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CONTROL MATRIX OF THE SINGULAR VALUE DECOMPOSITION OF THE RADON TRANSFORM

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Abstract. A formula for inverting the control submatrix of the singular value decomposition of the Radon transform is obtained. We prove that the pseudoinverse matrix is expressed in terms of the submatrix itself and its calculation does not require the use of linear algebra packages. The formula and proof are given. The practical value lies in the ability to control the components of the singular value decomposition of the Radon transform and to filter and analyze images. The numerical experiments on real-world images demonstrate the efficiency of the techniques proposed.

Keywords: control matrix, Radon transform, SVD, destriping

1 Introduction

In many problems of linear algebra and its applications the problem arises of calculating an economical block partition of inverse and generalized inverse matrices, especially when they are used repeatedly. Of particular interest are the cases where the inverse matrix is expressed analytically in terms of the original matrix itself. In the case of a block partition of a matrix, it is also advantageous to have submatrix inversion formulas. The article

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considers such matrices and their blocks that control the smoothness of restoration and the direction of projections in the formulas for the singular value decomposition of the Radon transform. The submatrices of the control matrix are involved in the task of tomography with a limited range of transmission angles.

The article is structured as follows. In section 2, we introduce a steering matrix and for its block we give the form of its generalized inverse. In section 3, we revisited the singular value decomposition of the Radon transform. We present material for solving the problem with a limited viewing angle (one projection is not available) and to solve the problem of destriping, that is, the removal of band noise from the image. Section 4 presents numerical experiments with an image obtained from a space station with an additively superimposed stripe structure. Destriping results are illustrated.

2 Control matrix and its properties

Consider a nonsingular block matrix M:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1}$$

Block partition of matrix M^{-1} has the form [1]

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & A^{-1}B(CA^{-1}B - D)^{-1} \\ (CA^{-1}B - D)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}. \tag{2}$$

provided that all inverse matrices included here exist. As we can see, blocks of matrix M^{-1} depend on all blocks A, B, C, D of matrix M.

In the case when the matrix is Msingular, the block decomposition for its generalized inverse matrix M^{+} is also known. However, this representation is more complex. We refer the reader to the works [2], [3], [4], [5] with a detailed exposition of formulas block decomposition of the matrix M^{+}.

In this paper, we are interested in the matrix

$$\Lambda_k = (\lambda_{ij}^{(k)}),\ i, j = 1, \ldots, n,\ k < n,$$

where

$$\lambda_{ij}^{(k)} = \frac{\sin\big(k(\omega_i - \omega_j)\big)}{k \sin(\omega_i - \omega_j)}.$$

and $\omega_i \in [0, 2\pi)$ are arbitrary angles. Geometry of parallel beams in each projection and evenly located projection angles in the interval $[0, \pi)$ is a classic case in computed tomography:

$$\omega_i = \frac{\pi}{n}(i - 1).$$

In what follows, we consider only uniformy sampled angles. Then

$$\lambda_{ij}^{(k)} = \frac{\sin\Big(k(i-j)\frac{\pi}{n}\Big)}{k \sin\Big((i-j)\frac{\pi}{n}\Big)}.$$

The matrix Λ_k is of rank k is circulant and has remarkable properties [6].

Proposition 1. *For $k < n$, we have*

$$\Lambda_k^2 = \frac{n}{k}\Lambda_k. \tag{3}$$

Proposition 2. *For $k < n$, matrix Λ_k is singular and its generalized inverse Λ_k is*

$$\Lambda_k^+ = \frac{k^2}{n^2}\Lambda_k. \tag{4}$$

Let's represent the matrix Λ_k in block form:

$$\Lambda_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \tag{5}$$

where matrix $A_k = (a_{ij}^{(k)})$, $i, j = 1, \ldots, n-1$ has size $(n-1)\times(n-1)$ and

$$a_{ij}^{(k)} \equiv \lambda_{ij}^{(k)}$$

Let's present the matrix Λ_k in more detail and illustratively.

$$\Lambda_k = \left| \begin{array}{cccc|c} 1 & \frac{\sin\left(k\frac{\pi}{n}\right)}{k\sin\left(\frac{\pi}{n}\right)} & \ldots & \frac{\sin\left(k\frac{\pi}{n}(n-2)\right)}{k\sin\left(\frac{\pi}{n}(n-2)\right)} & \frac{\sin\left(k\frac{\pi}{n}(n-1)\right)}{k\sin\left(\frac{\pi}{n}(n-1)\right)} \\ \frac{\sin\left(k\frac{\pi}{n}\right)}{k\sin\left(\frac{\pi}{n}\right)} & 1 & \ldots & \frac{\sin\left(k\frac{\pi}{n}(n-3)\right)}{k\sin\left(\frac{\pi}{n}(n-3)\right)} & \frac{\sin\left(k\frac{\pi}{n}(n-2)\right)}{k\sin\left(\frac{\pi}{n}(n-2)\right)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\sin\left(k\frac{\pi}{n}(n-2)\right)}{k\sin\left(\frac{\pi}{n}(n-2)\right)} & \frac{\sin\left(k\frac{\pi}{n}(n-3)\right)}{k\sin\left(\frac{\pi}{n}(n-3)\right)} & \ldots & 1 & \frac{\sin\left(k\frac{\pi}{n}\right)}{k\sin\left(\frac{\pi}{n}\right)} \\ \hdashline \frac{\sin\left(k\frac{\pi}{n}(n-1)\right)}{k\sin\left(\frac{\pi}{n}(n-1)\right)} & \frac{\sin\left(k\frac{\pi}{n}(n-2)\right)}{k\sin\left(\frac{\pi}{n}(n-2)\right)} & \ldots & \frac{\sin\left(k\frac{\pi}{n}\right)}{k\sin\left(\frac{\pi}{n}\right)} & 1 \end{array} \right|. \tag{6}$$

where dashed lines separate blocks A_k, B_k, C_k, D_k. In the problem of restoring images from projections in directions $(\omega_1, \ldots, \omega_{n-1})$ the control matrix A_k and its pseudoinverse A_k^+ are used. It turns out that in this case, too, an analytical inversion of this matrix is possible, which has the form:

Theorem 1.

$$A_k^+ = \frac{k^2}{n^2}A_k + \frac{k(2n-k)}{(n-k)^2}A_k - \frac{k^3(2n-k)}{n^2(n-k)^2})A_k^2 \tag{7}$$

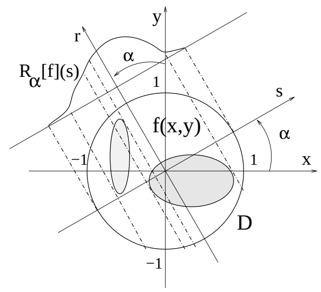


FIG. 1. The Radon transform R_α of function $f \in L^2(D)$

Proof. The pseudoinverse (generalized inverse) matrix A is uniquely determined by the Penrose equations [2]:

$$\begin{cases} AA^+A = A, \\ A^+AA^+ = A^+, \\ (A^+A)^* = A^+A, \\ (AA^+)^* = AA^+ \end{cases} \tag{8}$$

The proof of the relations (8) is carried out by a direct verification of the equations, that is, by substituting (7) into (8). □

We can hypothetically suggest that the power matrix series (7) can be extended to the case of a set of angles $(\omega_1, \ldots, \omega_{n-2})$, and further, which corresponds to the problem of tomography with a limited angle view. This is the subject of the present search. However, the formulation of the problem with one inaccessible direction ω_n from the full set of angles $\omega_1, \ldots, \omega_n$ has practical implications, as will be illustrated in the following sections.

3 Radon transfom and its SVD

The need to calculate the generalized inverse submatrix arises in the Radon problem when calculating its singular value decomposition in the case when one projection is not available and the problem is called with a limited angle. Here we have a soft setting, when the restrictions concern only one projection.

Definition 1. *The Radon transform R of the function f with a support in the form of the unit disk D is defined as the set of its integrals along a line with direction α and distance s from the origin*

$$R_\alpha[f](s) = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s\cos\alpha - t\sin\alpha, s\sin\alpha + t\cos\alpha)\, dt. \tag{9}$$

We denote a single projection (9) as $p(\alpha, s) = p_\alpha(s)$ and refer to it as a complete projection if it is known for all points in the interval $s \in [-1, 1]$.

The set of projection directions is represented by the n-vector $\Omega = (\omega_1, \ldots, \omega_n)$, and the corresponding set of "complete" (known for all s)

projections is denoted as

$$\mathcal{R}_\Omega[f] \equiv (p(\omega_1, s), \ldots, p(\omega_n, s)).$$

We follow the [7], [8], [9] approach based on the existence and uniqueness of the approximation f_Ω (minimal norm solution) functions f, consistent with projection data

$$\mathcal{R}_\Omega[f] = \mathcal{R}_\Omega[f_\Omega] \tag{10}$$

and having the form of a superposition of plane waves in given directions ω:

$$f(x, y) \approx f_\Omega(x, y) = \sum_{i=1}^{n} h_{\omega_i}(x \cos\omega_i + y \sin\omega_i), \tag{11}$$

Plane waves h_{ω_i} are back projected versions of functions of one variable. There is a known constructive algorithm for calculating plane waves that add up to f_Ω , based on the singular value decomposition of the ray transform. It is shown that the functions h have the form [6]

$$h_{\omega_i}(s) = \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{n} \eta_{ij}^{(k)} U_{k-1}(s) \int_{-1}^{1} p_{\omega_j}(t) U_{k-1}(t)\, dt, \tag{12}$$

where $p_{\omega_j}(t) = R_{\omega_j}[f](t)$, $U_{k-1}(t) = \frac{\sin(k \arccos t)}{\sin(\arccos t)}$, $k = 1, 2, \ldots$ are the Chebyshev polynomials of the second kind, $\eta_{ij}^{(k)}$ are entries of the matrix Λ_k^+ (generalized inverse), $\Lambda_k = (\lambda_{ij}^{(k)})$, $i, j = 1, \ldots, n$, $\lambda_{ij}^{(k)} = \frac{\sin\big(k(\omega_i - \omega_j)\big)}{k \sin(\omega_i - \omega_j)}$.

Consider the case of reconstruction from projections in directions

$$\Omega \setminus \omega_n = (\omega_1, \ldots, \omega_{n-1}).$$

Similar to the (10)-(12) formalism, we have:

$$\mathcal{R}_{\Omega\setminus\omega_n}[f] = \mathcal{R}_{\Omega\setminus\omega_n}[f_{\Omega\setminus\omega_n}] \tag{13}$$

$$f(x, y) \approx f_{\Omega\setminus\omega_n}(x, y) = \sum_{i=1}^{n-1} g_{\omega_i}(x \cos\omega_i + y \sin\omega_i), \tag{14}$$

$$g_{\omega_i}(s) = \frac{1}{\pi} \sum_{k=1}^{\infty} \sum_{j=1}^{n-1} \gamma_{ij}^{(k)} U_{k-1}(s) \int_{-1}^{1} p_{\omega_j}(t) U_{k-1}(t)\, dt, \tag{15}$$

where $\gamma_{ij}^{(k)}$ are entries of the matrix $\Gamma_k \equiv A_k^+$ (generalized inverse), $A_k = (\lambda_{ij}^{(k)})$, $i, j = 1, \ldots, n-1$.

4 Numerical example

When transmitting satellite images of the Earth's surface and space bodies, interference sometimes occurs in the form of overlays of sinusoidal bands. These bands together can be modeled as a single plane wave (or ridge function). Therefore, we can eliminate one plane wave from the singular decomposition representation (), get an approximation to the true image.

This restoration will involve the submatrix A_k matrix Λ_k and its generalized inverse A_k^+.

Let us consider an additive image formation model with distortions described as the equation

$$z(x, y) = u(x, y) + v(x, y) \tag{16}$$

where z is the observed image; u is the desired useful image, usually with many details, and v is some interfering structure, which is known to be a plane wave, or a ridge function in the direction of ω_n,

$$v(x, y) = v(x \cos \omega_n + y \sin \omega_n).$$

The problem is to estimate u given z . Due to the additivity of the model (16) and the linearity of the Radon transform, the expansion of the functions z, u, v into a superposition of ridge functions in the directions

$$\omega \equiv (\omega_1, \ldots, \omega_{n-1}) = \Omega \setminus \omega_{n-1}$$

leads to equality

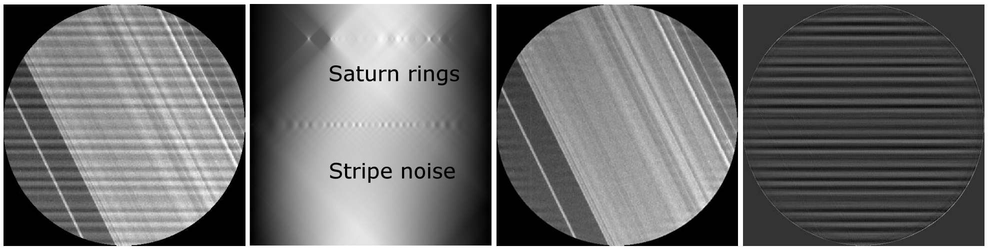
$$z_\omega(x, y) = u_\omega(x, y) + v_\omega(x, y) \tag{17}$$

Here $z_\omega, u_\omega, v_\omega$ are the minimum norm solutions obtained by reconstructing the images z, u, v from the datasets $\mathcal{R}_\omega[z], \mathcal{R}_\omega[u], \mathcal{R}_\omega[v]$ respectively. Since it is known that the distorting image (noise) v is a single ridge funztion and is well approximated by ridge functions in the directions $\omega = (\omega_1, \ldots, \omega_{n-1})$, we have $v \approx v_\omega$. Then, subtracting the equation (17) from (16), we approximately get

$$z(x, y) - z_\omega(x, y) \approx u(x, y) - u_\omega(x, y) \tag{18}$$

Therefore, the difference $u(x, y) - u_\omega(x, y)$ on the right side (18) is approximately equal to u with a shifted mean. As a result, an image is synthesized that is substantially free of v noise. Converting the image $z(x, y) - z_\omega(x, y)$ to the range of the renderer results in an image free of distortion caused by aliasing sinusoidal noise. To illustrate the problem of calculating SVD reconstruction for a real problem of removing band noise from an image, we use a test image from the book [10] as training examples. Example (Fig. 2 (a)) represent the rings of Saturn, subject to band noise. The image $z = u + v$ is the result of observation by the system of registration and transmission of remote sensing data. The distortions imposed by the bands are unknown. However, their directions are known in advance, they are parallel horizontal lines, resembling dark and light stripes, with a sinusoidal pattern of amplitude changes. The images are 512×512 pixels.

It's easy to show that a ridge function with a known direction can be uniquely determined, i.e. reconstructed from its projection taken in the same direction. This property is used by us to extract the ridge function from the additive model. To do this, we generate a Radon transform for a sufficiently large number of projections ($n = 512$), and then we calculate the reconstruction based on the singular value decomposition, excluding the horizontal ridge function, responsible for data corruption. The Radon



(a) (b) (c) (d)

FIG. 2. Space station Cassini remote sensing image of Saturn rings. (a) The test image $z = u + v$ is the sum of the remote sensing image u distorted with parallel stripe v; (b) The Radon transform, or sinogram; (c) The result of the SVD-based approximation $z_{\Omega\setminus(\pi/2)}$ of z in the directions Ω without filtered out horizontal projection; (e) The difference $z - z_{\Omega\setminus(\pi/2)}$ of images (a) and (c);

transform, or sinogram, is an image of $n \times n$. The projections are line by line from top to bottom, readings ("detectors with coordinate s") make up the horizontal axis (Fig. 2 (b)). The reconstruction or solution of the minimum norm z is shown in (Fig. 2 (c)), and the difference between the image z and its ridge approximation z_ω can be seen in (Fig. reffig:Saturn1 (d)).

5 Conclusion

The article considers the problem in the formulation, interesting in tomography on projections with a limited transmission angle and multiple inversion of the matrix that controls the directions of projections and the smoothness of the singular value decomposition of the Radon transform. Usually, standard pseudo-reversal programs are used, for example, in the Matlab package. Such a problem also arises in the processing of images in the Radon space, in order to remove any one noisy projection from the projection data. An analytical formula for the generalized inversion of the control matrix is obtained. An analytical formula for the generalized inversion of the control matrix is obtained. The generalization of the obtained results of the analytical inversion to the case of several directions is the subject of research in the future.

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