}$. In that case, G is a minimal non-supersolvable group and G has a Sylow tower of supersolvable type. Therefore, $|G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})| = p^n > p$ for $p = \max \pi(G)$ by [7, Lemma 2.1] and G has no subgroup of index p. Hence G has no $\mathbb{P}^<$-chain, i. e. $G \notin \mathcal{P}^<$, a contradiction. Consequently, $\mathcal{P}^< \cap \mathfrak{D} \subseteq \mathfrak{U}$ and $\mathcal{P}^< \cap \mathfrak{D} = \mathfrak{U}$.

(4) Simple non-abelian groups with $\mathbb{P}$-chains were enumerated in [8, Theorem 3.2]. From these groups only $PSL(2, 7)$, $PSL(2, 11)$ and $PSL(2, 2^n)$, where $2^n + 1$ is a Fermat prime, have $\mathbb{P}^<$-chains. □

Corollary 1. *A group G is supersolvable if and only if there are a $\mathbb{P}^<$-chain and a $\mathbb{P}^>$-chain in G.*

Proof. If G is supersolvable, then G has a Sylow tower of supersolvable type [1, VI.9.1(c)], and by Lemma 1 (1) (3), there are a $\mathbb{P}^>$-chain and a $\mathbb{P}^<$-chain in G. Conversely, assume that in G, there are a $\mathbb{P}^<$-chain and a $\mathbb{P}^>$-chain. In that case, $G \in \mathcal{P}^<$ by the definition of the class $\mathcal{P}^<$, and $G \in \mathfrak{D}$ by Lemma 1 (1). Thus, $G \in \mathfrak{D} \cap \mathcal{P}^<$ and G is supersolvable by Theorem 2 (3). □

Corollary 2. *A group G is supersolvable if and only if G has a Sylow tower of supersolvable type and there is a $\mathbb{P}^<$-chain in G.*

Proof. If G is supersolvable, then G has a Sylow tower of supersolvable type [1, VI.9.1(c)], and there is a $\mathbb{P}^<$-chain in G by Lemma 1 (3). Conversely, if G has a Sylow tower of supersolvable type [1, VI.9.1(c)] and there is a $\mathbb{P}^<$-chain in G, then $G \in \mathfrak{D} \cap \mathcal{P}^<$. According to Theorem 2 (3), G is supersolvable. □

Example 3. The group $G = C_5^2 \rtimes D_8$ [6, SmallGroup(200,43)] contains subgroups A and B such that $A \cong B \cong D_{10}^2$ and $|G : A| = |G : B| = 2$. These subgroups are supersolvable, therefore they have $\mathbb{P}^<$-chain. In G, there is no maximal subgroups of index 5, hence G contains no $\mathbb{P}^<$-chain. Thus, the normal subgroup-closed class $\mathcal{P}^<$ is not a Fitting class.

5 Characterisations of w- and v-supersolvable groups

In [8], the following concept was proposed.

Definition 2. *A subgroup H of a group G is $\mathbb{P}$-subnormal in G if either $G = H$ or there is a chain*

$$H = H_0 < H_1 < \ldots < H_{n-1} < H_n = G \tag{7}$$

such that $|H_{i+1} : H_i| \in \mathbb{P}$ for every i.

It follows from the Huppert Theorem that we can define the formation $\mathfrak{U}$ of all supersolvable groups as the class of all groups in which every subgroup is $\mathbb{P}$-subnormal. The classes of all groups with $\mathbb{P}$-subnormal Sylow subgroups and with $\mathbb{P}$-subnormal cyclic primary subgroups are denoted by $\mathrm{w}\mathfrak{U}$ and $\mathrm{v}\mathfrak{U}$, respectively. These classes were quit well investigated [7, 8, 9, 10]. In particular, $\mathrm{w}\mathfrak{U}$ and $\mathrm{v}\mathfrak{U}$ are subgroup-closed hereditary formations, $\mathfrak{U} \subset \mathrm{w}\mathfrak{U} \subset \mathrm{v}\mathfrak{U} \subset \mathfrak{D}$, and all inclusions are proper. Groups from $\mathrm{w}\mathfrak{U}$ is called w-*supersolvable*, groups from $\mathrm{v}\mathfrak{U}$ is called v-*supersolvable*.

In the context of this paper, the following concepts are quite natural.

Definition 3. *A subgroup H of a group G is $\mathbb{P}^{<}$-subnormal ($\mathbb{P}^{>}$-subnormal) in G, if either $G = H$ or there is a chain (7) such that $|H_{i+1} : H_i| \in \mathbb{P}$ and $|H_i : H_{i-1}| \le |H_{i+1} : H_i|$ (respectively, $|H_i : H_{i-1}| \ge |H_{i+1} : H_i|$) for every i.*

It is clear that every $\mathbb{P}^{<}$-subnormal and every $\mathbb{P}^{>}$-subnormal subgroup is $\mathbb{P}$-subnormal. The converse is not true in general, see Example 1 (3).

In this section, supersolvable, w- and v-supersolvable groups are characterized by the existence or unexistence of $\mathbb{P}^{<}$- or $\mathbb{P}^{>}$-subnormal subgroups in them.

Lemma 2. *Let H be a $\mathbb{P}^{<}$-subnormal subgroup of a group G, let K be a subgroup of G, and let N be a normal subgroup of G . The following statements hold.*

(1) *$(H \cap N)$ is $\mathbb{P}^{<}$-subnormal in N.*
(2) *HN is $\mathbb{P}^{<}$-subnormal in G.*
(3) *HN/N is $\mathbb{P}^{<}$-subnormal in G/N.*

Proof. For $\mathbb{P}$-subnormal subgroups, all statements are true, see [9, Lemma 1]. By hypothesis, there is a chain (7) with indices $p_1 \le p_2 \le \ldots \le p_n$.

(1) It is clear that there is a chain

$$H \cap N = H_0 \cap N \le H_1 \cap N \le \ldots \le H_{n-1} \cap N \le H_n \cap N = N.$$

Since N is normal in G, we deduce that $|H_i \cap N : H_{i-1} \cap N|$ divides $p_i = |H_i : H_{i-1}|$ for every i. We leave only one of the coincide subgroups in the chain and we get that $H \cap N$ is $\mathbb{P}^{<}$-subnormal in N.

(2) Since N is normal in G, there is a chain

$$HN = H_0N \le H_1N \le \ldots \le H_{n-1}N \le H_nN = GN.$$

In view of

$$|H_{i+1}N : H_iN| = \frac{|H_{i+1} : H_i|}{|H_{i+1} \cap N : H_i \cap N|} \in \{1 \cup \mathbb{P}\},$$

we leave only one of the coincide subgroups in the chain and we get that HN is $\mathbb{P}^{<}$-subnormal in G.

(3) Since N is normal in G, there is a chain

$$HN/N = H_0N/N \le H_1N/N \le \ldots \le H_{n-1}N/N \le H_nN/N = GN/N.$$

In view of

$$|H_{i+1}N/N : H_iN/N| = \frac{|H_{i+1} : H_i|}{|H_{i+1} \cap N : H_i \cap N|} \in \{1 \cup \mathbb{P}\},$$

we leave only one of the coincide subgroups in the chain and we get that HN/N is $\mathbb{P}^{<}$-subnormal in G/N. □

We can repeat the proof of Lemma 2 with replacing «$\mathbb{P}^{<}$-subnormality» by «$\mathbb{P}^{>}$-subnormality» to get the similar properties of $\mathbb{P}^{>}$-subnormal subgroups.

Lemma 3. *The following statements hold.*
(1) *If $G \in \mathfrak{D}$, then every $\mathbb{P}$-subnormal subgroup of G is $\mathbb{P}^{>}$-subnormal.*
(2) *In supersolvable group, every subgroup is $\mathbb{P}^{>}$-subnormal.*
(3) *If $G \in \overline{\mathfrak{D}}$, then every $\mathbb{P}$-subnormal subgroup of G is $\mathbb{P}^{<}$-subnormal.*

Proof. (1) Let H be a $\mathbb{P}$-subnormal subgroup of a group $G \in \mathfrak{D}$. Use induction on $|G : H|$. Assume that R is a Sylow r-subgroup of G for $r = \max \pi(G)$. In view of $G \in \mathfrak{D}$, R is normal in G by the definition of the class $\mathfrak{D}$. If $G = HR$, then H is $\mathbb{P}^{>}$-subnormal in G. Assume that HR is a proper subgroup of G. In view of [9, Lemma 4], HR is $\mathbb{P}$-subnormal in G and H is $\mathbb{P}$-subnormal in HR. By induction, HR is $\mathbb{P}^{>}$-subnormal in G, hence there is a $\mathbb{P}^{>}$-chain

$$HR <_{q_1} U_1 <_{q_2} U_2 <_{q_3} \ldots <_{q_{n-1}} U_{n-1} <_{q_n} U_n = G,$$

$q_1 \geq q_2 \geq \ldots \geq q_{n-1} \geq q_n$.
In a $\mathbb{P}$-chain from H to HR, all indices are equal to $r > q_1$. Therefore, H is $\mathbb{P}^{>}$-subnormal in G.

(2) It follows from Statement (1) because in a supersolvable group, every subgroup is $\mathbb{P}$-subnormal.

(3) We repeat the proof of Statement (1) with replacing «$r = \max \pi(G)$» by «$r = \min \pi(G)$» and we get that the statement is true. □

Theorem 3. *Let G be a group. The following statements hold.*

(1) *If G is supersolvable, then every Hall subgroup of G is $\mathbb{P}^{>}$-subnormal and $\mathbb{P}^{<}$-subnormal in G.*

(2) *G is supersolvable if and only if every Sylow subgroup of G is $\mathbb{P}^{<}$-subnormal in G.*

(3) *G is* w*-supersolvable if and only if every Sylow subgroup of G is $\mathbb{P}^{>}$-subnormal in G.*

Proof. (1) Let G be supersolvable and let H be a Hall subgroup of G. In view of Lemma 3 (2), H is $\mathbb{P}^{>}$-subnormal in G. Put

$$\pi = \pi(G) \setminus \pi(H) = \{q_1, q_2, \ldots q_n\}.$$

Then there is Sylow q_i-subgroups Q_i such that $HQ_1Q_2 \ldots Q_i$ is a subgroup of G for every i. Since G is supersolvable, it follows from the Huppert Theorem that a chain

$$H < HQ_1 < HQ_1Q_2 < \ldots < HQ_1Q_2 \ldots Q_{n-1} < G$$

can be compacted so that all indices of new chain are prime. It is true for any ordering π. For $q_1 < q_2 < \ldots < q_n$, a new chain shows that H is $\mathbb{P}^{<}$-subnormal in G.

(2) If G is supersolvable, then every Sylow subgroup of G is $\mathbb{P}^{<}$-subnormal in G by Statement (1). Conversely, assume that every Sylow subgroup of G is $\mathbb{P}^{<}$-subnormal in G. In that case, $G \in \mathrm{w}\mathfrak{U} \subset \mathfrak{D}$. In view of Lemma 2 (3) and by induction, G/N is supersolvable for every non-trivial normal subgroup N of G. Therefore G is a primitive group, and $G = R \rtimes M$, where $R = F(G) = C_G(R)$ is a Sylow r-subgroup of G for $r = \max \pi(G)$ and a unique minimal normal subgroup, and $M \lessdot G$. Let Q be a Sylow q-subgroup of M. Then $q \neq r$ and Q is a Sylow q-subgroup of G. By hypothesis, Q is $\mathbb{P}^{<}$-subnormal in G. Hence, there is a subgroup H of G such that $Q \leq H$ and $|G : H| = r$. Consequently, $M \leq H^g \leq G$ for some $g \in G$. Since $M \lessdot G$, we have $M = H^g$ and $|R| = |G : M| = r$. Therefore, $M \cong G/R = N_G(R)/C_G(R)$ is a cyclic group of order dividing $r - 1$, and G is supersolvable.

(3) Since every w-supersolvable group G has a Sylow tower of supersolvable type, then every Sylow subgroup of G is $\mathbb{P}^{>}$-subnormal in G by Lemma 3 (1). Conversely, if every Sylow subgroup of G is $\mathbb{P}^{>}$-subnormal in G, then every Sylow subgroup of G is $\mathbb{P}$-subnormal in G, and G is w-supersolvable by definition. □

Theorem 4. *Let G be a group. The following statements hold.*

(1) *If every cyclic primary subgroup of G is $\mathbb{P}^{<}$-subnormal in G, then G is supersolvable.*

(2) *G is* v*-supersolvable if and only if every cyclic primary subgroup of G is $\mathbb{P}^{>}$-subnormal in G.*

Proof. (1) Let every cyclic primary subgroup of G be $\mathbb{P}^{<}$-subnormal in G. In that case, $G \in \mathrm{v}\mathfrak{U} \subset \mathfrak{D}$, and by Lemma 1 (1), G contains a $\mathbb{P}^{>}$-chain. Let $q = \min \pi(G)$. By hypothesis, a subgroup of prime order q is $\mathbb{P}^{<}$-subnormal in G. Therefore G contains a $\mathbb{P}^{<}$-chain. By Corollary 1, G is supersolvable.

(2) If $G \in \mathrm{v}\mathfrak{U}$, then $G \in \mathfrak{D}$, and by Lemma 3 (1), every cyclic primary subgroup is $\mathbb{P}^{>}$-subnormal in G. Conversely, if every cyclic primary subgroup A of G is $\mathbb{P}^{>}$-subnormal in G, then A is $\mathbb{P}$-subnormal in G, and $G \in \mathrm{v}\mathfrak{U}$. □

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