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DUALITY ANALYSIS OF THE FRICTIONLESS CONTACT PROBLEM BETWEEN LINEAR ELASTIC BODY AND RIGID-PLASTIC FOUNDATION

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Abstract: We investigate a frictionless contact problem between a linear elastic body and a rigid-plastic foundation. The problem can be formulated as a variational inequality or constrained minimization problem of the potential energy functional. To release constraints, we apply the duality scheme and prove convergence to a saddle point for the modified Lagrange functional. Finally, we present numerical results using the finite element method and provide the corresponding mechanical interpretations.

Keywords: variational inequality, contact problem, modified Lagrange functional

1 Introduction

Mechanical contact problems and their simulations are of great interest to industry. In particular, their analysis is primarily carried out by using the weak formulation, which is usually in the form of a variational inequality or constrained minimization problem. The theory of variational inequalities has undergone significant development in recent decades and is now a powerful mathematical tool providing existence, uniqueness and convergence results

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for various nonlinear boundary value problems in contact mechanics, see for instance [1, 2, 3, 4, 5, 6].

In this paper, we consider a mathematical model which describes the frictionless contact between an elastic body and a foundation made of a rigid-plastic material with yield limit ξ. The variational formulation and analysis of this problem are presented in [7, 8]. To solve this problem, we employ a duality method based on modified Lagrange functionals. Its use is explained by the fact that, for the classical Lagrange functionals, it is possible to prove the convergence of the iterative Uzawa method for finding a saddle point by a primal variable only, under the simultaneous condition that the step in the dual variable is sufficiently small [10, 11]. The application of modified Lagrange functionals to solve variational inequalities in mechanics has been introduced and studied by the authors in papers [12, 13, 14] for problems with a crack, and in [15, 16] and the references therein for contact problems. In the finite-dimensional case, similar constructions were proposed in the monographs of R.T. Rockafellar [17], D.P. Bertsekas [18] and B.T. Polyak [19].

In this paper we show the form of the saddle point of the classical Lagrange functional and research its existence under the assumption of additional regularity. Since the saddle points of the classical and modified Lagrange functionals coincide, the modified approach allows us to prove convergence to the saddle point of the Uzawa method and to implement the gradient method to solve the dual problem. To demonstrate the effectiveness of the proposed method, numerical results are presented using the finite element method.

The rest of the paper is organized as follows. Section 2 describes the initial boundary value problem and the corresponding variational formulation. In Section 3, we introduce the dual formulation of the problem and analyze convergence of Uzawa method to the saddle point. Finally, Section 4 presents the results of numerical experiments and gives mechanical interpretations.

2 Problem statement

We consider a two-dimensional contact problem between an elastic body Ω and a rigid-plastic foundation. The boundary $\Gamma = \partial\Omega$ is assumed to be Lipschitz continuous and is partitioned into three disjoint and measurable parts: Γ_d, Γ_n and Γ_c, such that the measure of Γ_d is positive. Suppose that on part Γ_d of the boundary the elastic body is rigidly fixed, on part Γ_n the surface tractions $\boldsymbol{p} \in [L_2(\Gamma_n)]^2$ are applied, and Γ_c represents the possible contact part (see Fig. 1). Moreover, volume forces $\boldsymbol{f} \in [L_2(\Omega)]^2$ act in Ω.

We denote by $\boldsymbol{u} = (u_1, u_2)$ the displacement field of elastic body Ω, $\varepsilon(\boldsymbol{u})$ the linearized strain tensor, $\sigma(\boldsymbol{u})$ the stress tensor and by $\boldsymbol{\nu}$ the outward unit normal on Γ. Assume that strain and stress tensors related by the linear Hooke's law

$$\sigma(\boldsymbol{u}) = \lambda \operatorname{tr}(\varepsilon(\boldsymbol{u}))\boldsymbol{I} + 2\mu\varepsilon(\boldsymbol{u}),$$

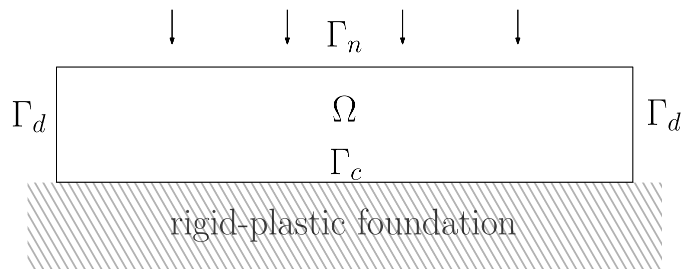


Рис. 1. Contact with rigid-plastic foundation.

where $\lambda \geq 0$ and $\mu \geq 0$ denote the Lamé constants and $\boldsymbol{I}$ is the identity matrix.

We consider the following boundary value problem in Ω. Find a displacement field $\boldsymbol{u}$ such that

$$-\mathrm{div}\ \sigma(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_d, \tag{2}$$

$$\sigma(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{p} \quad \text{on} \quad \Gamma_n, \tag{3}$$

$$\sigma_\tau(\boldsymbol{u}) = 0,\ -\xi \leq \sigma_\nu(\boldsymbol{u}) \leq 0,\ -\sigma_\nu(\boldsymbol{u}) = \begin{cases} 0 & \text{if} \quad u_\nu < 0, \\ \xi & \text{if} \quad u_\nu > 0. \end{cases} \quad \text{on} \quad \Gamma_c, \tag{4}$$

where $u_\nu = u_i\nu_i$, $\sigma_\nu(\boldsymbol{u}) = \sigma_{ij}(\boldsymbol{u})\nu_i\nu_j$ are normal components of the displacement vector $\boldsymbol{u}$ and stress vector $(\sigma_{1j}(\boldsymbol{u})\nu_j, \sigma_{2j}(\boldsymbol{u})\nu_j)$ on Γ_c. We assume that there is no friction on Γ_c, so the tangential stress $\sigma_\tau(\boldsymbol{u}) = \sigma_{ij}(\boldsymbol{u})\nu_j - \sigma_\nu(\boldsymbol{u})\boldsymbol{\nu}$ is equal to zero. Note that we use Einstein's summation convention to sum over repeated indices $i, j = 1, 2$.

Boundary value problem (1)-(4) belongs to the class of free boundary problems. Condition (4) models the contact with a foundation made of a rigid-plastic material, where the function $\xi \in L_2(\Gamma_c)$, such that $\xi(x) > 0$ a.e. $\boldsymbol{x} \in \Gamma_c$, can be interpreted as the yield limit. This means that foundation behaves like a rigid body as long as the inequality $-\xi < \sigma_\nu(\boldsymbol{u}) \leq 0$ holds. When the threshold $\sigma_\nu(\boldsymbol{u}) = -\xi$ is reached, it allows penetration and offers no additional resistance as surface plastic flow begins [7, 8].

For the weak formulation of the problem, we introduce the Hilbert space of virtual displacements

$$\mathbb{V} = \left\{ \boldsymbol{v} \in \mathbf{H}^1(\Omega) := [H^1(\Omega)]^2 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_d \right\}.$$

Next, we define the symmetric bilinear form $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, the linear form $b : \mathbb{V} \to \mathbb{R}$, and the function $j : \mathbb{V} \to \mathbb{R}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int\limits_\Omega \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, dx,\ b(\boldsymbol{v}) = \int\limits_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx + \int\limits_{\Gamma_n} \boldsymbol{p} \cdot \boldsymbol{v}\, ds,\ j(\boldsymbol{v}) = \int\limits_{\Gamma_c} \xi v_\nu^+\, ds,$$

where $v_\nu^+ = \max\{v_\nu, 0\}$, $\sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) = \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})$, $i, j = 1, 2$. The function j is continuous, convex and non-differentiable. Thus, j is weakly lower semi-continuous on $\mathbb{V}$.

Using these definitions we obtain the following variational formulation of problem (1)-(4). Find a displacement field $\boldsymbol{u} \in \mathbb{V}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geq b(\boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \mathbb{V}. \tag{5}$$

Variational inequality (5) can be formulated as the minimization problem

$$\begin{cases} J(\boldsymbol{v}) \to \min, \\ \boldsymbol{v} \in \mathbb{V}. \end{cases} \tag{6}$$

The functional to be minimized is defined as $J(\boldsymbol{v}) = \Pi(\boldsymbol{v}) + j(\boldsymbol{v})$, where $\Pi(\boldsymbol{v})$ is the potential energy functional

$$\Pi(\boldsymbol{v}) = \frac{1}{2} a(\boldsymbol{v}, \boldsymbol{v}) - b(\boldsymbol{v}),$$

consisting of the internal energy of deformation $\frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v})$ and the work of the external loads $b(\boldsymbol{v})$. Function $j(\boldsymbol{v})$ describes the work of the contact forces.

With the assumption $\operatorname{meas}(\Gamma_d) > 0$, the functional $\Pi(\boldsymbol{v})$ is convex, coercive and weakly lower semi-continuous on $\mathbb{V}$. Therefore using Stampacchia's theorem [9], there exists a unique solution $\boldsymbol{u} = \arg\min_{\boldsymbol{v} \in \mathbb{V}} J(\boldsymbol{v})$ of the problem (6).

3 Dual formulation and analysis

To solve the extremal problem (6), we apply a modified duality scheme that allows us to smooth the non-differentiable functional $J(\boldsymbol{v})$. Using the fact that $v_\nu^+ = (v_\nu + |v_\nu|)/2$, we can rewrite problem (6) in an equivalent form

$$\begin{cases} \bar{J}(\boldsymbol{v}, w) \to \min, \\ (\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c), \ w = 0 \text{ on } \Gamma_c. \end{cases} \tag{7}$$

Here functional defined as

$$\bar{J}(\boldsymbol{v}, w) = \Pi(\boldsymbol{v}) + \frac{1}{2} \int_{\Gamma_c} \xi v_\nu \, ds + \frac{1}{2} \int_{\Gamma_c} \xi |v_\nu - w| \, ds.$$

To release the equality constraint $w = 0$, we introduce the classical Lagrange functional on the space $\mathbb{V} \times [L_2(\Gamma_c)]^2$

$$L(\boldsymbol{v}, w, l) = \bar{J}(\boldsymbol{v}, w) + \int_{\Gamma_c} lw \, ds.$$

The corresponding saddle-point problem consists in funding a point $(\boldsymbol{v}^*, w^*, l^*) \in \mathbb{V} \times [L_2(\Gamma_c)]^2$ which satisfies the two-sided inequality

$$L(\boldsymbol{v}^*, w^*, l) \leq L(\boldsymbol{v}^*, w^*, l^*) \leq L(\boldsymbol{v}, w, l^*) \quad \forall (\boldsymbol{v}, w, l) \in \mathbb{V} \times [L_2(\Gamma_c)]^2. \tag{8}$$

Let us show that $\boldsymbol{v}^*$ is the solution to problem (7). From the left inequality of (8) we have

$$\bar{J}(\boldsymbol{v}^*, w^*) + \int_{\Gamma_c} l w^* \, ds \le \bar{J}(\boldsymbol{v}^*, w^*) + \int_{\Gamma_c} l^* w^* \, ds \quad \forall l \in L_2(\Gamma_c)$$

so that

$$\int_{\Gamma_c} w^*(l - l^*) \, ds \le 0 \quad \forall l \in L_2(\Gamma_c).$$

Taking $l = 2l^*$ and $l = 0$, we deduce that $w^* = 0$. Using the right inequality of (8), we find that

$$\bar{J}(\boldsymbol{v}^*, w^*) + \int_{\Gamma_c} l^* w^* \, ds \le \min_{\boldsymbol{v}, w} \Big\{ \bar{J}(\boldsymbol{v}, w) + \int_{\Gamma_c} l^* w \, ds \Big\} \le \min_{\boldsymbol{v} \in \mathbb{V}, w=0} \bar{J}(\boldsymbol{v}, w).$$

Thus, $\{\boldsymbol{v}^*, w^*\} = \{\boldsymbol{v}^*, 0\}$ is a solution to the problem (7) and $\boldsymbol{v}^* = \boldsymbol{u}$. In general, the opposite statement is not true. The solution of the problem (7) is not guaranteed to correspond to a certain saddle point of $L(\boldsymbol{v}, w, l)$ [10].

Theorem 1. *Suppose that the solution $\boldsymbol{u}$ of the problem (6) belongs to space $\mathbf{H}^2(\Omega)$. Then $(\boldsymbol{u}, 0, -\sigma_\nu(\boldsymbol{u}) - \frac{\xi}{2})$ is the unique saddle point of $L(\boldsymbol{v}, w, l)$ on the space $\mathbb{V} \times [L_2(\Gamma_c)]^2$.*

Proof. If $\boldsymbol{u} \in \mathbf{H}^2(\Omega)$, then from the trace theorem we have unique $\sigma_\nu(\boldsymbol{u}) \in H^{1/2}(\Gamma_c) \subset L_2(\Gamma_c)$ [4]. Therefore $-\sigma_\nu(\boldsymbol{u}) - \frac{\xi}{2} \in L_2(\Gamma_c)$. It is clear that the left part of (8) is satisfied because $w^* = 0$. It remains to show that

$$L(\boldsymbol{v}, w, -\sigma_\nu - \frac{\xi}{2}) - L(\boldsymbol{u}, 0, -\sigma_\nu - \frac{\xi}{2}) \ge 0, \tag{9}$$

where $\sigma_\nu \equiv \sigma_\nu(\boldsymbol{u})$.

We rewrite it as

$$\frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v}) - b(\boldsymbol{v}) + \frac{1}{2}\int_{\Gamma_c} (\xi v_\nu + \xi |v_\nu - w|) \, ds + \int_{\Gamma_c} (-\sigma_\nu - \frac{\xi}{2}) w \, ds - \frac{1}{2}a(\boldsymbol{u}, \boldsymbol{u}) +$$

$$+ b(\boldsymbol{u}) - \frac{1}{2}\int_{\Gamma_c} (\xi u_\nu + \xi |u_\nu|) \, ds =$$

$$= a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - b(\boldsymbol{v} - \boldsymbol{u}) + \frac{1}{2}a(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \frac{1}{2}\int_{\Gamma_c} (\xi v_\nu + \xi |v_\nu - w|) \, ds +$$

$$+ \int_{\Gamma_c} (-\sigma_\nu - \frac{\xi}{2}) w \, ds - \int_{\Gamma_c} \xi u_\nu^+ \, ds.$$

Using the Green formula for $a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})$ and the fact that $\xi u_\nu^+ = -\sigma_\nu u_\nu$, $\sigma_\tau = 0$ on Γ_c we obtain

$$\frac{1}{2}a(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + \int_{\Gamma_c} \sigma_\nu(v_\nu - u_\nu)\,ds + \frac{1}{2}\int_{\Gamma_c} (\xi v_\nu + \xi|v_\nu - w|)\,ds + \\ + \int_{\Gamma_c} (-\sigma_\nu - \frac{\xi}{2})w\,ds - \int_{\Gamma_c} (-\sigma_\nu u_\nu)\,ds = \\ = \frac{1}{2}a(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + \int_{\Gamma_c} \sigma_\nu v_\nu\,ds + \frac{1}{2}\int_{\Gamma_c} (\xi v_\nu + \xi|v_\nu - w|)\,ds - \int_{\Gamma_c} (\sigma_\nu + \frac{\xi}{2})w\,ds.$$

In order to show the fulfillment of (9), we need to consider 3 cases:

1. For $w = v_\nu$.

$$\frac{1}{2}a(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + \int_{\Gamma_c} \sigma_\nu v_\nu\,ds + \frac{1}{2}\int_{\Gamma_c} \xi v_\nu\,ds - \int_{\Gamma_c} (\sigma_\nu + \frac{\xi}{2})v_\nu\,ds = \frac{1}{2}a(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) \geq 0.$$

2. For $w < v_\nu$. Further we omit the bilinear form for simplicity.

$$\int_{\Gamma_c} \sigma_\nu v_\nu\,ds + \frac{1}{2}\int_{\Gamma_c} (\xi v_\nu + \xi(v_\nu - w))\,ds + \int_{\Gamma_c} (-\sigma_\nu - \frac{\xi}{2})w\,ds = \\ = \int_{\Gamma_c} (\sigma_\nu v_\nu + \xi v_\nu - \xi w - \sigma_\nu w)\,ds = \int_{\Gamma_c} (v_\nu - w)(\sigma_\nu + \xi)\,ds \geq 0.$$

3. For $w > v_\nu$.

$$\int_{\Gamma_c} \sigma_\nu v_\nu\,ds + \frac{1}{2}\int_{\Gamma_c} (\xi v_\nu - \xi(v_\nu - w))\,ds + \int_{\Gamma_c} (-\sigma_\nu - \frac{\xi}{2})w\,ds = \\ = \int_{\Gamma_c} \sigma_\nu(v_\nu - w)\,ds \geq 0.$$

Denote by $p = -\sigma_\nu(\boldsymbol{u}) - \frac{\xi}{2}$. Above, we showed that $(\boldsymbol{u}, 0, p)$ is saddle point of L on the space $\mathbb{V} \times [L_2(\Gamma_c)]^2$.

Now we show uniqueness of saddle point. Let $(\boldsymbol{u}^*, 0, p^*)$ be another saddle point of L on $\mathbb{V} \times [L_2(\Gamma_c)]^2$. From the second inequality of (8) we have

$$L(\boldsymbol{u}, 0, p) \leq L(\boldsymbol{v}, w, p) \quad \forall(\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c). \tag{10}$$

Analogously,

$$L(\boldsymbol{u}^*, 0, p^*) \leq L(\boldsymbol{v}, w, p^*) \quad \forall(\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c). \tag{11}$$

Taking $(\boldsymbol{v}, w) = (\boldsymbol{u}^*, 0)$ in (10) and $(\boldsymbol{v}, w) = (\boldsymbol{u}, 0)$ in (11) we obtain

$$\bar{J}(\boldsymbol{u}, 0) \leq \bar{J}(\boldsymbol{u}^*, 0) \leq \bar{J}(\boldsymbol{u}, 0). \tag{12}$$

Hence $\bar{J}(\boldsymbol{u}, 0) = \bar{J}(\boldsymbol{u}^*, 0)$. Using this equality it can be shown that $a(\boldsymbol{u}^* - \boldsymbol{u}, \boldsymbol{u}^* - \boldsymbol{u}) = 0$, i.e., $\boldsymbol{u}^* = \boldsymbol{u}$.

For arbitrary real number $\lambda > 0$ and $(\boldsymbol{h}, \theta) \in \mathbb{V} \times L_2(\Gamma_c)$ take $(\boldsymbol{v}, w) = (\boldsymbol{u} + \lambda \boldsymbol{h}, \lambda\theta)$ in (10), $(\boldsymbol{v}, w) = (\boldsymbol{u} - \lambda \boldsymbol{h}, -\lambda\theta)$ in (11). The sum of obtained inequalities can be expressed as

$$\int\limits_{\Gamma_c} \xi |u_\nu|\, ds \leq a(\lambda \boldsymbol{h}, \lambda \boldsymbol{h}) + \frac{1}{2} \int\limits_{\Gamma_c} \xi |u_\nu + \lambda h_\nu - \lambda\theta|\, ds + \\ + \frac{1}{2} \int\limits_{\Gamma_c} \xi |u_\nu - \lambda h_\nu + \lambda\theta|\, ds + \int\limits_{\Gamma_c} p\lambda\theta\, ds - \int\limits_{\Gamma_c} p^* \lambda\theta\, ds.$$

Since $\boldsymbol{h}$ is arbitrary, we can take such function that $h_\nu = \theta$ on Γ_c, then

$$0 \leq \lambda^2 a(\boldsymbol{h}, \boldsymbol{h}) + \int\limits_{\Gamma_c} (p - p^*)\lambda\theta\, ds \quad \forall \theta \in L_2(\Gamma_c). \tag{13}$$

Dividing (13) by λ and tending to zero, we have

$$\int\limits_{\Gamma_c} (p - p^*)\theta\, ds \geq 0 \quad \forall \theta \in L_2(\Gamma_c),$$

which implies that $p^* = p$. The theorem is proved. □

Remark 1. *Conditions under which the solution of the problem (6) has additional regularity were not studied and remained outside the scope of this article. Therefore, the theorem 1 has a conditional nature, and the* $\mathbf{H}^2$*-regularity assumption is sufficient to prove it.*

Introduce the modified Lagrange functional on the space $\mathbb{V} \times [L_2(\Gamma_c)]^2$

$$M(\boldsymbol{v}, w, l) = \bar{J}(\boldsymbol{v}, w) + \int\limits_{\Gamma_c} lw\, ds + \frac{r}{2} \int\limits_{\Gamma_c} w^2\, ds, \;\; r > 0 - const$$

and give the definition of a saddle point for $M(\boldsymbol{v}, w, l)$.

Definition 1. *The point* $(\boldsymbol{v}^*, w^*, l^*) \in \mathbb{V} \times [L_2(\Gamma_c)]^2$ *is called the saddle point of the modified Lagrange functional* $M(\boldsymbol{v}, w, l)$ *if for any* $(\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c)$ *and* $l \in L_2(\Gamma_c)$ *the two-sided inequality*

$$M(\boldsymbol{v}^*, w^*, l) \leq M(\boldsymbol{v}^*, w^*, l^*) \leq M(\boldsymbol{v}, w, l^*)$$

holds.

Define the dual functional

$$\underline{M}(l) = \inf_{(\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c)} M(\boldsymbol{v}, w, l) \;\forall l \in L_2(\Gamma_c).$$

Dual functional has the other representation

$$\underline{M}(l) = \inf_{m \in L_2(\Gamma_c)} \Big\{ \chi(m) + \int\limits_{\Gamma_c} lm\, ds + \frac{r}{2} \int\limits_{\Gamma_c} m^2\, ds \Big\},$$

where

$$\chi(m) = \begin{cases} \inf\limits_{(\boldsymbol{v},w)\in K_m} \bar{J}(\boldsymbol{v},w), \text{ if } \mathbb{K}_m = \{(\boldsymbol{v},w)\in \mathbb{V}\times L_2(\Gamma_c) : w = m\} \neq \varnothing, \\ +\infty, \text{ otherwise.} \end{cases}$$

Functional $\chi(m)$ is called sensitivity functional. It can be proved that $\chi(m)$ is a proper weakly lower semicontinuous functional in $L_2(\Gamma_c)$ [12].

For an arbitrary point $l \in L_2(\Gamma_c)$ let us denote the functional

$$F_l(m) = \chi(m) + \int\limits_{\Gamma_c} lm\,ds + \frac{r}{2}\int\limits_{\Gamma_c} m^2\,ds.$$

Obviously $F_l(m)$ is a strongly convex functional. It means that for any l there is a unique element $m(l) = \operatorname*{arg\,min}\limits_{m\in L_2(\Gamma_c)} F_l(m)$.

It can be shown that sets of saddle points for both the modified and classical Lagrange functionals coincide [18, 20]. Typically, the search for saddle points is conducted using classical Lagrange functional. Nonetheless, these methods occasionally do not converge due to the linear nature of classical functionals with respect to the dual variable [10]. Considering the coincidence of saddle points, we propose the following algorithm for finding a saddle point. Set the initial element $l^0 \in L_2(\Gamma_c)$. Next, for $k = 0, 1, \ldots$ we perform two steps:

(**i**) Knowing point l^0, we find

$$(\boldsymbol{v}^{k+1}, w^{k+1}) = \operatorname*{arg\,min}\limits_{(\boldsymbol{v},w)\in\mathbb{V}\times L_2(\Gamma_c)} M(\boldsymbol{v}, w, l^k). \tag{14}$$

(**ii**) Determine l^{k+1} by the formula

$$l^{k+1} = l^k + rw^{k+1}. \tag{15}$$

For the dual functional $\underline{M}(l)$ can be proved the next important statement, similar as in [15].

Theorem 2. *The dual functional $\underline{M}(l)$ is Gateaux differentiable in $L_2(\Gamma_c)$ and its derivative $\nabla\underline{M}(l)$ satisfies a Lipschitz condition with a constant $\frac{1}{r}$, that is*

$$\|\nabla\underline{M}(l_1) - \nabla\underline{M}(l_2)\|_{L_2(\Gamma_c)} \leq \frac{1}{r}\|l_1 - l_2\|_{L_2(\Gamma_c)} \quad \forall\, l_1, l_2 \in L_2(\Gamma_c).$$

In addition, the equality $\nabla\underline{M}(l) = m(l)$ is correct.

Proof. Since the sensitivity functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\Gamma_c)$, then the functional

$$F_l(m) = \chi(m) + \int\limits_{\Gamma_c} lm\,ds + \frac{r}{2}\int\limits_{\Gamma_c} m^2\,ds$$

is also a weakly lower semicontinuous on $L_2(\Gamma_c)$. From the convexity of $\chi(m)$ it follow that its epigraph $\operatorname{epi}\chi = \{(m,a) \in L_2(\Gamma_c)\times\mathbb{R} : \chi(m) \leq a\}$ is a closed convex set on $L_2(\Gamma_c)\times\mathbb{R}$, where $\mathbb{R}=(-\infty,+\infty)$. Then $\operatorname{epi}\chi$ can be strictly separated by a hyperplane from an arbitrary fixed point of the strict hypograph $\mathrm{hyp}_S=\{(m,a)\in L_2(\Gamma_c)\times\mathbb{R}:\chi(m)>a\}$. This means that there exist such $\Psi\in L_2(\Gamma_c)$, $\alpha\in\mathbb{R}$, for which the inequality holds

$$\int\limits_{\Gamma_c}\Psi m\,ds+\chi(m)+\alpha\geq 0\ \forall m\in L_2(\Gamma_c).$$

From here it follows that $F_l(m)\to\infty$ under $\|m\|_{L_2(\Gamma_c)}\to\infty$, which means $F_l(m)$ has the property of coercivity. From the coercivity and weak lower semicontinuity of the functional $F_l(m)$ it follows the existence of the element $m(l)=\underset{m\in L_2(\Gamma_c)}{\arg\min}\,F_l(m)$. Moreover, the element $m(l)$ is unique for any l due to the strong convexity of $F_l(m)$ in $L_2(\Gamma_c)$.

From the strong convexity of $F_l(m)$ it follows that for element $m(l)$ such inequality

$$\chi(m(l))+\int\limits_{\Gamma_c}lm(l)\,ds+\frac{r}{2}\int\limits_{\Gamma_c}m(l)^2\,ds+\frac{r}{2}\|m-m(l)\|^2_{L_2(\Gamma_c)}\leq$$
$$\leq\chi(m)+\int\limits_{\Gamma_c}lm\,ds+\frac{r}{2}\int\limits_{\Gamma_c}m^2\,ds\ \forall l\in L_2(\Gamma_c)$$

is correct.

Let us take two arbitrary elements $\hat{l}$, $\hat{\hat{l}}$ from $L_2(\Gamma_c)$. Denote $\hat{m}=m(\hat{l})$, $\hat{\hat{m}}=m(\hat{\hat{l}})$. We write the last inequality for $m=\hat{m}$, and then for $m=\hat{\hat{m}}$.

$$\begin{gathered}\chi(\hat{m})+\int\limits_{\Gamma_c}\hat{l}\hat{m}\,ds+\frac{r}{2}\int\limits_{\Gamma_c}\hat{m}^2\,ds+\frac{r}{2}\|\hat{\hat{m}}-\hat{m}\|^2_{L_2(\Gamma_c)}\leq\\ \leq\chi(\hat{\hat{m}})+\int\limits_{\Gamma_c}\hat{l}\hat{\hat{m}}\,ds+\frac{r}{2}\int\limits_{\Gamma_c}\hat{\hat{m}}^2\,ds,\\ \chi(\hat{\hat{m}})+\int\limits_{\Gamma_c}\hat{\hat{l}}\hat{\hat{m}}\,ds+\frac{r}{2}\int\limits_{\Gamma_c}\hat{\hat{m}}^2\,ds+\frac{r}{2}\|\hat{\hat{m}}-\hat{m}\|^2_{L_2(\Gamma_c)}\leq\\ \leq\chi(\hat{m})+\int\limits_{\Gamma_c}\hat{\hat{l}}\hat{m}\,ds+\frac{r}{2}\int\limits_{\Gamma_c}\hat{m}^2\,ds.\end{gathered}\tag{16}$$

Adding these two inequalities, we get

$$r\|\hat{m}-\hat{\hat{m}}\|^2_{L_2(\Gamma_c)}\leq\int\limits_{\Gamma_c}(\hat{l}-\hat{\hat{l}})(\hat{\hat{m}}-\hat{m})\,ds.\tag{17}$$

Finally, we use the Cauchy-Schwarz inequality in (17) and obtain

$$\|\hat{m}-\hat{\hat{m}}\|_{L_2(\Gamma_c)}\leq\frac{1}{r}\|\hat{l}-\hat{\hat{l}}\|_{L_2(\Gamma_c)}.\tag{18}$$

In addition, it follows from (16) that

$$\int\limits_{\Gamma_c} \hat{\hat{l}}(\hat{\hat{m}} - \hat{m})\, ds + \frac{r}{2} \int\limits_{\Gamma_c} (\hat{\hat{m}}^2 - \hat{m}^2)\, ds \le \chi(\hat{m}) - \chi(\hat{\hat{m}}) \le$$

$$\le \int\limits_{\Gamma_c} \hat{l}(\hat{\hat{m}} - \hat{m})\, ds + \frac{r}{2} \int\limits_{\Gamma_c} (\hat{\hat{m}}^2 - \hat{m}^2)\, ds.$$

Therefore, the limit equality $\lim\limits_{\hat{\hat{l}} \to \hat{l}} \chi(\hat{\hat{m}}) = \chi(\hat{m})$ holds. It means that the dual functional $\underline{M}(l)$ is continuous in $L_2(\Gamma_c)$. Hence, the subdifferential $\partial(-\underline{M}(l))$ of the convex functional $-\underline{M}(l)$ is not an empty set for any $l \in L_2(\Gamma_c)$. To prove the differentiability of $\underline{M}(l)$, it is enough to show that $\partial(-\underline{M}(l))$ consists of only one element [21]. This element will be the derivative of the dual functional $\underline{M}(l)$. Let the element $l \in L_2(\Gamma_c)$ be fixed and $-t \in \partial(-\underline{M}(l))$. Then for any $\zeta \in L_2(\Gamma_c)$ we have

$$\underline{M}(\zeta) \le \underline{M}(l) + \int\limits_{\Gamma_c} t(\zeta - l)\, ds. \tag{19}$$

From (19) and $m(l) = \mathop{\arg\min}\limits_{m \in L_2(\Gamma_c)} F_l(m)$, it follows that for real $\beta > 0$ such inequality is correct

$$\beta^{-1} \int\limits_{\Gamma_c} (m(\zeta) - t)(\zeta - l) \le 0.$$

Let us put $\zeta = l + \beta p$, where p is an arbitrary element from $L_2(\Gamma_c)$. Then the last inequality takes the form

$$\beta^{-1}\beta \int\limits_{\Gamma_c} (m(l + \beta p) - t)p \le 0.$$

Taking into account (18), under $\beta \to 0$ we have

$$\int\limits_{\Gamma_c} (m(l) - t)p \le 0 \quad \forall p \in L_2(\Gamma_c).$$

It means that $t = m(l)$. From the uniqueness of the element $m(l)$ for a fixed l it follows that $\nabla \underline{M}(l) = m(l)$. Together with (18), this ensures the Gateaux differentiability of the functional $\underline{M}(l)$ and the satisfaction of the Lipschitz condition for its derivative. Theorem has been proved. □

Let us consider the dual problem

$$\begin{cases} \underline{M}(l) \to \sup, \\ l \in L_2(\Gamma_c). \end{cases} \tag{20}$$

Since the derivative of the $\underline{M}(l)$ satisfies the Lipschitz condition, then to solve the problem (20) one can apply the well-known gradient method

$$l^{k+1} = l^k + r\nabla \underline{M}(l^k),\ k = 0, 1, \dots \tag{21}$$

Introduce the mapping $\mathbb{P}(l) = l + r\nabla \underline{M}(l)$, $\forall l \in L_2(\Gamma_c)$.

Theorem 3. *Let the set of optimal solutions*

$$\mathbb{Y} = \{y \in L_2(\Gamma_c) :\ \underline{M}(y) = \sup_{\kappa \in L_2(\Gamma_c)} \underline{M}(\kappa)\}$$

of the dual problem (20) be non-empty. Then $\mathbb{P}(l) = l$ *for any* $l \in \mathbb{Y}$ *and, moreover,*

$$\|\mathbb{P}(l) - \mathbb{P}(z)\|_{L_2(\Gamma_c)} < \|l - z\|_{L_2(\Gamma_c)}$$

for any $z \notin \mathbb{Y}$.

Proof. The functional $\underline{M}(l)$ is concave and differentiable. Therefore, the condition $l \in \mathbb{Y}$ is equivalent to the condition $\nabla \underline{M}(l) = 0$. Hence $\mathbb{P}(l) = l$ for any $l \in \mathbb{Y}$. Let $l \in \mathbb{Y}$, $z \in L_2(\Gamma_c) \setminus \mathbb{Y}$. From Theorem 2 it follows $\nabla \underline{M}(l) = m(l)$. Then

$$\|\mathbb{P}(l) - \mathbb{P}(z)\|^2_{L_2(\Gamma_c)} = \|l - z + r(m(l) - m(z))\|^2_{L_2(\Gamma_c)} =$$
$$= \|l - z\|^2_{L_2(\Gamma_c)} + 2r \int_{\Gamma_c} (l - z)(m(l) - m(z))\, ds + r^2 \|m(l) - m(z)\|^2_{L_2(\Gamma_c)}.$$

From here and inequality (17) we have

$$\|\mathbb{P}(l) - \mathbb{P}(z)\|^2_{L_2(\Gamma_c)} \le \|l - z\|^2_{L_2(\Gamma_c)} - 2r^2 \|m(l) - m(z)\|^2_{L_2(\Gamma_c)} +$$
$$+ r^2 \|m(l) - m(z)\|^2_{L_2(\Gamma_c)} = \|l - z\|^2_{L_2(\Gamma_c)} - r^2 \|m(l) - m(z)\|^2_{L_2(\Gamma_c)},$$

that is $\|\mathbb{P}(l) - \mathbb{P}(z)\|^2_{L_2(\Gamma_c)} < \|l - z\|^2_{L_2(\Gamma_c)}$. Theorem has been proved. □

We will further show that formula (15) coincides with the gradient method (21) for solving the dual problem (20). We have

$$\underline{M}(l) = \inf_{(\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c)} \Big\{ \bar{J}(\boldsymbol{v}, w) + \int_{\Gamma_c} l w\, ds + \frac{r}{2} \int_{\Gamma_c} w^2\, ds \Big\} =$$
$$= \inf_{m \in L_2(\Gamma_c)} \Big\{ \chi(m) + \int_{\Gamma_c} l m\, ds + \frac{r}{2} \int_{\Gamma_c} m^2\, ds \Big\} = \bar{J}(\bar{\boldsymbol{v}}, \bar{w}) + \int_{\Gamma_c} l \bar{m}\, ds + \frac{r}{2} \int_{\Gamma_c} \bar{m}^2\, ds,$$

where $(\bar{\boldsymbol{v}}, \bar{w}) = \operatorname*{arg\,min}_{(\boldsymbol{v}, w) \in \mathbb{V} \times L_2(\Gamma_c)} \bar{J}(\boldsymbol{v}, w)$, $\bar{m} = \bar{w}$.

Therefore, the formula of the gradient method (21) can be rewritten as follows

$$l^{k+1} = l^k + r\nabla \underline{M}(l^k) = l^k + r m(l^k) = l^k + r\bar{m}^{k+1} = l^k + r\bar{w}^{k+1}.$$

This shows that formulas (15) and (21) are equivalent.

Theorem 4. *Let the set $\mathbb{Y}$ be non-empty. Then for sequence $\{l^k\}$, where $l^k = l^{k-1} + r\nabla\underline{M}(l^{k-1})$, $k = 1, 2, \dots$, the limit equality*

$$\lim_{k\to\infty} \underline{M}(l^k) = \max_{l\in L_2(\Gamma_c)} \underline{M}(l)$$

is fulfilled for an arbitrary starting $l^0 \in L_2(\Gamma_c)$.

Proof. Let $l^* \in \mathbb{Y}$. From theorem 3 it follows

$$\|l^{k+1} - l^*\|_{L_2(\Gamma_c)} < \|l^k - l^*\|_{L_2(\Gamma_c)}, \; k = 0, 1, \dots$$

Hence, non-negative number sequence $\{\|l^k - l^*\|_{L_2(\Gamma_c)}\}$ is decreasing and bounded from below, so it has a limit. Moreover, it is known that $\lim\limits_{k\to\infty} m(l^k) = 0$ [12, 19].

Then

$$\underline{\lim_{k\to\infty}}\, \underline{M}(l^k) = \underline{\lim_{k\to\infty}} \inf_{m\in L_2(\Gamma_c)} \Big\{\chi(m) + \int\limits_{\Gamma_c} l^k m\, ds + \frac{r}{2}\int\limits_{\Gamma_c} m^2\, ds\Big\} =$$

$$= \underline{\lim_{k\to\infty}} \Big\{\chi(m(l^k)) + \int\limits_{\Gamma_c} l^k m(l^k)\, ds + \frac{r}{2}\int\limits_{\Gamma_c} m(l^k)^2\, ds\Big\} = \underline{\lim_{k\to\infty}}\, \chi(m(l^k)) \geq \chi(0).$$

On the other hand, following the definition of the functional $\underline{M}(l)$, we have

$$\underline{M}(l^k) = \chi(m(l^k)) + \int\limits_{\Gamma_c} l^k m(l^k)\, ds + \frac{r}{2}\int\limits_{\Gamma_c} m(l^k)^2\, ds =$$

$$= \inf_{m\in L_2(\Gamma_c)} \Big\{\chi(m) + \int\limits_{\Gamma_c} l^k m\, ds + \frac{r}{2}\int\limits_{\Gamma_c} m^2\, ds\Big\} \leq \chi(0).$$

Then

$$\overline{\lim_{k\to\infty}} \Big\{\chi(m(l^k)) + \int\limits_{\Gamma_c} l^k m(l^k)\, ds + \frac{r}{2}\int\limits_{\Gamma_c} m(l^k)^2\, ds\Big\} \leq \chi(0).$$

Therefore $\lim\limits_{k\to\infty} \underline{M}(l^k)$ exists and

$$\lim_{k\to\infty} \underline{M}(l^k) = \chi(0) = \max_{l\in L_2(\Gamma_c)} \underline{M}(l).$$

Theorem has been proved. □

Theorem 5. *Let the set of saddle points of the modified Lagrange functional $M(\boldsymbol{v}, w, l)$ be non-empty, and let the set of points $(\boldsymbol{v}^{k+1}, w^{k+1}, l^k)$ belong to the space $\mathbf{H}^2(\Omega) \times [H^{1/2}(\Gamma_c)]^2$ and be bounded in it. Then any limit point of the sequence $\{(\boldsymbol{v}^{k+1}, w^{k+1}, l^k)\}$, generated by the Uzawa method (14), (15), is a saddle point of $M(\boldsymbol{v}, w, l)$.*

Proof. From the conditions of the theorem it follows that the sequence $\{(\boldsymbol{v}^{k+1}, w^{k+1}, l^k)\}$ is compact in $\mathbf{H}^1(\Omega) \times [H^{1/2}(\Gamma_c)]^2$. It means, that the sequence $\{l^k\}$ has at least one limit point $\bar{l}$ in $L_2(\Gamma_c)$. Let $\bar{l} = \lim\limits_{i\to\infty} l^{k_i}$ and $l^* \in \mathbb{Y}$ is arbitrary. From the continuity of the operator $\mathbb{P}$ the validity of the relations follows

$$\|\mathbb{P}(\bar{l}) - \mathbb{P}(l^*)\|_{L_2(\Gamma_c)} = \lim_{i\to\infty} \|\mathbb{P}(l^{k_i}) - \mathbb{P}(l^*)\|_{L_2(\Gamma_c)} = \\ = \lim_{i\to\infty} \|l^{k_i+1} - l^*\|_{L_2(\Gamma_c)} = \|\bar{l} - l^*\|_{L_2(\Gamma_c)}.$$

The last equality in the chain of equalities is a consequence of the fact that the sequence $\{\|l^k - l^*\|_{L_2(\Gamma_c)}\}$ is convergent. From theorem 3 now it follows $\bar{l} \in \mathbb{Y}$. So we have $\lim\limits_{i\to\infty} \|l^{k_i} - \bar{l}\|_{L_2(\Gamma_c)} = 0$. Besides, $\|l^j - \bar{l}\|_{L_2(\Gamma_c)} \le \|l^{k_i} - \bar{l}\|_{L_2(\Gamma_c)}$ for all $j \ge k_i$, $i = 1, 2, \ldots$ By tending i to infinity, while maintaining the inequality $j \ge k_i$, we obtain $\lim\limits_{j\to\infty} l^j = \bar{l}$. So $\{l^j\}$ is a sequence converging to the optimal solution of the dual problem (20).

Now consider the sequence $\{(\boldsymbol{v}^{k+1}, w^{k+1})\}$. It is compact in $\mathbb{V} \times L_2(\Gamma_c)$. Therefore it has at least one limit point $(\overline{\boldsymbol{v}}, \overline{w})$. Without loss of generality, we can assume that $\{(\boldsymbol{v}^{k+1}, w^{k+1})\}$ is a strongly convergent sequence and

$$\lim_{k\to\infty} (\boldsymbol{v}^{k+1}, w^{k+1}) = (\overline{\boldsymbol{v}}, \overline{w})$$

in space $\mathbb{V} \times L_2(\Gamma_c)$. Since $\bar{J}(\boldsymbol{v}^k, w^k) = \chi(m(l^{k-1}))$, then

$$\bar{J}(\bar{\boldsymbol{v}}, \bar{w}) = \lim_{k\to\infty} \bar{J}(\boldsymbol{v}^{k+1}, w^{k+1}) = \lim_{k\to\infty} \chi(m(l^k)) = \chi(0).$$

Thus, $(\overline{\boldsymbol{v}}, \overline{w}, \bar{l}) = (\overline{\boldsymbol{v}}, 0, \bar{l})$ is a saddle point of the modified Lagrange functional $M(\boldsymbol{v}, w, l)$. □

The minimization problem with respect to $(\boldsymbol{v}, w)$ can be rewritten as follows [15]

$$\inf_{(\boldsymbol{v},w)\in\mathbb{V}\times L_2(\Gamma_c)} M(\boldsymbol{v}, w, l) = \\ = \inf_{(\boldsymbol{v},w)\in\mathbb{V}\times L_2(\Gamma_c)} \left\{ \mathit{\Pi}(\boldsymbol{v}) + \frac{1}{2}\int\limits_{\Gamma_c} \xi v_\nu \, ds + \frac{1}{2}\int\limits_{\Gamma_c} \xi |v_\nu - w| \, ds + \int\limits_{\Gamma_c} \left(lw + \frac{r}{2} w^2 \right) ds \right\} \\ = \inf_{\boldsymbol{v}\in\mathbb{V}} \left\{ \mathit{\Pi}(\boldsymbol{v}) + \frac{1}{2}\int\limits_{\Gamma_c} \xi v_\nu \, ds + \int\limits_{\Gamma_c} \inf_{w\in L_2(\Gamma_c)} \left(\frac{1}{2}\xi |v_\nu - w| + lw + \frac{r}{2} w^2 \right) ds \right\}.$$

Let us define the last expression under the integral as a function of v_ν

$$\Xi(v_\nu) = \inf_{w\in L_2(\Gamma_c)} \left(\frac{1}{2}\xi |v_\nu - w| + lw + \frac{r}{2} w^2 \right).$$

It has an explicit representation in the form

$$\Xi(v_\nu) = \begin{cases} -0.5\xi v_\nu - \dfrac{(0.5\xi + l)^2}{2r}, & v_\nu < w = -\dfrac{0.5\xi + l}{r}, \\ l v_\nu + \dfrac{r}{2} v_\nu^2, & -\dfrac{0.5\xi + l}{r} \leq v_\nu = w \leq \dfrac{0.5\xi - l}{r}, \\ 0.5\xi v_\nu - \dfrac{(0.5\xi - l)^2}{2r}, & v_\nu > w = \dfrac{0.5\xi - l}{r}. \end{cases}$$

It is possible to show that $\Xi(v_\nu)$ is a continuously differentiable convex function and its derivative at the points $-(0.5\xi + l)/r$, $(0.5\xi - l)/r$ is equal to -0.5ξ, 0.5ξ respectively.

This gives us the problem of minimizing the continuously differentiable functional $\tilde{M}(\boldsymbol{v}, l)$ on the space $\mathbb{V}$ in (14)

$$\inf_{\boldsymbol{v} \in \mathbb{V}} \tilde{M}(\boldsymbol{v}, l) = \inf_{\boldsymbol{v} \in \mathbb{V}} \Big\{ \Pi(\boldsymbol{v}) + \frac{1}{2} \int\limits_{\Gamma_c} \xi v_\nu \, ds + \int\limits_{\Gamma_c} \Xi(v_\nu) \, ds \Big\}.$$

The considered approach allows us to replace the problem of minimizing nondifferentiable functional (7) with a smooth problem of finding saddle point of the modified Lagrange functional. This fact makes it possible to use effective generalized Newton methods for its numerical solution.

4 Numerical experiments

This section is devoted to the numerical solution of the plane strain problem based on the finite element method.

Example 1. The body $\Omega = (0, 5) \times (0, 1)$ (in m) is made of an elastic isotropic, homogeneous material characterized by Young's modulus $\mathrm{E} = 2000$ $\mathrm{N} \cdot \mathrm{m}^{-2}$ and Poisson's ratio $\mu = 0.3$. It is fixed along $\Gamma_d = (\{0\} \times [0, 1]) \cup (\{5\} \times [0, 1])$ and loaded by surface tractions of density $\boldsymbol{p} = (0, -20 \sin(\pi x_1/5))$ $\mathrm{N} \cdot \mathrm{m}^{-1}$ on $\Gamma_n = (0, 5) \times \{1\}$. Part $\Gamma_c = (0, 5) \times \{0\}$ represents the potential contact region. The volume forces will be neglected, i.e. $\boldsymbol{f} = 0$ in Ω.

We construct a regular mesh with step size $h = \frac{1}{N}$ by dividing body Ω into $5N \times N$ 4-node quadrilateral finite elements (Q1). The mesh size h varies to obtain problems of different sizes.

To solve the discrete problem of minimizing a piecewise quadratic functional, we use the generalized Newton method [15, 22, 23] and choose the following stopping criterion for it

$$\frac{\|\dot{\boldsymbol{u}}_h^{i+1} - \dot{\boldsymbol{u}}_h^{i}\|_2}{\|\dot{\boldsymbol{u}}_h^{k+1}\|_2} < 10^{-10}, \ i = 0, 1, \ldots,$$

where $\dot{\boldsymbol{u}}_h$ is vector of displacement values at the mesh nodes and $\|\cdot\|_2$ denotes the Euclidean norm. The parameter r is taken equal to 10^8, which ensures fast convergence of the gradient method in the Uzawa algorithm. We choose

$$\frac{\|\tilde{l}_h^{k+1} - \tilde{l}_h^{k}\|_2}{\|\tilde{l}_h^{k+1}\|_2} < 10^{-8}$$

as stopping criterion for it. Here $\tilde{l}_h$ is the vector whose components are the values of dual variable at the contact nodes.

All experiments are implemented in Python, using the scikit-fem library ver. 10.0.0 [24] for performing finite element assembly and CuPy library [25] for GPU-accelerated computing. Computation was carried out on IBM Power System AC922 (8335-GTH) server with NVIDIA Tesla V100 GPUs.

Table 1 presents results for the different mesh sizes with $\xi = 2\ \mathrm{N} \cdot \mathrm{m}^{-1}$, where n_p, n_d are the numbers of primal (displacements) and dual (stresses) variables respectively.

Таблица 1. The number of Uzawa iterations for different meshes.

h	n_p	n_d	Uzawa it	$J(\boldsymbol{u}_h)$, $\mathrm{N} \cdot \mathrm{m}$
1/8	738	41	3	-3.906382
1/16	2754	81	3	-3.954664
1/32	10626	161	3	-3.969465
1/64	41730	321	3	-3.974141
1/128	165378	641	3	-3.975677
1/256	658434	1281	3	-3.976198
1/512	2627586	2561	3	-3.976381

It can be seen that the number of iterations of the Uzawa method does not depend on h and the values of the energy functional $J(\boldsymbol{u}_h)$ stabilize with decreasing h.

Fig. 2 shows the dependence of the relative error on the mesh size h. We use the numerical solution $\boldsymbol{u}^*$ corresponding to $h = 1/512$ as the reference solution for computing the solution errors, since the exact solution $\boldsymbol{u}$ is not available. The results highlight the linear asymptotic convergence of the numerical solutions, where the energy norm has the form $\|\boldsymbol{v}\|_E = \sqrt{a(\boldsymbol{v}, \boldsymbol{v})}$.

The number of Uzawa iterations with respect to ξ is shown in Table 2. It can be seen that with an increase in ξ, the number of iterations slightly increases. Calculations show that increasing the yield limit makes the foundation more rigid and for a sufficiently large ξ, it starts to behave like a rigid foundation. In Figs. 3a, 3b we can observe that the foundation is deformed when the normal stress reaches the yield limit.

Таблица 2. Dependence of the number of Uzawa iterations on ξ with $h = 1/128$.

ξ	4	8	12	16	18	20
Uzawa it	3	3	3	4	4	4

The resulting deformations of the body and the von Mises stresses are shown in Fig. 4.

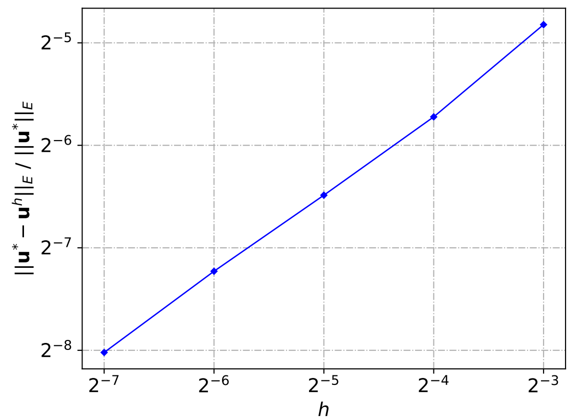


Рис. 2. Relative errors with respect to the mesh size h using $\log_2$ scales.

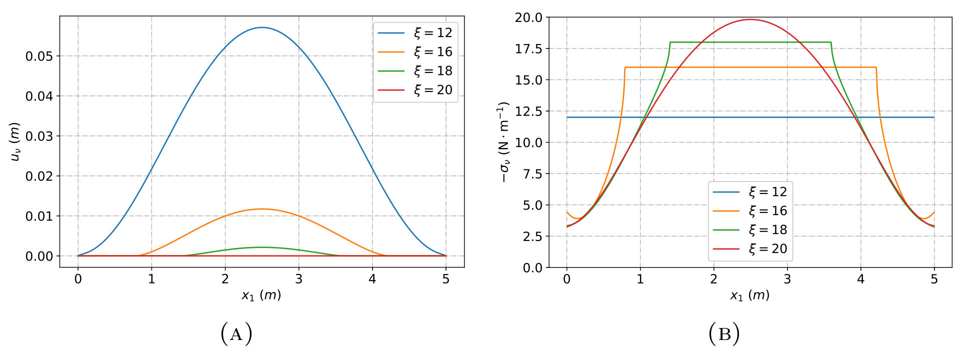


Рис. 3. Normal displacements (A) and stresses (B) on Γ_c for different ξ.

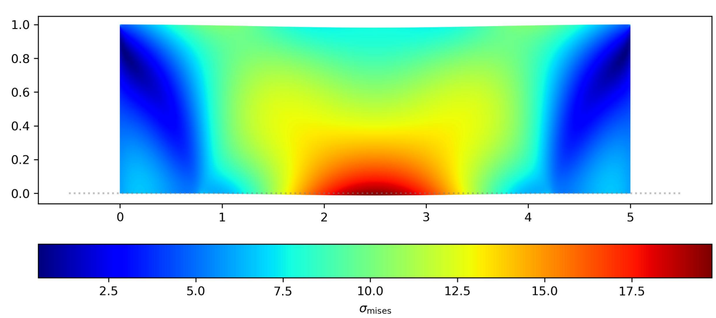


Рис. 4. Deformation and Von mises stress with $\xi = 16$.

Example 2. The problem consists of an infinitely long elastic cylinder with radius R=8 resting on rigid-plastic foundation

$S = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \leq 0\}$, where d is initial gap between them. The body has the same material parameters as in the first example. Taking into account the symmetry of the problem we define

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 | -8 \leq x_1 \leq 8,\ R - \sqrt{R^2 - x_1^2} \leq x_2 \leq 8\right\}.$$

We impose a downward vertical displacement of 0.1 on the top of the half cylinder $\Gamma_d = (-8, 8) \times \{8\}$ and neglect surface forces $\boldsymbol{p}$. Contact part is

$$\Gamma_c = \left\{(x_1, x_2) \in \mathbb{R}^2 | -2 \leq x_1 \leq 2,\ x_2 = d,\ d = R - \sqrt{R^2 - x_1^2}\right\}.$$

The remaining part of the boundary Γ_n is stress free. Domain is discretized using nonuniform mesh consisting of 30578 quadratic triangular elements and 15599 nodes as shown in Fig. 5. Number of nodes on Γ_c is 407, parameter r is taken 10^8.

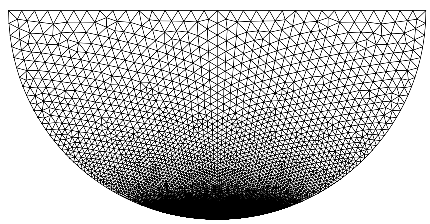
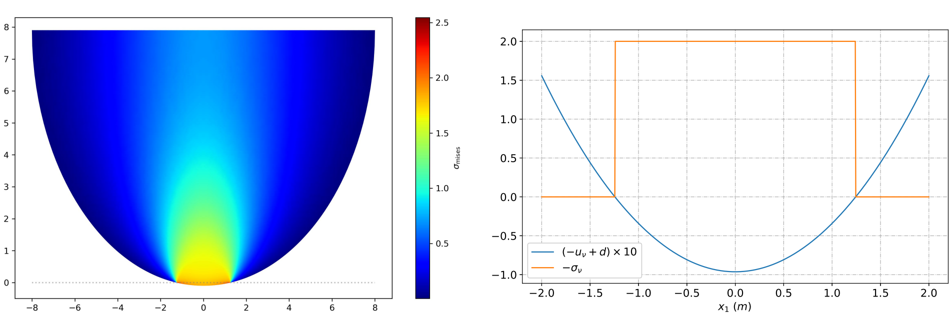


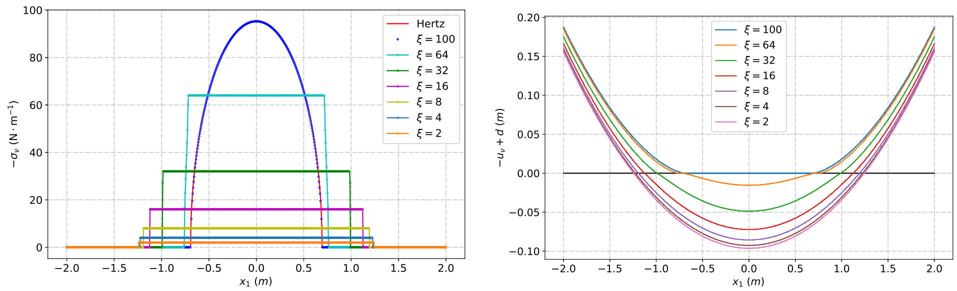
Рис. 5. The mesh of the elastic body.

The resulting deformations of the body and von Mises stresses are depicted in Fig. 6a. In Fig. 6b the normal position of the body on Γ_c with an amplification factor 10 is plotted against the value of normal stress. We can observe that the foundation is deformed when the stress reaches the yield limit $\xi = 2$. In this example, it took only 2 iterations for the Uzawa algorithm to converge.



(A) (B)

Рис. 6. Deformed configuration and von Mises stress (A), normal stress, displacements for $\xi = 2$ (B) on Γ_c.



(A) (B)

Рис. 7. Normal stress (A) and displacements (B) for different ξ on Γ_c.

At each internal step, 5 iterations of the generalized Newton method were performed. Numerical calculations correspond to the boundary conditions (4).

Figs. 7a and 7b show the normal stress and displacements for different ξ on Γ_c. We can see that with an increase of ξ the penetration of elastic body into foundation decreases and for $\xi = 100$ there is no penetration since $-\sigma_\nu < \xi$. With further increase of ξ foundation behaves as a rigid one and in the limiting case we get Hertz contact problem which has explicit analytic solution. In Fig. 7a we can observe a convergence of the calculated normal stresses to the Hertz analytical solution with increase of ξ.

Conclusion

In this paper, a variational method for solving the frictionless contact problem between an elastic body and a rigid-plastic foundation is presented. The method is based on finding the saddle point of the modified Lagrange functional. We derive the form of the saddle point and prove the convergence of the Uzawa method to it. Furthermore, we show that the considered approach smooths the minimized functional of the problem. After finite element approximation of the problem, Uzawa's algorithm reduces to an iterative process in which a continuously differentiable piecewise quadratic function is minimized at each step using the generalized Newton method. Finally, numerical experiments are conducted to demonstrate the performance of the algorithm.

A continuation of our work would involve investigating the effect of introducing the assumption of friction in the contact zone. Moreover, it would be interesting to consider models with nonlinear stress-strain dependence.

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