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EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE INITIAL BOUNDARY VALUE PROBLEM FOR ONE-DIMENSIONAL ISOTHERMAL EQUATIONS OF COMPRESSIBLE VISCOUS MULTICOMPONENT MEDIA DYNAMICS

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Abstract: An initial boundary value problem is considered for one-dimensional equations of the dynamics of compressible multicomponent media. A global theorem of existence and uniqueness of a strong solution is proved without restrictions on the structure of the viscosity matrix except for the standard properties of symmetry and positivity.

Keywords: compressible viscous medium, multicomponent flows, viscosity matrix, boundary value problem, existence and uniqueness theorem.

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1 Introduction

Modelling the motion of multi-component media and solving the mathematical problems arising is of great interest to both physics and mathematics. However it is little-studied, and no unified approach to this field has been developed so far, and no mathematical theory has been constructed concerning the existence, uniqueness and properties of solutions of the initial-boundary value problems that arise in the process of modelling. A detailed survey of this problem area can be obtained from the monographs [1, 2] and the papers [3, 4]. In this paper we choose to study one of the numerous versions of modelling the motion of multi-component fluid mixtures, namely, a homogeneous mixture of viscous compressible fluids with multiple velocities. This means that all the components (constituents) of the mixture are present in the same phase at every point of space, but each of them has its own local velocity of motion; the components interact via momentum exchange and viscous friction (and also heat exchange in heatconducting models). The model under consideration is a generalization of the well-known Navier—Stokes system of equations describing the motions of compressible viscous single-component media and includes the continuity and momentum equations, and also the energy equation(s) in heatconducting models. The characteristic feature of these equations, in addition to their nonlinearity, is the presence of higher order derivatives of the velocities of all components in the momentum (and energy) conservation laws, due to the composite structure of the viscous stress tensors [1, 2, 5, 6, 7, 8, 9]. This specificity of multicomponent flows can be described using the concept of viscosity matrix. Unlike the single-component case in which the viscosity is scalar-valued, in the multicomponent case the viscosities form a matrix whose entries describe viscous friction. Diagonal entries describe viscous friction within each component, and non-diagonal entries describe friction between the components. In the case of a diagonal viscosity matrix, the momentum equations are possibly connected via the lower order terms only. In the paper the more complicated case of off-diagonal viscosity matrices is under consideration. The aim of the paper is to analyze the existence and uniqueness of solution to an initial-boundary value problem for isothermal (non-heatconducting) equations of compressible viscous multicomponent media in the case of one-dimensional motions in a bounded domain with non permeable boundaries.

As mentioned above, the multi-velocity model of compressible viscous multicomponent media dynamics under consideration is a generalization of the well-known Navier—Stokes system of equations and hence the mathematical results for multicomponent media appeared after the progress achieved in the Navier—Stokes theory, for which an invaluable contribution has been made by one-dimensional results [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The unique solvability of the considered one-dimensional equations of compressible viscous multicomponent media in the polytropic case is investigated in [36, 37]. Similar issues for related models of multicomponent media are discussed in [38, 39, 40, 41, 42, 43, 43, 44, 45, 46, 47, 48, 49, 50]. Spectral analysis of some linear models of compressible viscous multicomponent media is carried out in [51, 52].

The structure of the paper is as follows. Section 2 contains the statement of the initial-boundary value problem and the formulation of the main result which is Theorem 1 concerning the existence and uniqueness of the solution. In Section 3, we study the solvability of an approximate initial-boundary value problem which is obtained from the original one via Galerkin method. In Section 4, the solutions to the approximate problem are estimated uniformly in the approximation parameter. Basing on these estimates, in Section 5 the limit transition is made and the local-in-time existence of a solution to the original initial-boundary value problem is verified. In order to continue the local solution, in Section 6 we prove a priori estimates, in which the constants do not depend on the local existence interval. In Section 7, the uniqueness of the solution to the initial-boundary value problem is proved.

2 Statement of the initial-boundary value problem and formulation of the existence and uniqueness theorem

We consider the initial boundary value problem for one-dimensional isothermal equations of the dynamics of compressible viscous multicomponent media. In the closure $\overline{Q}_T$ of a domain $Q_T = (0,T)\times(0,1)$ $(T>0)$ the sought values are the densities $\rho_i(t,x)>0$ and the velocities $u_i(t,x)$ of each component with the number $i=1,\dots,N$ $(N\in\mathbb{N},\ N\geqslant 2)$, which satisfy the following system of differential equations, initial and boundary conditions:

$$\frac{\partial\rho_i}{\partial t}+\frac{\partial(\rho_i v)}{\partial x}=0,\quad i=1,\dots,N, \tag{1}$$

$$\rho_i\left(\frac{\partial u_i}{\partial t}+v\frac{\partial u_i}{\partial x}\right)+\alpha_i K\frac{\partial\rho}{\partial x}=\sum_{j=1}^{N}\nu_{ij}\frac{\partial^2 u_j}{\partial x^2},\quad i=1,\dots,N, \tag{2}$$

$$\rho_i|_{t=0}=\rho_{0i}(x),\quad u_i|_{t=0}=u_{0i}(x),\quad i=1,\dots,N, \tag{3}$$

$$u_i|_{x=0}=u_i|_{x=1}=0,\quad i=1,\dots,N. \tag{4}$$

Here v is the average velocity, $v=\sum_{j=1}^{N}\alpha_j u_j$, $\alpha_j=\mathrm{const}\in(0,1)$, $\sum_{j=1}^{N}\alpha_j=1$, ρ is the total density, $\rho=\sum_{j=1}^{N}\rho_j$, constant viscosity coefficients ν_{ij} form the matrix $\mathbf{N}>0$, the coefficient $K>0$, the initial data $\rho_{0i}(x)$, $u_{0i}(x)$, $i=1,\dots,N$ are given.

The given model corresponds to the so-called diffusion approximation appropriate at description of motions in which relative velocities of components are small in comparison with the general (average) velocity of the flow (mixture).

Definition 1. *By a strong solution to problem (1)–(4) we mean $2N$ functions $(\rho_1, \ldots, \rho_N, u_1, \ldots, u_N)$ such that for all $i = 1, \ldots, N$*

$$
\begin{aligned}
&\rho_i > 0, \quad \rho_i \in L_\infty\big(0, T; W_2^1(0,1)\big), \quad \frac{\partial \rho_i}{\partial t} \in L_\infty\big(0, T; L_2(0,1)\big), \\
&u_i \in L_\infty\big(0, T; W_2^1(0,1)\big) \bigcap L_2\big(0, T; W_2^2(0,1)\big), \quad \frac{\partial u_i}{\partial t} \in L_2(Q_T),
\end{aligned} \tag{5}
$$

equations (1), (2) are satisfied almost everywhere in Q_T, the initial conditions (3) are valid for almost all $x \in (0,1)$, and the boundary conditions (4) are accepted for almost all $t \in (0,T)$.

Theorem 1. *Let the initial data in (3) satisfy the conditions*

$$
\rho_{0i} > 0, \quad \rho_{0i} \in W_2^1(0,1), \quad u_{0i} \in \overset{\circ}{W}{}_2^1(0,1), \quad i = 1, \ldots, N. \tag{6}
$$

Then there exists a unique strong solution to problem (1)–(4) in the sense of Definition 1.

Proof of Theorem 1 is given in Sections 3–7.

3 Construction of Galerkin approximations

Let us first prove the solvability of an approximate initial-boundary value problem, obtained from problem (1)–(4) by applying the Galerkin method (in the spatial variable x) to equations (2).

Theorem 2. *On the assumptions of Theorem 1 for all $m \in \mathbb{N}$ there is a time-interval $(0, t^m) \subset (0, T)$, where there exists a solution to the problem*

$$
\frac{\partial \rho_i^m}{\partial t} + \frac{\partial(\rho_i^m v^m)}{\partial x} = 0, \quad i = 1, \ldots, N, \tag{7}
$$

$$
\int_0^1 \left(\rho_i^m \frac{\partial u_i^m}{\partial t} + \rho_i^m v^m \frac{\partial u_i^m}{\partial x} + \alpha_i K \frac{\partial \rho^m}{\partial x} - \sum_{j=1}^N \nu_{ij} \frac{\partial^2 u_j^m}{\partial x^2} \right) \sin(\pi k x)\, dx = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, m, \tag{8}
$$

$$
\rho_i^m|_{t=0} = \rho_{0i}(x), \quad i = 1, \ldots, N, \tag{9}
$$

$$
u_i^m = \sum_{s=1}^m \xi_{is}^m(t) \sin(\pi s x), \quad u_i^m|_{t=0} = \sum_{s=1}^m \xi_{0is}^m \sin(\pi s x), \quad i = 1, \ldots, N, \tag{10}
$$

$$
\xi_{0is}^m := \xi_{is}^m(0) = 2 \int_0^1 u_{0i}(x) \sin(\pi s x)\, dx, \quad i = 1, \ldots, N, \quad s = 1, \ldots, m,
$$

where $v^m = \sum_{j=1}^N \alpha_j u_j^m$, $\rho^m = \sum_{j=1}^N \rho_j^m$, and we have

$$
\begin{aligned}
&\rho_i^m > 0, \quad \rho_i^m \in L_\infty\big(0, t^m; W_2^1(0,1)\big) \bigcap W_\infty^1\big(0, t^m; L_2(0,1)\big), \\
&\xi_{is}^m \in C^1[0, t^m], \quad i = 1, \ldots, N, \quad s = 1, \ldots, m.
\end{aligned} \tag{11}
$$

Proof. We fix $t^m < T$. We omit the upper index m in the notations of the solutions up to the beginning of Section 5. Consider the set

$$V = \left\{ \boldsymbol{\xi} \in (C[0,t^m])^{mN} \middle| \ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \ \ \|\boldsymbol{\xi}\|_{(C[0,t^m])^{mN}} \leqslant c \right\},$$

where

$$\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N), \quad \boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{im}), \quad i = 1, \ldots, N,$$

$$\boldsymbol{\xi}_0 = (\boldsymbol{\xi}_{01}, \ldots, \boldsymbol{\xi}_{0N}), \quad \boldsymbol{\xi}_{0i} = (\xi_{0i1}, \ldots, \xi_{0im}), \quad i = 1, \ldots, N,$$

$$c^2 = e \frac{\max\limits_{1\leqslant i\leqslant N} \sup\limits_{[0,1]} \rho_{0i}}{\min\limits_{1\leqslant i\leqslant N} \inf\limits_{[0,1]} \rho_{0i}} \|\boldsymbol{\xi}_0\|^2_{\mathbb{R}^{mN}} + 1.$$

We construct the operator $A : V \to (C[0,t^m])^{mN}$, $\operatorname{Im} A \subset (C^1[0,t^m])^{mN}$, $A(\boldsymbol{\xi}) = \boldsymbol{\psi}$, where $\boldsymbol{\psi} = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_N)$, $\boldsymbol{\psi}_i = (\psi_{i1}, \ldots, \psi_{im})$, $i = 1, \ldots, N$, by the following algorithm. First, we find the functions

$$\rho_i > 0, \quad \rho_i \in L_\infty\big(0,t^m; W_2^1(0,1)\big) \bigcap W_\infty^1\big(0,t^m; L_2(0,1)\big), \quad i = 1, \ldots, N$$

as solutions to the Cauchy problems (7), (9), where $v = \sum\limits_{j=1}^N \alpha_j u_j$, and u_j, $j = 1, \ldots, N$ are given by (10) (see [53]). Moreover, the inequalities

$$\left(\inf_{[0,1]} \rho_{0i}\right) e^{-\sum\limits_{j=1}^N \int\limits_0^t \sup\limits_{[0,1]} \left|\frac{\partial u_j}{\partial x}\right| d\tau} \leqslant \rho_i(t,x) \leqslant \left(\sup_{[0,1]} \rho_{0i}\right) e^{\sum\limits_{j=1}^N \int\limits_0^t \sup\limits_{[0,1]} \left|\frac{\partial u_j}{\partial x}\right| d\tau}, \quad i = 1, \ldots, N \tag{12}$$

hold, which, in view of the inclusion $\boldsymbol{\xi} \in V$, give the estimates

$$\left(\inf_{[0,1]} \rho_{0i}\right) e^{-\pi m^2 cNt} \leqslant \rho_i(t,x) \leqslant \left(\sup_{[0,1]} \rho_{0i}\right) e^{\pi m^2 cNt}, \quad i = 1, \ldots, N. \tag{13}$$

Next, we find $\boldsymbol{\psi}$ from the Cauchy problem for the system of mN ordinary first order differential (linear) equations:

$$\int\limits_0^1 \left(\rho_i \frac{\partial U_i}{\partial t} + \rho_i v \frac{\partial U_i}{\partial x} + \alpha_i K \frac{\partial \rho}{\partial x} - \sum_{j=1}^N \nu_{ij} \frac{\partial^2 U_j}{\partial x^2}\right) \sin(\pi k x)\, dx = 0, \quad i = 1, \ldots, N, \quad k = 1, \ldots, m, \tag{14}$$

$$\boldsymbol{\psi}(0) = \boldsymbol{\xi}_0, \tag{15}$$

where $U_i = \sum\limits_{s=1}^m \psi_{is}(t) \sin(\pi s x)$, $i = 1, \ldots, N$. The inequality

$$\det M(t) \neq 0,$$

where

$$M(t) = \begin{pmatrix} M_1(t) & 0 & \dots & 0 \\ 0 & M_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_N(t) \end{pmatrix},$$

$$M_i(t) = \left\{ \int\limits_0^1 \rho_i(t,x) \sin(\pi k x) \sin(\pi s x)\, dx \right\}_{k,s=1}^{m}, \quad i = 1, \dots, N,$$

which is valid due to the positivity of ρ_i, $i = 1, \dots, N$, admits to solve system (14) with respect to derivatives, which justifies the existence of $\boldsymbol{\psi} \in (C^1[0,t^m])^{mN}$. Thus, for arbitrary $t^m \in (0,T]$ we can define the operator $A : V \to (C^1[0,t^m])^{mN} \subset (C[0,t^m])^{mN}$, $A(\boldsymbol{\xi}) = \boldsymbol{\psi}$, whose fixed point (if it exists), together with the corresponding functions ρ_i, u_i, $i = 1, \dots, N$, is a solution to problem (7)–(10).

We show that for sufficiently small t^m the operator A satisfies the assumptions of the Schauder theorem (see [21, P. 31]), i. e.

(1) V is a convex closed bounded set (which is obvious in the case under consideration);
(2) $A :\ V \to V$;
(3) A is a completely continuous operator.

We first show that $A(V) \subset V$. Hereinafter, $C_i(\cdot)$, $i \in \mathbb{N}$, denote quantities which accept positive finite values depending on arguments indicated in the brackets or in comments. Multiplying equations (14) by $\psi_{ik}(t)$, summarizing with respect to i, k and integrating with respect to x, we obtain due to (7), that

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{N} \int\limits_0^1 \rho_i U_i^2\, dx \right) + \sum_{i,j=1}^{N} \nu_{ij} \int\limits_0^1 \left(\frac{\partial U_i}{\partial x}\right)\left(\frac{\partial U_j}{\partial x}\right) dx =$$

$$= K \sum_{i=1}^{N} \alpha_i \int\limits_0^1 \rho \left(\frac{\partial U_i}{\partial x}\right) dx,$$

and taking into account the inequalities (to obtain them, we use (13) and the fact $\mathbf{N} > 0$)

$$\sum_{i,j=1}^{N} \nu_{ij} \int\limits_0^1 \left(\frac{\partial U_i}{\partial x}\right)\left(\frac{\partial U_j}{\partial x}\right) dx \geqslant C_1(\mathbf{N}) \sum_{i=1}^{N} \int\limits_0^1 \left(\frac{\partial U_i}{\partial x}\right)^2 dx,$$

$$K \sum_{i=1}^{N} \alpha_i \int\limits_0^1 \rho \left(\frac{\partial U_i}{\partial x}\right) dx \leqslant \frac{C_1}{2} \sum_{i=1}^{N} \int\limits_0^1 \left(\frac{\partial U_i}{\partial x}\right)^2 dx + C_2,$$

where $C_2 = \dfrac{K^2N^3}{2C_1}\left(\max\limits_{1\leqslant i\leqslant N}\sup\limits_{[0,1]}\rho_{0i}\right)^2 e^{2\pi m^2 cNt^m}$, we obtain the estimate

$$\frac{d}{dt}\left(\sum_{i=1}^{N}\int_0^1 \rho_i U_i^2\,dx\right) + C_1\sum_{i=1}^{N}\int_0^1\left(\frac{\partial U_i}{\partial x}\right)^2 dx \leqslant 2C_2,$$

which, in its turn, leads to

$$\sum_{i=1}^{N}\int_0^1 \rho_i U_i^2\,dx \leqslant \sum_{i=1}^{N}\int_0^1 \rho_{0i} U_{0i}^2\,dx + 2C_2 t^m, \tag{16}$$

where $U_{0i} = \sum\limits_{s=1}^{m}\psi_{is}(0)\sin(\pi sx) = \sum\limits_{s=1}^{m}\xi_{0is}\sin(\pi sx)$, $i = 1,\dots,N$. Using (13) for the second time, we obtain from (16) the inequality

$$\|\boldsymbol{\psi}\|^2_{(C[0,t^m])^{mN}} \leqslant e^{\pi m^2 cNt^m}\frac{\max\limits_{1\leqslant i\leqslant N}\sup\limits_{[0,1]}\rho_{0i}}{\min\limits_{1\leqslant i\leqslant N}\inf\limits_{[0,1]}\rho_{0i}}\|\boldsymbol{\xi}_0\|^2_{\mathbb{R}^{mN}} + \frac{4C_2 e^{\pi m^2 cNt^m}}{\min\limits_{1\leqslant i\leqslant N}\inf\limits_{[0,1]}\rho_{0i}}t^m.$$

Choosing

$$t^m < \min\left(T, \frac{1}{\pi m^2 cN}, \frac{\min\limits_{1\leqslant i\leqslant N}\inf\limits_{[0,1]}\rho_{0i}}{4eC_3}\right), \tag{17}$$

where $C_3 = \dfrac{K^2N^3e^2}{2C_1}\left(\max\limits_{1\leqslant i\leqslant N}\sup\limits_{[0,1]}\rho_{0i}\right)^2$, we obtain that $C_2 \leqslant C_3$, and arrive at the required estimate

$$\|\boldsymbol{\psi}\|_{(C[0,t^m])^{mN}} \leqslant c.$$

Thus, if (17) holds, the operator A maps the set V to itself.

Let us prove the compactness of the operator A. Multiplying (14) by $\dfrac{d\psi_{ik}(t)}{dt}$, summarizing with respect to i, k and integrating with respect to x, we get the relation

$$\sum_{i=1}^{N}\int_0^1 \rho_i\left(\frac{\partial U_i}{\partial t}\right)^2 dx = \sum_{i=1}^{N}\int_0^1\Bigg(-\rho_i v\left(\frac{\partial U_i}{\partial x}\right)\left(\frac{\partial U_i}{\partial t}\right) + $$
$$+\alpha_i K\rho\left(\frac{\partial^2 U_i}{\partial t\partial x}\right) - \sum_{j=1}^{N}\nu_{ij}\left(\frac{\partial^2 U_i}{\partial t\partial x}\right)\left(\frac{\partial U_j}{\partial x}\right)\Bigg)dx. \tag{18}$$

Let us estimate the terms on the right-hand side of (18) by using (13), the Cauchy inequality, and the inequalities $\|\boldsymbol{\xi}\|_{(C[0,t^m])^{mN}} \leqslant c$, $\|\boldsymbol{\psi}\|_{(C[0,t^m])^{mN}} \leqslant c$,

$$\left\|\frac{\partial U_i}{\partial x}\right\|_{L_2(0,1)} \leqslant C_4(m)\|U_i\|_{L_2(0,1)}, \ \left\|\frac{\partial^2 U_i}{\partial t \partial x}\right\|_{L_2(0,1)} \leqslant C_4 \left\|\frac{\partial U_i}{\partial t}\right\|_{L_2(0,1)}, \ i=1,...,N:$$

$$-\sum_{i=1}^{N}\int_0^1 \rho_i v \left(\frac{\partial U_i}{\partial x}\right)\left(\frac{\partial U_i}{\partial t}\right) dx \leqslant$$
$$\leqslant \frac{1}{6}\sum_{i=1}^{N}\int_0^1 \rho_i \left(\frac{\partial U_i}{\partial t}\right)^2 dx + C_5\left(C_4, \left\{\sup_{[0,1]}\rho_{0i}\right\}_{i=1}^{N}, N, c, m, t^m\right),$$

$$K\sum_{i=1}^{N}\alpha_i\int_0^1 \rho\left(\frac{\partial^2 U_i}{\partial t\partial x}\right) dx \leqslant \frac{1}{6}\sum_{i=1}^{N}\int_0^1 \rho_i\left(\frac{\partial U_i}{\partial t}\right)^2 dx +$$
$$+ C_6\left(C_4, \left\{\inf_{[0,1]}\rho_{0i}\right\}_{i=1}^{N}, \left\{\sup_{[0,1]}\rho_{0i}\right\}_{i=1}^{N}, K, N, c, m, t^m\right),$$

$$-\sum_{i,j=1}^{N}\nu_{ij}\int_0^1 \left(\frac{\partial^2 U_i}{\partial t\partial x}\right)\left(\frac{\partial U_j}{\partial x}\right) dx \leqslant$$
$$\leqslant \frac{1}{6}\sum_{i=1}^{N}\int_0^1 \rho_i\left(\frac{\partial U_i}{\partial t}\right)^2 dx + C_7\left(C_4, \left\{\inf_{[0,1]}\rho_{0i}\right\}_{i=1}^{N}, \mathbf{N}, N, c, m, t^m\right).$$

Thus, from (18) we obtain the inequality

$$\frac{1}{2}\sum_{i=1}^{N}\int_0^1 \rho_i\left(\frac{\partial U_i}{\partial t}\right)^2 dx \leqslant C_5 + C_6 + C_7.$$

Integrating the last inequality in time and applying (13), we obtain the estimate

$$\sum_{i=1}^{N}\left\|\frac{\partial U_i}{\partial t}\right\|_{L_2(Q_{t^m})} \leqslant C_8\left(C_5, C_6, C_7, \left\{\inf_{[0,1]}\rho_{0i}\right\}_{i=1}^{N}, N, c, m, t^m\right), \tag{19}$$

where $Q_{t^m} = (0, t^m) \times (0, 1)$. Thus, we have obtained the estimate for $\boldsymbol{\psi}$ in $(W_2^1(0, t^m))^{mN}$. Consequently, A is a compact operator.

We establish the continuity of the operator A from V in $(C[0, t^m])^{mN}$. Let $\boldsymbol{\xi}^{(1,2)} \in V$, $\boldsymbol{\psi}^{(1,2)} = A(\boldsymbol{\xi}^{(1,2)})$, $u_i^{(1,2)} = \sum_{s=1}^{m} \xi_{is}^{(1,2)} \sin(\pi s x)$, $U_i^{(1,2)} = \sum_{s=1}^{m} \psi_{is}^{(1,2)} \sin(\pi s x)$, $i = 1, \ldots, N$. Let $\rho_i^{(1,2)}$, $i = 1, \ldots, N$ be the solutions to the Cauchy problems (7), (9), with v instead of $v^{(1,2)} = \sum_{j=1}^{N} \alpha_j u_j^{(1,2)}$ respectively. Denote $\rho_i = \rho_i^{(1)} - \rho_i^{(2)}$, $u_i = u_i^{(1)} - u_i^{(2)}$, $U_i = U_i^{(1)} - U_i^{(2)}$, $i = 1, ..., N$, $v = v^{(1)} - v^{(2)}$, $\rho = \rho^{(1)} - \rho^{(2)}$, where $\rho^{(1,2)} = \sum_{j=1}^{N} \rho_j^{(1,2)}$. Differentiating the equations

$$\frac{\partial \rho_i^{(1,2)}}{\partial t} + \frac{\partial \left(\rho_i^{(1,2)} v^{(1,2)}\right)}{\partial x} = 0, \quad i = 1, \ldots, N \tag{20}$$

(see Remark 1 below) with respect to x, multiplying by $\frac{\partial \rho_i^{(1,2)}}{\partial x}$, integrating with respect to x, t, using the initial conditions

$$\rho_i^{(1,2)}|_{t=0} = \rho_{0i}, \quad i = 1, \ldots, N, \tag{21}$$

the inequality (see (13))

$$\left(\inf_{[0,1]} \rho_{0i}\right) e^{-\pi m^2 cNt} \leqslant \rho_i^{(1,2)} \leqslant \left(\sup_{[0,1]} \rho_{0i}\right) e^{\pi m^2 cNt}, \quad i = 1, \ldots, N \tag{22}$$

and Gronwall inequality we get

$$\left\| \frac{\partial \rho_i^{(1,2)}}{\partial x} \right\|_{L_2(0,1)} \leqslant C_9 \left(\left\{ \|\rho_{0i}\|_{W_2^1(0,1)} \right\}_{i=1}^{N}, N, c, m, t^m \right), \quad i = 1, \ldots, N. \tag{23}$$

Let us note that (20), (21) lead to the equalities

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial \left(\rho_i v^{(1)}\right)}{\partial x} + \frac{\partial \left(\rho_i^{(2)} v\right)}{\partial x} = 0, \quad \rho_i|_{t=0} = 0, \quad i = 1, \ldots, N. \tag{24}$$

Multiplying (24) by ρ_i, and integrating with respect to x, we arrive at the relations

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \rho_i^2\, dx\right) = -\int_0^1 \left(\frac{1}{2}\rho_i^2 \left(\frac{\partial v^{(1)}}{\partial x}\right) + \rho_i^{(2)} \rho_i \left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial \rho_i^{(2)}}{\partial x}\right) \rho_i v\right) dx$$
$$\leqslant \frac{1}{2}\left(\sup_{[0,1]} \left|\frac{\partial v^{(1)}}{\partial x}\right| \int_0^1 \rho_i^2\, dx + \sup_{[0,1]} \rho_i^{(2)} \int_0^1 \left(\rho_i^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) dx + \right.$$
$$\left. + \sup_{[0,1]} v^2 \int_0^1 \left(\frac{\partial \rho_i^{(2)}}{\partial x}\right)^2 dx + \int_0^1 \rho_i^2\, dx\right) \leqslant$$
$$\leqslant C_{10}\left(C_9, \left\{\sup_{[0,1]} \rho_{0i}\right\}_{i=1}^{N}, N, c, m, t^m\right) \left(\int_0^1 \rho_i^2\, dx + \sum_{j=1}^{N} \int_0^1 u_j^2\, dx\right). \tag{25}$$

Here we have used obvious relations

$$\sum_{j=1}^{N}\int_0^1 u_j^2\,dx = \frac{1}{2}\sum_{j=1}^{N}\sum_{s=1}^{m}\xi_{js}^2(t), \quad \sup_{[0,1]} v^2 \leqslant \sum_{i,j=1}^{N}\sum_{s,l=1}^{m}|\xi_{il}(t)\xi_{js}(t)|,$$
$$\int_0^1\left(\frac{\partial v}{\partial x}\right)^2 dx = \frac{\pi^2}{2}\sum_{i,j=1}^{N}\alpha_i\alpha_j\sum_{s=1}^{m}s^2\xi_{js}(t)\xi_{is}(t). \tag{26}$$

From (25), applying the Gronwall inequality and the initial conditions in (24), we obtain the inequalities

$$\int_0^1 \rho_i^2\,dx \leqslant C_{11}(C_{10},t^m)\sum_{j=1}^{N}\int_0^t\int_0^1 u_j^2\,dxd\tau, \quad i=1,\ldots,N \tag{27}$$

for all $t\in(0,t_m]$. Further, from the equations for $U_i^{(1,2)}$, $i=1,...,N$ (see (14)) due to (20) we get

$$\frac{1}{2}\sum_{i=1}^{N}\int_0^1 \rho_i^{(1)}U_i^2\,dx + \sum_{i,j=1}^{N}\nu_{ij}\int_0^t\int_0^1\left(\frac{\partial U_i}{\partial x}\right)\left(\frac{\partial U_j}{\partial x}\right)dxd\tau =$$
$$= K\sum_{i=1}^{N}\alpha_i\int_0^t\int_0^1\rho\left(\frac{\partial U_i}{\partial x}\right)dxd\tau - \sum_{i=1}^{N}\int_0^t\int_0^1\rho_iU_i\left(\frac{\partial U_i^{(2)}}{\partial\tau}\right)dxd\tau -$$
$$-\sum_{i=1}^{N}\int_0^t\int_0^1\rho_i^{(1)}vU_i\left(\frac{\partial U_i^{(2)}}{\partial x}\right)dxd\tau - \sum_{i=1}^{N}\int_0^t\int_0^1\rho_iv^{(2)}U_i\left(\frac{\partial U_i^{(2)}}{\partial x}\right)dxd\tau \tag{28}$$

for all $t\in(0,t_m]$. The first term on the left-hand side of (28) satisfies the estimate

$$\frac{1}{2}\sum_{i=1}^{N}\int_0^1\rho_i^{(1)}U_i^2\,dx \geqslant \frac{e^{-\pi m^2cNt^m}\min\limits_{1\leqslant i\leqslant N}\inf\limits_{[0,1]}\rho_{0i}}{2}\sum_{i=1}^{N}\int_0^1 U_i^2\,dx. \tag{29}$$

For the second term on the left-hand side of (28) we have

$$\sum_{i,j=1}^{N}\int_0^t\int_0^1\nu_{ij}\left(\frac{\partial U_i}{\partial x}\right)\left(\frac{\partial U_j}{\partial x}\right)dxd\tau \geqslant C_1\sum_{i=1}^{N}\int_0^t\int_0^1\left(\frac{\partial U_i}{\partial x}\right)^2dxd\tau. \tag{30}$$

For the first term on the right-hand side of (28) we obtain the relation

$$K\sum_{i=1}^{N}\alpha_i\int\limits_0^t\int\limits_0^1\rho\left(\frac{\partial U_i}{\partial x}\right)dxd\tau\leqslant\frac{C_1}{2}\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1\left(\frac{\partial U_i}{\partial x}\right)^2dxd\tau+$$
$$+C_{12}\left(C_1,C_{11},K,N,t^m\right)\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1u_i^2\,dxd\tau. \quad (31)$$

For the second term on the right-hand side of (28) the inequality

$$-\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1\rho_iU_i\left(\frac{\partial U_i^{(2)}}{\partial\tau}\right)dxd\tau\leqslant$$
$$\leqslant\frac{\varepsilon}{2}\sum_{i=1}^{N}\sup_{[0,t]\times[0,1]}U_i^2\int\limits_0^t\int\limits_0^1\left(\frac{\partial U_i^{(2)}}{\partial\tau}\right)^2dxd\tau+\frac{C_{11}Nt^m}{2\varepsilon}\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1u_i^2\,dxd\tau\leqslant$$
$$\leqslant\varepsilon mC_8^2\sum_{i=1}^{N}\sup_{[0,t]}\int\limits_0^1U_i^2\,dx+\frac{C_{11}Nt^m}{2\varepsilon}\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1u_i^2\,dxd\tau$$

holds, and taking

$$\varepsilon mC_8^2=\frac{e^{-\pi m^2cNt^m}\min\limits_{1\leqslant i\leqslant N}\inf\limits_{[0,1]}\rho_{0i}}{4N},$$

we obtain that

$$-\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1\rho_iU_i\left(\frac{\partial U_i^{(2)}}{\partial\tau}\right)dxd\tau\leqslant\frac{e^{-\pi m^2cNt^m}\min\limits_{1\leqslant i\leqslant N}\inf\limits_{[0,1]}\rho_{0i}}{4N}\sum_{i=1}^{N}\sup_{[0,t]}\int\limits_0^1U_i^2\,dx+$$
$$+C_{13}\left(C_8,C_{11},\left\{\inf_{[0,1]}\rho_{0i}\right\}_{i=1}^{N},N,c,m,t^m\right)\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1u_i^2\,dxd\tau. \quad (32)$$

The third term on the right-hand side of (28) can be estimated as follows

$$-\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1\rho_i^{(1)}vU_i\left(\frac{\partial U_i^{(2)}}{\partial x}\right)dxd\tau\leqslant$$
$$\leqslant C_{14}\left(\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1u_i^2\,dxd\tau+\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1U_i^2\,dxd\tau\right), \quad (33)$$

where $C_{14} = C_{14}\left(\left\{\sup\limits_{[0,1]} \rho_{0i}\right\}_{i=1}^{N}, N, c, m, t^m\right)$. Finally, the last term on the right-hand side of (28) satisfies the relation

$$-\sum_{i=1}^{N} \int_0^t \int_0^1 \rho_i v^{(2)} U_i \left(\frac{\partial U_i^{(2)}}{\partial x}\right) dx d\tau \leqslant$$
$$\leqslant C_{15} \left(\sum_{i=1}^{N} \int_0^t \int_0^1 u_i^2 \, dx d\tau + \sum_{i=1}^{N} \int_0^t \int_0^1 U_i^2 \, dx d\tau\right), \quad (34)$$

where $C_{15} = C_{15}(C_{11}, N, c, m, t^m)$. Thus, from (28), using (29)–(34), the inequality

$$\sum_{i=1}^{N} \int_0^1 U_i^2 \, dx \leqslant C_{16} \left(\sum_{i=1}^{N} \int_0^t \int_0^1 u_i^2 \, dx d\tau + \sum_{i=1}^{N} \int_0^t \int_0^1 U_i^2 \, dx d\tau\right)$$

follows, where $C_{16} = C_{16}\left(C_{12}, \ldots, C_{15}, \left\{\inf\limits_{[0,1]} \rho_{0i}\right\}_{i=1}^{N}, N, c, m, t^m\right)$, which, in view of the Gronwall inequality, implies the estimate

$$\sum_{i=1}^{N} \int_0^1 U_i^2 \, dx \leqslant C_{17}(C_{16}, t^m) \sum_{i=1}^{N} \int_0^{t^m} \int_0^1 u_i^2 \, dx dt,$$

and, consequently,

$$\|\boldsymbol{\psi}^{(1)} - \boldsymbol{\psi}^{(2)}\|_{(C[0,t^m])^{mN}} \leqslant C_{18}(C_{17}, t^m) \|\boldsymbol{\xi}^{(1)} - \boldsymbol{\xi}^{(2)}\|_{(C[0,t^m])^{mN}}.$$

The last inequality justifies the continuity of the operator A.

Since the operator A satisfies the assumptions of the Schauder theorem listed above, in V there exists a fixed point $\boldsymbol{\xi}$ of the operator A which, together with the corresponding functions ρ_i, $i = 1, \ldots, N$, is a solution to problem (7)–(10). Theorem 2 is proved.

4 Uniform estimates of Galerkin approximations

Let us obtain estimates of solutions to the approximate initial-boundary value problem (7)–(10), which would be uniform with respect to the parameter m and would allow us to pass to the limit as $m \to \infty$. Let us denote

$$\alpha(t) = \sum_{i=1}^{N} \int_0^t \int_0^1 \left(\rho_i \left(\frac{\partial u_i}{\partial t}\right)^2 + \left(\frac{\partial^2 u_i}{\partial x^2}\right)^2\right) dx d\tau, \quad \alpha'(t) \geqslant 0. \quad (35)$$

Equations (7) imply inequalities (12) which, in turn, imply the estimates

$$C_{19}^{-1} e^{-C_{19}\alpha(t)} \leqslant \rho_i(x,t) \leqslant C_{19} e^{C_{19}\alpha(t)}, \quad i = 1, \ldots, N, \quad (36)$$

where $C_{19} = C_{19}\left(\left\{\sup\limits_{[0,1]}\rho_{0i}\right\}_{i=1}^N, \left\{\inf\limits_{[0,1]}\rho_{0i}\right\}_{i=1}^N, N, T\right)$. We note that (7) imply the equalities

$$\rho_i \frac{\partial^2}{\partial t \partial x}\left(\frac{1}{\rho_i}\right) + \rho_i v \frac{\partial^2}{\partial x^2}\left(\frac{1}{\rho_i}\right) = \frac{\partial^2 v}{\partial x^2}, \quad i = 1, \ldots, N, \tag{37}$$

which yield

$$\frac{d}{dt}\left(\int\limits_0^1 \rho_i \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho_i}\right)\right)^2 dx\right) = 2\int\limits_0^1 \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho_i}\right)\right)\left(\frac{\partial^2 v}{\partial x^2}\right) dx, \quad i = 1, \ldots, N. \tag{38}$$

From (38), the relations

$$\int\limits_0^1 \rho_i \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho_i}\right)\right)^2 dx \leqslant \int\limits_0^1 \rho_{0i}\left(\left(\frac{1}{\rho_{0i}}\right)'\right)^2 dx + $$
$$+ \int\limits_0^t \int\limits_0^1 \left(\rho_i \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho_i}\right)\right)^2 + \frac{1}{\rho_i}\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right) dx d\tau, \quad i = 1, \ldots, N \tag{39}$$

follow.

Remark 1. *In order to obtain* (39)*, we need* (*in* (37), (38)) *an additional smoothness of* ρ_i*,* $i = 1, \ldots, N$ *in comparison with* (11)*, however the formulation of relations* (39) *does not require any additional regularity. This means that* (39) *can be obtained via regularization of* ρ_{0i}*,* $i = 1, \ldots, N$*, derivation of* (39) *for the solutions of the corresponding problems, and then the limit via the regularization parameter. The derivation of relations* (23) *should be understood in a similar way.*

Using (35), (36) and the Gronwall inequality, from (39) we obtain the estimates

$$\int\limits_0^1 \left(\frac{\partial}{\partial x}\left(\frac{1}{\rho_i}\right)\right)^2 dx + \int\limits_0^1 \left(\frac{\partial \rho_i}{\partial x}\right)^2 dx \leqslant C_{20} e^{C_{20}\alpha(t)}, \quad i = 1, \ldots, N, \tag{40}$$

where $C_{20} = C_{20}\left(C_{19}, \left\{\|\rho_{0i}\|_{W_2^1(0,1)}\right\}_{i=1}^N, \left\{\inf\limits_{[0,1]}\rho_{0i}\right\}_{i=1}^N, N, T\right)$. Further, multiplying (8) by $\psi'_{ik} + \pi^2 k^2 \psi_{ik}$, summarizing with respect to i, k, and taking into account (7), (10), we obtain the relation (here, we also use the symmetry of the matrix $\mathbf{N}$)

$$\sum_{i=1}^N \int\limits_0^1 \rho_i \left(\frac{\partial u_i}{\partial t}\right)^2 dx + \sum_{i,j=1}^N \nu_{ij} \int\limits_0^1 \left(\frac{\partial^2 u_i}{\partial x^2}\right)\left(\frac{\partial^2 u_j}{\partial x^2}\right) dx +$$

$$+\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{N}\int_{0}^{1}\rho_i\left(\frac{\partial u_i}{\partial x}\right)^2dx\right)+\frac{1}{2}\frac{d}{dt}\left(\sum_{i,j=1}^{N}\nu_{ij}\int_{0}^{1}\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right)dx\right)=$$
$$=-\sum_{i=1}^{N}\int_{0}^{1}\rho_i v\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_i}{\partial x}\right)dx-K\int_{0}^{1}\left(\frac{\partial \rho}{\partial x}\right)\left(\frac{\partial v}{\partial t}\right)dx+$$
$$+K\int_{0}^{1}\left(\frac{\partial \rho}{\partial x}\right)\left(\frac{\partial^2 v}{\partial x^2}\right)dx-\sum_{i=1}^{N}\int_{0}^{1}\left(\frac{\partial \rho_i}{\partial x}\right)\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_i}{\partial x}\right)dx+$$
$$+2\sum_{i=1}^{N}\int_{0}^{1}\rho_i v\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial^2 u_i}{\partial x^2}\right)dx. \quad (41)$$

The left-hand side of (41) satisfies the estimate (since $\mathbf{N}>0$)

$$\sum_{i=1}^{N}\int_{0}^{1}\rho_i\left(\frac{\partial u_i}{\partial t}\right)^2dx+\sum_{i,j=1}^{N}\nu_{ij}\int_{0}^{1}\left(\frac{\partial^2 u_i}{\partial x^2}\right)\left(\frac{\partial^2 u_j}{\partial x^2}\right)dx+$$
$$+\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{N}\int_{0}^{1}\rho_i\left(\frac{\partial u_i}{\partial x}\right)^2dx\right)+\frac{1}{2}\frac{d}{dt}\left(\sum_{i,j=1}^{N}\nu_{ij}\int_{0}^{1}\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right)dx\right)\geqslant$$
$$\geqslant C_{21}(C_1)\alpha'(t)+\beta'(t), \quad (42)$$

where

$$\beta(t)=\frac{1}{2}\sum_{i=1}^{N}\int_{0}^{1}\rho_i\left(\frac{\partial u_i}{\partial x}\right)^2dx+\frac{1}{2}\sum_{i,j=1}^{N}\nu_{ij}\int_{0}^{1}\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right)dx.$$

We separately consider each term on the right-hand side of (41). For the first term on the right-hand side of (41) we have

$$-\sum_{i=1}^{N}\int_{0}^{1}\rho_i v\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_i}{\partial x}\right)dx\leqslant$$
$$\leqslant\sum_{i=1}^{N}\sup_{[0,1]}|v|\left(\int_{0}^{1}\rho_i\left(\frac{\partial u_i}{\partial t}\right)^2dx\right)^{1/2}\left(\int_{0}^{1}\rho_i\left(\frac{\partial u_i}{\partial x}\right)^2dx\right)^{1/2}\leqslant$$
$$\leqslant\frac{C_{21}}{10}\alpha'(t)+C_{22}\,\beta^2(t)e^{C_{22}\alpha(t)}, \quad (43)$$

where $C_{22} = C_{22}(C_{19}, C_{21}, N)$. For the second and third terms on the right-hand side of (41) we obtain

$$-K\int_0^1 \left(\frac{\partial\rho}{\partial x}\right)\left(\frac{\partial v}{\partial t}\right) dx \leqslant \frac{C_{21}}{10}\alpha'(t) + C_{23}e^{C_{23}\alpha(t)}, \tag{44}$$

$$K\int_0^1 \left(\frac{\partial\rho}{\partial x}\right)\left(\frac{\partial^2 v}{\partial x^2}\right) dx \leqslant \frac{C_{21}}{10}\alpha'(t) + C_{24}e^{C_{24}\alpha(t)} \tag{45}$$

respectively, where $C_{23} = C_{23}(C_{19}, C_{20}, C_{21}, K, N)$, $C_{24} = C_{24}(C_{20}, C_{21}, K, N)$. For the fourth term on the right-hand side of (41), using the interpolation estimate

$$\sup_{[0,1]}\left|\frac{\partial u_i}{\partial x}\right| \leqslant \sqrt{2}\left(\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx\right)^{1/2}\left(\int_0^1 \left(\frac{\partial^2 u_i}{\partial x^2}\right)^2 dx\right)^{1/2},$$

we deduce

$$-\sum_{i=1}^N \int_0^1 \left(\frac{\partial\rho_i}{\partial x}\right)\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_i}{\partial x}\right) dx \leqslant$$
$$\leqslant \sum_{i=1}^N \sup_{[0,1]}\left|\frac{\partial u_i}{\partial x}\right|\left(\int_0^1 \frac{1}{\rho_i}\left(\frac{\partial\rho_i}{\partial x}\right)^2 dx\right)^{1/2}\left(\int_0^1 \rho_i\left(\frac{\partial u_i}{\partial t}\right)^2 dx\right)^{1/2} \leqslant$$
$$\leqslant \frac{C_{21}}{10}\alpha'(t) + C_{25}\,\beta(t)e^{C_{25}\alpha(t)}, \quad C_{25} = C_{25}(C_{19}, C_{20}, C_{21}, N). \tag{46}$$

Finally, for the last term on the right-hand side of (41) we obtain the relation

$$2\sum_{i=1}^N \int_0^1 \rho_i v\left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial^2 u_i}{\partial x^2}\right) dx \leqslant$$
$$\leqslant 2\sum_{i=1}^N \sup_{[0,1]}|v| \sup_{[0,1]}\sqrt{\rho_i}\left(\int_0^1 \rho_i\left(\frac{\partial u_i}{\partial x}\right)^2 dx\right)^{1/2}\left(\int_0^1 \left(\frac{\partial^2 u_i}{\partial x^2}\right)^2 dx\right)^{1/2} \leqslant$$
$$\leqslant \frac{C_{21}}{10}\alpha'(t) + C_{26}\,\beta^2(t)e^{C_{26}\alpha(t)}, \quad C_{26} = C_{26}(C_{19}, C_{21}, N). \tag{47}$$

Thus, from (43)–(46) it follows that the right-hand side of (41)

$$-\sum_{i=1}^N \int_0^1 \rho_i v\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_i}{\partial x}\right) dx - K\int_0^1 \left(\frac{\partial\rho}{\partial x}\right)\left(\frac{\partial v}{\partial t}\right) dx +$$

$$+ K \int_0^1 \left(\frac{\partial \rho}{\partial x}\right)\left(\frac{\partial^2 v}{\partial x^2}\right) dx - \sum_{i=1}^N \int_0^1 \left(\frac{\partial \rho_i}{\partial x}\right)\left(\frac{\partial u_i}{\partial t}\right)\left(\frac{\partial u_i}{\partial x}\right) dx +$$

$$+ 2 \sum_{i=1}^N \int_0^1 \rho_i v \left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial^2 u_i}{\partial x^2}\right) dx \leqslant$$

$$\leqslant \frac{C_{21}}{2}\alpha'(t) + C_{27}(1+\beta^2(t))e^{C_{27}\alpha(t)}, \quad C_{27} = C_{27}(C_{22}, \ldots, C_{26}). \quad (48)$$

Combining relations (42) and (48), from (41) we obtain the inequality

$$\left(\frac{C_{21}}{2}\alpha(t) + \beta(t)\right)' \leqslant C_{28} e^{C_{28}\left(C_{21}\frac{\alpha(t)}{2} + \beta(t)\right)}, \quad C_{28} = C_{28}(C_{21}, C_{27}). \quad (49)$$

We take any $C_{29} > \beta(0)$, for example, $C_{29} = 2\beta(0)$. Then for

$$t_0 = \min\left(T, \frac{e^{-C_{28}\beta(0)} - e^{-C_{28}C_{29}}}{C_{28}^2}\right) \quad (50)$$

we derive from (49) that the estimate

$$\sup_{0 \leqslant t \leqslant t_0} (\alpha + \beta) \leqslant \left(1 + \frac{2}{C_{21}}\right) C_{30}, \quad C_{30} = \frac{1}{C_{28}} \ln\left(\frac{1}{e^{-C_{28}\beta(0)} - C_{28}^2 t_0}\right)$$

holds, which together with (7), (36) and (40) yields that

$$\sum_{i=1}^N \left(\|\rho_i\|_{L_\infty\left(0,t_0;W_2^1(0,1)\right)} + \|u_i\|_{L_\infty\left(0,t_0;W_2^1(0,1)\right)} + \|u_i\|_{L_2\left(0,t_0;W_2^2(0,1)\right)} + \right.$$

$$\left. + \left\|\frac{\partial \rho_i}{\partial t}\right\|_{L_\infty\left(0,t_0;L_2(0,1)\right)} + \left\|\frac{\partial u_i}{\partial t}\right\|_{L_2(Q_{t_0})} + \left\|\frac{1}{\rho_i}\right\|_{L_\infty(Q_{t_0})} \right) \leqslant C_{31}, \quad (51)$$

where $Q_{t_0} = (0, t_0) \times (0, 1)$, $C_{31} = C_{31}(C_{19}, C_{20}, C_{21}, C_{30}, N)$.

5 Convergence of Galerkin approximations

We have constructed the solutions $(\rho_1^m, \ldots, \rho_N^m, u_1^m, \ldots, u_N^m)$ to problems (7)–(10) for all $m \in \mathbb{N}$, and now extend them, if necessary, for $(0, t_0)$, and we can use estimate (51) for them. Based on this estimate, we can extract a subsequence from the mentioned sequence (keeping the same notation, below this procedure is implied as necessary) such that

$$\rho_i^m \to \rho_i \ \text{ weakly* in } \ L_\infty\big(0, t_0; W_2^1(0,1)\big),$$

$u_i^m \to u_i$ weakly* in $L_\infty\big(0, t_0; W_2^1(0,1)\big)$ and weakly in $L_2\big(0, t_0; W_2^2(0,1)\big)$ as $m \to \infty$ for all $i = 1, \ldots, N$. The other properties listed in (5) are also satisfied by this sequence in Q_{t_0} uniformly with respect to m. Consequently, the limit functions belong to the corresponding classes. We show that $(\rho_1, \ldots, \rho_N, u_1, \ldots, u_N)$ is a strong solution to problem (1)–(4) on $(0, t_0)$.

By the Arzela—Ascoli theorem (see [54, Theorem 1.70, P. 58]) and the uniform estimates for ρ_i^m, u_i^m in $L_\infty\big(0,t_0;W_2^1(0,1)\big)$ and for $\dfrac{\partial\rho_i^m}{\partial t}$, $\dfrac{\partial u_i^m}{\partial t}$ in $L_2(Q_{t_0})$ (cf. (51)) we have

$$\rho_i^m\to\rho_i \ \text{ as } \ m\to\infty \ \text{ strongly in } \ C\big([0,t_0];L_2(0,1)\big), \quad i=1,\ldots,N, \tag{52}$$

$$u_i^m\to u_i \ \text{ as } \ m\to\infty \ \text{ strongly in } \ C\big([0,t_0];L_2(0,1)\big), \quad i=1,\ldots,N. \tag{53}$$

Since $\dfrac{\partial u_i^m}{\partial t}$ are bounded in $L_2(Q_{t_0})$, an uniform estimate for $\dfrac{\partial^2 u_i^m}{\partial t\partial x}$ in $L_2(0,t_0;W_2^{-1}(0,1))$ is valid, which, together with the estimate for $\dfrac{\partial u_i^m}{\partial x}$ in $L_2(0,t_0;W_2^1(0,1))$ means via Lions–Aubin's lemma (see [54, Theorem 1.71, P. 59]) that

$$\frac{\partial u_i^m}{\partial x}\to\frac{\partial u_i}{\partial x} \ \text{ as } \ m\to\infty \ \text{ strongly in } \ L_2(Q_{t_0}), \quad i=1,\ldots,N. \tag{54}$$

Hence (53) leads to the relations

$$u_i^m\to u_i \ \text{ as } \ m\to\infty \ \text{ strongly in } \ L_2\big(0,t_0;C[0,1]\big), \quad i=1,\ldots,N. \tag{55}$$

Thus, the limit functions ρ_i, u_i, $i=1,\ldots,N$, satisfy (almost everywhere in Q_{t_0}) the continuity equations (1), in which $v=\sum\limits_{j=1}^N\alpha_j u_j$, the initial data (3) for almost all $x\in(0,1)$ and the boundary conditions (4) for almost all $t\in(0,t_0)$.

The boundedness of $\dfrac{\partial u_i^m}{\partial t}$ in $L_2(Q_{t_0})$ implies the weak convergence of $\dfrac{\partial u_i^m}{\partial t}$ to $\dfrac{\partial u_i}{\partial t}$ in $L_2(Q_{t_0})$, which, together with (52) and the boundedness of $\rho_i^m\dfrac{\partial u_i^m}{\partial t}$ in $L_2(Q_{t_0})$ implies

$$\rho_i^m\frac{\partial u_i^m}{\partial t}\to\rho_i\frac{\partial u_i}{\partial t} \ \text{ as } \ m\to\infty \ \text{ weakly in } \ L_2(Q_{t_0}), \quad i=1,\ldots,N.$$

Further, from (52) and (54) it follows that

$$\rho_i^m\frac{\partial u_i^m}{\partial x}\to\rho_i\frac{\partial u_i}{\partial x} \ \text{ as } \ m\to\infty \ \text{ strongly in } \ L_2\big(0,t_0;L_1(0,1)\big), \ \ i=1,\ldots,N,$$

and hence (55) yields that the convergences

$$\left(\rho_i^m\frac{\partial u_i^m}{\partial x}\right)u_j^m\to\left(\rho_i\frac{\partial u_i}{\partial x}\right)u_j \ \text{ as } \ m\to\infty \ \text{ strongly in } \ L_1(Q_{t_0}) \tag{56}$$

are valid for all $i,j=1,\ldots,N$.

By (8), for any functions of the form $(i=1,\ldots,N)$

$$\varphi_i=\sum_{k=1}^M\eta_{ik}(t)\sin(\pi kx), \quad \eta_{ik}\in C[0,t_0], \quad k=1,\ldots,M, \quad M\leqslant m, \tag{57}$$

we have the equalities

$$\int_0^{t_0}\int_0^1 \left(\rho_i^m \frac{\partial u_i^m}{\partial t} + \rho_i^m v^m \frac{\partial u_i^m}{\partial x} + \alpha_i K \frac{\partial \rho^m}{\partial x} - \sum_{j=1}^N \nu_{ij} \frac{\partial^2 u_j^m}{\partial x^2}\right)\varphi_i \, dxdt = 0,$$
$$i = 1, \dots, N,$$

passing in which to the limit as $m \to \infty$ (by the proved convergences), we find (since the set of functions φ_i, $i = 1, \dots, N$ of form (57) is everywhere dense in $L_2(Q_{t_0})$) that the momentum equations (2) hold for the limit functions ρ_i, u_i, $i = 1, \dots, N$ almost everywhere Q_{t_0}, with $\rho = \sum_{i=1}^N \rho_i$.

Thus, we have proved the existence of a solution to the initial-boundary value problem (1)–(4) in small time. In order to continue the local solution defined on the interval $(0, t_0)$ into the entire target interval $(0, T)$, it is necessary to obtain a priori estimates for this local solution which contain constants depending on the input data of the problem and on the value T, but not on the parameter t_0 (see, for example, [21, P. 40]).

6 Global a priori estimates

During the further study of the unique solvability of problem (1)–(4), the use of the Lagrangian mass coordinates is convenient. Let us accept t and $y(t,x) = \int_0^x \rho(t,s)\, ds$ as new independent variables. Then system (1), (2) takes the form

$$\frac{\partial \widetilde{\rho}_i}{\partial t} + \widetilde{\rho}\widetilde{\rho}_i \frac{\partial \widetilde{v}}{\partial y} = 0, \quad i = 1, \dots, N, \quad \widetilde{v} = \sum_{j=1}^N \alpha_j \widetilde{u}_j, \tag{58}$$

$$\frac{\widetilde{\rho}_i}{\widetilde{\rho}} \frac{\partial \widetilde{u}_i}{\partial t} + \alpha_i K \frac{\partial \widetilde{\rho}}{\partial y} = \sum_{j=1}^N \nu_{ij} \frac{\partial}{\partial y}\left(\widetilde{\rho}\frac{\partial \widetilde{u}_j}{\partial y}\right), \quad i = 1, \dots, N, \quad \widetilde{\rho} = \sum_{j=1}^N \widetilde{\rho}_j. \tag{59}$$

The domain Q_T is transformed into the rectangular $\Pi_T = (0, T) \times (0, d)$, where $d = \int_0^1 \rho_0 \, dx > 0$, $\rho_0 = \sum_{j=1}^N \rho_{0j}$, and the initial and boundary conditions take the form

$$\widetilde{\rho}_i|_{t=0} = \widetilde{\rho}_{0i}(y), \quad \widetilde{u}_i|_{t=0} = \widetilde{u}_{0i}(y), \quad y \in [0, d], \quad i = 1, \dots, N, \tag{60}$$

$$\widetilde{u}_i|_{y=0} = \widetilde{u}_i|_{y=d} = 0, \quad t \in [0, T], \quad i = 1, \dots, N. \tag{61}$$

Let us construct a priori estimates. First of all, we note that the summation of (58) with respect to i gives

$$\frac{\partial \widetilde{\rho}}{\partial t} + \widetilde{\rho}^2 \frac{\partial \widetilde{v}}{\partial y} = 0, \tag{62}$$

and hence
$$\frac{\partial}{\partial t}\left(\frac{\widetilde{\rho}_i}{\widetilde{\rho}}\right) = 0, \quad i = 1, \ldots, N.$$
Hence, due to (60) we get the equalities
$$\frac{\widetilde{\rho}_i(t,y)}{\widetilde{\rho}(t,y)} = \frac{\widetilde{\rho}_{0i}(y)}{\widetilde{\rho}_0(y)} \quad \text{as} \quad (t,y) \in [0,T] \times [0,d] \tag{63}$$
for all $i = 1, \ldots, N$, where $\widetilde{\rho}_0 = \sum\limits_{j=1}^{N} \widetilde{\rho}_{0j}$. In the Eulerian coordinates the ratios ρ_i/ρ, $i = 1, \ldots, N$, satisfy the transport equations, and we only have the inequalities
$$0 < \inf_{[0,1]} \frac{\rho_{0i}}{\rho_0} \leqslant \frac{\rho_i(t,x)}{\rho(t,x)} \leqslant \sup_{[0,1]} \frac{\rho_{0i}}{\rho_0} \leqslant 1 \quad \text{for} \quad (t,x) \in [0,T] \times [0,1]. \tag{64}$$

Let us multiply equations (2) by u_i, integrate over x and sum with respect to i. In view of (1), (4) and the condition $\mathbf{N} > 0$, the following relations hold
$$\sum_{i=1}^{N} \int_0^1 \Big(\rho_i \frac{\partial u_i}{\partial t} + \rho_i v \frac{\partial u_i}{\partial x}\Big) u_i \, dx = \frac{1}{2} \frac{d}{dt} \left(\sum_{i=1}^{N} \int_0^1 \rho_i u_i^2 \, dx \right),$$
$$\sum_{i=1}^{N} \alpha_i K \int_0^1 u_i \frac{\partial \rho}{\partial x} \, dx = -K \int_0^1 \rho \frac{\partial v}{\partial x} \, dx = K \frac{d}{dt} \int_0^1 \big(\rho \ln \rho - (\ln d + 1)\rho + d\big) dx,$$
$$\sum_{i,j=1}^{N} \nu_{ij} \int_0^1 \left(\frac{\partial^2 u_j}{\partial x^2}\right) u_i \, dx = - \sum_{i,j=1}^{N} \nu_{ij} \int_0^1 \left(\frac{\partial u_i}{\partial x}\right) \left(\frac{\partial u_j}{\partial x}\right) dx \leqslant$$
$$\leqslant -C_1 \sum_{i=1}^{N} \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx, \tag{65}$$
and hence we get the inequality
$$\frac{d}{dt} \int_0^1 \left(\frac{1}{2} \sum_{i=1}^{N} \rho_i u_i^2 + K \left(\rho \ln \rho - (\ln d + 1)\rho + d\right) \right) dx +$$
$$+ C_1 \sum_{i=1}^{N} \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx \leqslant 0. \tag{66}$$
Integrating (66) with respect to t, and using (3), we obtain the bound
$$\int_0^1 \left(\frac{1}{2} \sum_{i=1}^{N} \rho_i u_i^2 + K \big(\rho \ln \rho - (\ln d + 1)\rho + d\big) \right) dx + C_1 \sum_{i=1}^{N} \int_0^t \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx d\tau$$

$$\leqslant \int_0^1 \left(\frac{1}{2}\sum_{i=1}^N \rho_{0i}u_{0i}^2 + K\big(\rho_0\ln\rho_0 - (\ln d+1)\rho_0 + d\big)\right)dx,$$

which, due to (64), implies the estimate

$$\sum_{i=1}^N \int_0^1 \rho u_i^2 dx + \int_0^1 \big(\rho\ln\rho - (\ln d+1)\rho + d\big)dx + \\ + \sum_{i=1}^N \int_0^T \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dxdt \leqslant C_{32}, \quad (67)$$

where $C_{32} = C_{32}\left(C_1, \left\{\inf\limits_{[0,1]} \frac{\rho_{0i}}{\rho_0}\right\}_{i=1}^N, \left\{\sup\limits_{[0,1]} \rho_{0i}\right\}_{i=1}^N, \{\|u_{0i}\|_{L_2(0,1)}\}_{i=1}^N, K, N, d\right)$. Let us rewrite (67), using the Lagrangian mass coordinates, in the form

$$\sum_{i=1}^N \int_0^d \widetilde{u}_i^2 dy + \int_0^d \frac{\widetilde{\rho}\ln\widetilde{\rho} - (\ln d+1)\widetilde{\rho} + d}{\widetilde{\rho}}\, dy + \\ + \sum_{i=1}^N \int_0^T \int_0^d \widetilde{\rho}\left(\frac{\partial \widetilde{u}_i}{\partial y}\right)^2 dydt \leqslant C_{32}. \quad (68)$$

Let us note that, from estimates (67), in view of (4), the inequality

$$\sum_{i=1}^N \int_0^T \left(\sup_{[0,1]} |u_i|\right)^2 dt \leqslant C_{32} \quad (69)$$

obviously follows.

Let us rewrite equations (2), using (1), in the form

$$\sum_{j=1}^N \widetilde{\nu}_{ij}\left(\frac{\partial(\rho_j u_j)}{\partial t} + \frac{\partial(\rho_j v u_j)}{\partial x}\right) + K\left(\sum_{j=1}^N \widetilde{\nu}_{ij}\alpha_j\right)\frac{\partial \rho}{\partial x} = \frac{\partial^2 u_i}{\partial x^2}, \quad i = 1, \ldots, N, \quad (70)$$

where $\widetilde{\nu}_{ij}$ are the entries of the symmetric matrix $\widetilde{\mathbf{N}} = \mathbf{N}^{-1} > 0$. We multiply (70) by α_i and sum with respect to i, and we obtain

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \widetilde{K}\rho - vV\right), \quad (71)$$

where $V = \sum\limits_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i\rho_j u_j$, $\widetilde{K} = K\sum\limits_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i\alpha_j > 0$. We denote

$$\gamma(t,x) = \int\limits_0^t \left(\frac{\partial v}{\partial x} - \widetilde{K}\rho - vV\right) d\tau + \int\limits_0^x V_0\, ds, \tag{72}$$

where $V_0(x) = V(0,x) = \sum\limits_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i\rho_{0j}u_{0j}$. In view of (67), we have

$$\sup_{[0,T]} \int\limits_0^1 \left|\frac{\partial\gamma}{\partial x}\right| dx = \sup_{[0,T]} \int\limits_0^1 |V|\, dx \leqslant C_{33}(C_{32}, \mathbf{N}, N, d),$$

$$\sup_{[0,T]} \left|\int\limits_0^1 \gamma\, dx\right| \leqslant C_{34}\left(C_{32}, \{\|\rho_{0i}u_{0i}\|_{L_1(0,1)}\}_{i=1}^N, \widetilde{\mathbf{N}}, \widetilde{K}, N, T, d\right),$$

and hence, using Poincaré's inequality (see [54, Lemma 1.43, P. 44]), we get

$$\sup_{[0,T]} \int\limits_0^1 |\gamma|\, dx \leqslant C_{35}(C_{33}, C_{34}),$$

and we arrive at the boundedness of γ in $L_\infty\big(0,T; W_1^1(0,1)\big)$. Using this and the fact $W_1^1(0,1) \hookrightarrow L_\infty(0,1)$, we obtain the estimate

$$\|\gamma\|_{L_\infty(Q_T)} \leqslant C_{36}\,(C_{33}, C_{35}).$$

Let us note that, in view of (1), (71) and (72), the following relations hold

$$\frac{\partial(\rho e^\gamma)}{\partial t} + v\frac{\partial(\rho e^\gamma)}{\partial x} = -\widetilde{K}e^\gamma\rho^2 \leqslant 0,$$

and hence

$$\rho(t,x)e^{\gamma(t,x)} \leqslant \left(\sup_{[0,1]}\rho_0\right) e^{\int\limits_0^1 |V_0|\,dx},$$

so that the estimate

$$\rho(t,x) \leqslant C_{37} \quad \text{as} \quad (t,x) \in [0,T]\times[0,1] \tag{73}$$

is valid, where $C_{37} = C_{37}\left(C_{36}, \left\{\sup\limits_{[0,1]}\rho_{0i}\right\}_{i=1}^N, \{\|\rho_{0i}u_{0i}\|_{L_1(0,1)}\}_{i=1}^N, \widetilde{\mathbf{N}}, N\right)$.

Let us use equations (1), (2) in the form (58), (59). We rewrite equations (59) as

$$\sum_{j=1}^{N} \widetilde{\nu}_{ij}\frac{\widetilde{\rho}_j}{\widetilde{\rho}}\frac{\partial \widetilde{u}_j}{\partial t} + K\left(\sum_{j=1}^{N}\widetilde{\nu}_{ij}\alpha_j\right)\frac{\partial\widetilde{\rho}}{\partial y} = \frac{\partial}{\partial y}\left(\widetilde{\rho}\frac{\partial\widetilde{u}_i}{\partial y}\right), \quad i = 1,\ldots,N, \tag{74}$$

and then multiply (74) by α_i and sum with respect to i. In view of (63), we obtain

$$\sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\frac{\partial \widetilde{u}_j}{\partial t} + \widetilde{K}\frac{\partial \widetilde{\rho}}{\partial y} = \frac{\partial}{\partial y}\left(\widetilde{\rho}\frac{\partial \widetilde{v}}{\partial y}\right). \tag{75}$$

We extract from (62) that

$$\widetilde{\rho}\frac{\partial \widetilde{v}}{\partial y} = -\frac{\partial \ln \widetilde{\rho}}{\partial t} \tag{76}$$

and substitute this into (75), then we get

$$\frac{\partial^2 \ln \widetilde{\rho}}{\partial t \partial y} + \widetilde{K}\frac{\partial \widetilde{\rho}}{\partial y} = -\sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\frac{\partial \widetilde{u}_j}{\partial t}.$$

We multiply this equality by $\dfrac{\partial \ln \widetilde{\rho}}{\partial y}$ and integrate with respect to y, then we obtain the relation

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^d \left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right)^2 dy\right) + \widetilde{K}\int_0^d \widetilde{\rho}\left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right)^2 dy =$$

$$= -\sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \int_0^d \left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\frac{\partial \widetilde{u}_j}{\partial t}\right)\left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right) dy. \tag{77}$$

Let us transform the right-hand side of (77) via the integration by parts and using (76):

$$-\sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \int_0^d \left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\frac{\partial \widetilde{u}_j}{\partial t}\right)\left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right) dy =$$

$$= -\frac{d}{dt}\left(\sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \int_0^d \left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}u_j\right)\left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right) dy\right) +$$

$$+\sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \int_0^d \left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\right)' \widetilde{\rho}\,\widetilde{u}_j \left(\frac{\partial \widetilde{v}}{\partial y}\right) dy + \sum_{i,j=1}^{N} \widetilde{\nu}_{ij}\alpha_i \int_0^d \frac{\widetilde{\rho}_{0j}\widetilde{\rho}}{\widetilde{\rho}_0}\left(\frac{\partial \widetilde{u}_j}{\partial y}\right)\left(\frac{\partial \widetilde{v}}{\partial y}\right) dy. \tag{78}$$

Thus, after integration of (77) with respect to t, taking into account (73) and (78), we get

$$\int_0^d \left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right)^2 dy + 2\widetilde{K}\int_0^t \int_0^d \widetilde{\rho}\left(\frac{\partial \ln \widetilde{\rho}}{\partial y}\right)^2 dy d\tau \leqslant \int_0^d \left((\ln \widetilde{\rho}_0)'\right)^2 dy -$$

$$-2\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\alpha_i\int\limits_0^d\left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}u_j\right)\left(\frac{\partial\ln\widetilde{\rho}}{\partial y}\right)dy+2\sum_{i,j=1}^{N}\widetilde{\nu}_{ij}\alpha_i\int\limits_0^d\left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\widetilde{u}_{0j}\right)(\ln\widetilde{\rho}_0)'dy+$$

$$+2\sqrt{C_{37}}\sum_{i,j=1}^{N}|\widetilde{\nu}_{ij}|\int\limits_0^t\left\|\left(\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\right)'\right\|_{L_2(0,d)}\|\widetilde{u}_j\|_{L_\infty(0,d)}\left\|\sqrt{\widetilde{\rho}}\frac{\partial\widetilde{v}}{\partial y}\right\|_{L_2(0,d)}d\tau+$$

$$+2\sum_{i,j=1}^{N}|\widetilde{\nu}_{ij}|\sup_{[0,d]}\frac{\widetilde{\rho}_{0j}}{\widetilde{\rho}_0}\int\limits_0^t\left\|\widetilde{\rho}\left(\frac{\partial\widetilde{u}_j}{\partial y}\right)\left(\frac{\partial\widetilde{v}}{\partial y}\right)\right\|_{L_1(0,d)}d\tau.$$

Using estimates (68), (69) and (73), we derive from this the inequality

$$\int\limits_0^d\left(\frac{\partial\ln\widetilde{\rho}}{\partial y}\right)^2dy+\int\limits_0^T\int\limits_0^d\widetilde{\rho}\left(\frac{\partial\ln\widetilde{\rho}}{\partial y}\right)^2dydt\leqslant C_{38},\tag{79}$$

where $C_{38}=C_{38}\Bigg(C_{32},C_{37},\left\{\left\|\frac{\widetilde{\rho}_{0i}}{\widetilde{\rho}_0}\right\|_{W_2^1(0,d)}\right\}_{i=1}^{N},\|(\ln\widetilde{\rho}_0)'\|_{L_2(0,d)},\widetilde{K},\widetilde{\mathbf{N}},N,$ $\{\|\widetilde{u}_{0i}\|_{L_2(0,d)}\}_{i=1}^{N}\Bigg)$.

From (60)–(62) it follows obviously that for any $t\in[0,T]$ there exists a point $\delta(t)\in[0,d]$ such that

$$\widetilde{\rho}(t,\delta(t))=d.\tag{80}$$

Hence, we can use the representation

$$\ln\widetilde{\rho}(t,y)=\ln\widetilde{\rho}(t,\delta(t))+\int\limits_{\delta(t)}^{y}\frac{\partial\ln\widetilde{\rho}(t,s)}{\partial s}ds,$$

from which, via Hölder's inequality, and using (79) and (80), we get

$$|\ln\widetilde{\rho}(t,y)|\leqslant|\ln d|+\sqrt{d}\left\|\frac{\partial\ln\widetilde{\rho}}{\partial y}\right\|_{L_2(0,d)}\leqslant C_{39}(C_{38},d).$$

This leads immediately to

$$\widetilde{\rho}(t,y)\geqslant C_{40}(C_{39})\quad\text{as}\quad(t,y)\in[0,T]\times[0,d].\tag{81}$$

From (63), (73) and (81) we obtain that for all $i=1,\dots,N$

$$C_{41}\leqslant\widetilde{\rho}_i(t,y)\leqslant C_{37}\quad\text{as}\quad(t,y)\in[0,T]\times[0,d],\tag{82}$$

where $C_{41}=C_{41}\left(C_{40},\left\{\inf_{[0,d]}\frac{\widetilde{\rho}_{0i}}{\widetilde{\rho}_0}\right\}_{i=1}^{N}\right)$. Hence, for all $i=1,\dots,N$ we have

$$C_{41}\leqslant\rho_i(t,x)\leqslant C_{37}\quad\text{as}\quad(t,x)\in[0,T]\times[0,1].\tag{83}$$

From (63), (73) and (79) now it follows that

$$\int_0^1 \left(\frac{\partial \rho_i}{\partial x}\right)^2 dx \leqslant C_{42}, \quad i = 1, \dots, N, \tag{84}$$

where $C_{42} = C_{42}\left(C_{37}, C_{38}, \left\{\sup_{[0,d]} \frac{\widetilde{\rho}_{0i}}{\widetilde{\rho}_0}\right\}_{i=1}^N, \left\{\left\|\left(\frac{\widetilde{\rho}_0}{\widetilde{\rho}_{0i}}\right)'\right\|_{L_2(0,d)}\right\}_{i=1}^N\right)$, and hence

$$\int_0^1 \left(\frac{\partial \rho}{\partial x}\right)^2 dx \leqslant C_{43}(C_{42}, N). \tag{85}$$

We square equations (2), divide by ρ_i and sum the result with respect to i, then we get

$$\sum_{i=1}^N \rho_i \left(\frac{\partial u_i}{\partial t}\right)^2 + \sum_{i=1}^N \frac{1}{\rho_i}\left(\sum_{j=1}^N \nu_{ij}\frac{\partial^2 u_j}{\partial x^2}\right)^2 - 2\sum_{i=1}^N \left(\frac{\partial u_i}{\partial t}\right)\left(\sum_{j=1}^N \nu_{ij}\frac{\partial^2 u_j}{\partial x^2}\right) = \\ = \sum_{i=1}^N \rho_i \left(v\frac{\partial u_i}{\partial x} + \frac{\alpha_i K}{\rho_i}\frac{\partial \rho}{\partial x}\right)^2. \tag{86}$$

Let us introduce the function $\theta(t)$ via the relation:

$$\theta(t) = \sum_{i,j=1}^N \nu_{ij} \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right) dx + \\ + \sum_{i=1}^N \int_0^t \int_0^1 \left(\rho_i \left(\frac{\partial u_i}{\partial \tau}\right)^2 + \frac{1}{\rho_i}\left(\sum_{j=1}^N \nu_{ij}\frac{\partial^2 u_j}{\partial x^2}\right)^2\right) dx d\tau.$$

Then (86) and inequalities (65), (83) and (85) give the estimate (here the symmetry of the matrix $\mathbf{N}$ is used)

$$\theta'(t) \leqslant C_{44} + C_{45}\left(\sum_{j=1}^N \|u_j\|^2_{L_\infty(0,1)}\right)\left(\sum_{i,j=1}^N \nu_{ij}\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right) dx\right) \leqslant \\ \leqslant C_{44} + C_{45}\left(\sum_{j=1}^N \|u_j\|^2_{L_\infty(0,1)}\right)\theta(t),$$

where $C_{44} = C_{44}(C_{41}, C_{43}, K, N)$, $C_{16} = C_{16}(C_1, C_{37}, N)$, from which, via Gronwall's lemma (see also (69)), it follows that

$$\theta(t) \leqslant C_{46}\left(C_{32}, C_{44}, C_{45}, \{\|u'_{0i}\|_{L_2(0,1)}\}_{i=1}^N, \mathbf{N}, N, T\right). \tag{87}$$

It follows immediately from (87) that

$$\sum_{i=1}^{N}\left(\int_0^1\left(\frac{\partial u_i}{\partial x}\right)^2 dx+\int_0^T\int_0^1\left(\frac{\partial u_i}{\partial t}\right)^2 dxdt+\int_0^T\int_0^1\left(\frac{\partial^2 u_i}{\partial x^2}\right)^2 dxdt\right)\leqslant$$
$$\leqslant C_{47}(C_1,C_{37},C_{41},C_{46},N). \quad (88)$$

Finally, from the continuity equations (1) and the estimates (83), (84) and (88) we obtain that

$$\int_0^1\left(\frac{\partial \rho_i}{\partial t}\right)^2 dx\leqslant C_{48}(C_{37},C_{42},C_{47},N),\quad i=1,\ldots,N.$$

Thereby, we have obtained all estimates which are necessary (and sufficient) to continue the local solution to the initial-boundary value problem (1)–(4) from the interval $(0,t_0)$ into the entire target interval $(0,T)$. In order to conclude the proof of Theorem 1 we need to justify the uniqueness of the solution to the initial-boundary value problem (1)–(4).

7 Uniqueness of the solution

We assume that $\left(\rho_1^{(1)},...,\rho_N^{(1)},u_1^{(1)},...,u_N^{(1)}\right)$ and $\left(\rho_1^{(2)},...,\rho_N^{(2)},u_1^{(2)},...,u_N^{(2)}\right)$ are two solutions to the initial-boundary value problem (1)–(4), and let $v^{(1,2)}=\sum_{j=1}^{N}\alpha_j u_j^{(1,2)}$, $\rho^{(1,2)}=\sum_{j=1}^{N}\rho_j^{(1,2)}$. We set $\rho_i=\rho_i^{(1)}-\rho_i^{(2)}$, $u_i=u_i^{(1)}-u_i^{(2)}$, $i=1,...,N$, $v=v^{(1)}-v^{(2)}$, $\rho=\rho^{(1)}-\rho^{(2)}$.

From (1), (3) we have (see (24))

$$\frac{\partial \rho_i}{\partial t}+\partial_x\left(\rho_i v^{(1)}\right)+\partial_x\left(\rho_i^{(2)} v\right)=0,\quad \rho_i|_{t=0}=0,\quad i=1,\ldots,N. \quad (89)$$

Multiplying (89) by $2\rho_i$ and integrating with respect to x, we obtain

$$\frac{d}{dt}\left(\int_0^1 \rho_i^2\,dx\right)=-\int_0^1\left(\rho_i^2\left(\frac{\partial v^{(1)}}{\partial x}\right)+2\rho_i^{(2)}\rho_i\left(\frac{\partial v}{\partial x}\right)+2\rho_i v\left(\frac{\partial \rho_i^{(2)}}{\partial x}\right)\right)dx,$$
$$i=1,\ldots,N. \quad (90)$$

The terms on the right-hand side of (90) can be estimated as follows:

$$-\int_0^1 \rho_i^2\left(\frac{\partial v^{(1)}}{\partial x}\right)dx\leqslant\left(\sum_{j=1}^{N}\left\|\frac{\partial u_j}{\partial x}^{(1)}\right\|_{L_\infty(0,1)}\right)\left(\int_0^1 \rho_i^2\,dx\right),\quad i=1,\ldots,N,$$

$$- 2 \int\limits_0^1 \rho_i^{(2)} \rho_i \left(\frac{\partial v}{\partial x} \right) dx \leqslant \left\| \rho_i^{(2)} \right\|^2_{L_\infty(Q_T)} \left(\int\limits_0^1 \rho_i^2 \, dx \right) + \\ + N \sum_{j=1}^N \int\limits_0^1 \left(\frac{\partial u_j}{\partial x} \right)^2 dx, \quad i = 1, \ldots, N,$$

$$- 2 \int\limits_0^1 \rho_i v \left(\frac{\partial \rho_i^{(2)}}{\partial x} \right) dx \leqslant \int\limits_0^1 \rho_i^2 \, dx + \left\| \frac{\partial \rho_i^{(2)}}{\partial x} \right\|^2_{L_\infty\left(0,T; L_2(0,1)\right)} \|v\|^2_{L_\infty(0,1)} \leqslant \\ \leqslant \int\limits_0^1 \rho_i \, dx + N \left\| \frac{\partial \rho_i^{(2)}}{\partial x} \right\|^2_{L_\infty\left(0,T; L_2(0,1)\right)} \left(\sum_{j=1}^N \int\limits_0^1 \left(\frac{\partial u_j}{\partial x} \right)^2 dx \right), \; i = 1, \ldots, N.$$

By the inclusions

$$\rho_i^{(2)} \in L_\infty(Q_T), \quad \frac{\partial \rho_i^{(2)}}{\partial x} \in L_\infty\big(0, T; L_2(0,1)\big), \quad i = 1, \ldots, N, \\ \frac{\partial u_i^{(1)}}{\partial x} \in L_2\big(0, T; L_\infty(0,1)\big), \quad i = 1, \ldots, N,$$

we obtain the estimates

$$\frac{d}{dt} \left(\int\limits_0^1 \rho_i^2 \, dx \right) \leqslant C_{49}(t) \int\limits_0^1 \rho_i^2 \, dx + C_{50} \sum_{j=1}^N \int\limits_0^1 \left(\frac{\partial u_j}{\partial x} \right)^2 dx, \quad i = 1, \ldots, N,$$

where $C_{49} = C_{49}\left(\left\{ \left\| \frac{\partial u_i^{(1)}}{\partial x} \right\|_{L_\infty(0,1)} \right\}_{i=1}^N, \left\{ \|\rho_i^{(2)}\|_{L_\infty(Q_T)} \right\}_{i=1}^N \right), C_{49} \in L_2(0,T)$, $C_{50} = C_{50}\left(\left\{ \left\| \frac{\partial \rho_i^{(2)}}{\partial x} \right\|_{L_\infty\left(0,T; L_2(0,1)\right)} \right\}_{i=1}^N, N \right)$. We can apply the Gronwall inequality to get inequalities

$$\int\limits_0^1 \rho_i^2 \, dx \leqslant C_{51} \sum_{j=1}^N \int\limits_0^t \int\limits_0^1 \left(\frac{\partial u_j}{\partial x} \right)^2 dx d\tau, \quad i = 1, \ldots, N, \tag{91}$$

where $C_{51} = C_{51}\left(C_{50}, \|C_{49}\|_{L_1(0,T)} \right)$.

Further, from equations (2) and boundary conditions (4) we obtain (see (28))

$$\frac{1}{2} \frac{d}{dt} \left(\sum_{i=1}^N \int\limits_0^1 \rho_i^{(1)} u_i^2 \, dx \right) + \sum_{i,j=1}^N \nu_{ij} \int\limits_0^1 \left(\frac{\partial u_i}{\partial x} \right) \left(\frac{\partial u_j}{\partial x} \right) dx =$$

$$= K\int_0^1 \rho\left(\frac{\partial v}{\partial x}\right)dx - \sum_{i=1}^N \int_0^1 \rho_i u_i \left(\frac{\partial u_i^{(2)}}{\partial t}\right)dx -$$

$$-\sum_{i=1}^N \int_0^1 \rho_i^{(1)} v u_i \left(\frac{\partial u_i^{(2)}}{\partial x}\right)dx - \sum_{i=1}^N \int_0^1 \rho_i v^{(2)} u_i \left(\frac{\partial u_i^{(2)}}{\partial x}\right)dx,$$

which implies

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^N \int_0^1 \rho_i^{(1)} u_i^2\,dx + \sum_{i,j=1}^N \nu_{ij}\int_0^t\int_0^1 \left(\frac{\partial u_i}{\partial x}\right)\left(\frac{\partial u_j}{\partial x}\right)dx d\tau\right) =$$

$$= K\int_0^1 \rho\left(\frac{\partial v}{\partial x}\right)dx - \sum_{i=1}^N \int_0^1 \rho_i u_i \left(\frac{\partial u_i^{(2)}}{\partial t}\right)dx -$$

$$-\sum_{i=1}^N \int_0^1 \rho_i^{(1)} v u_i \left(\frac{\partial u_i^{(2)}}{\partial x}\right)dx - \sum_{i=1}^N \int_0^1 \rho_i v^{(2)} u_i \left(\frac{\partial u_i^{(2)}}{\partial x}\right)dx. \quad (92)$$

We estimate the terms on the right-hand side of (92) as follows:

$$K\int_0^1 \rho\left(\frac{\partial v}{\partial x}\right)dx \leqslant \frac{C_1}{4}\sum_{i=1}^N \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx + C_{52}\left(C_1, K, N\right)\sum_{i=1}^N \int_0^1 \rho_i^2\,dx,$$

$$-\sum_{i=1}^N \int_0^1 \rho_i u_i \left(\frac{\partial u_i^{(2)}}{\partial t}\right)dx \leqslant \frac{C_1}{4}\sum_{i=1}^N \int_0^1 \left(\frac{\partial u_i}{\partial x}\right)^2 dx +$$

$$+ C_{53}\left(C_1, \left\{\left\|\frac{\partial u_i^{(2)}}{\partial t}\right\|_{L_2(0,1)}\right\}_{i=1}^N\right)\sum_{i=1}^N \int_0^1 \rho_i^2\,dx,$$

$$-\sum_{i=1}^N \int_0^1 \rho_i^{(1)} v u_i \left(\frac{\partial u_i^{(2)}}{\partial x}\right)dx \leqslant$$

$$\leqslant C_{54}\left(\left\{\left\|\frac{\partial u_i^{(2)}}{\partial x}\right\|_{L_\infty(0,1)}\right\}_{i=1}^N, N\right)\frac{1}{2}\sum_{i=1}^N \int_0^1 \rho_i^{(1)} u_i^2\,dx,$$

$$-\sum_{i=1}^N \int_0^1 \rho_i v^{(2)} u_i \left(\frac{\partial u_i^{(2)}}{\partial x}\right)dx \leqslant C_{55}\left(\left\{\|u_i^{(2)}\|_{L_\infty(Q_T)}\right\}_{i=1}^N, N\right)\sum_{i=1}^N \int_0^1 \rho_i^2\,dx +$$

$$+ C_{56}\left(\left\{\left\|\frac{1}{\rho_i^{(1)}}\right\|_{L_\infty(Q_T)}\right\}_{i=1}^{N}, \left\{\left\|\frac{\partial u_i^{(2)}}{\partial x}\right\|_{L_\infty(0,1)}\right\}_{i=1}^{N}\right)\frac{1}{2}\sum_{i=1}^{N}\int\limits_0^1 \rho_i^{(1)} u_i^2\,dx,$$

where C_{53}, $C_{56} \in L_1(0,T)$, $C_{54} \in L_2(0,T)$. Hence, from (92), using the relation

$$\sum_{i=1}^{N}\int\limits_0^1 \rho_i^2\,dx \leqslant C_{57}(C_1, C_{51}, N)\frac{C_1}{2}\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1\left(\frac{\partial u_i}{\partial x}\right)^2 dxd\tau$$

proved above (see (91)), we deduce

$$\frac{1}{2}\sum_{i=1}^{N}\int\limits_0^1 \rho_i^{(1)} u_i^2\,dx + \frac{C_1}{2}\sum_{i=1}^{N}\int\limits_0^t\int\limits_0^1\left(\frac{\partial u_i}{\partial x}\right)^2 dxd\tau \leqslant$$
$$\leqslant \int\limits_0^t C_{58}(C_{52},\ldots,C_{57})\left(\frac{1}{2}\sum_{i=1}^{N}\int\limits_0^1 \rho_i^{(1)} u_i^2\,dx + \frac{C_1}{2}\sum_{i=1}^{N}\int\limits_0^\tau\int\limits_0^1\left(\frac{\partial u_i}{\partial x}\right)^2 dxds\right)d\tau,$$

where $C_{58} \in L_1(0,T)$, which yields the identities $\rho_i \equiv 0$, $u_i \equiv 0$, $i = 1,\ldots,N$. Theorem 1 is proved.

References

[1] K.R. Rajagopal, L. Tao, *Mechanics of mixtures*, World Scientific, Singapore, 1995. Zbl 0941.74500

[2] R.I. Nigmatulin, *Dynamics of multiphase media: V. 1-2*, Hemisphere, New York, 1990.

[3] A.E. Mamontov, D.A. Prokudin, *Viscous compressible homogeneous multi-fluids with multiple velocities: barotropic existence theory*, Sib. Èlektron. Mat. Izv., **14** (2017), 388–397. Zbl 1379.35248

[4] A.E. Mamontov, D.A. Prokudin, *Viscous compressible multi-fluids: modeling and multi-D existence*, Methods Appl. Anal., **20**:2 (2013), 179–196. Zbl 1290.35203

[5] A.D Kirwan, M. Massoudi, *The heat flux vector(s) in a two component fluid mixture*, Fluids, **5**:2 (2020), Article ID 77.

[6] R.T. Lee, K.T. Yang, Y.C. Chiou, *A novel model for a mixed-film lubrication with oil-in-water emulsions*, Tribology Intern., **66** (2013), 241–248.

[7] J. Málek, K.R. Rajagopal, *A thermodynamic framework for a mixture of two liquids*, Nonlinear Anal., Real World Appl., **9**:4 (2008), 1649–1660. Zbl 1154.76311

[8] R.J. Atkin, R.E. Craine, *Continuum theories of mixtures: Applications*, J. Inst. Math. Appl., **17**:2 (1976), 153–207. Zbl 0355.76004

[9] I. Müller, *A thermodynamic theory of mixtures of fluids*, Arch. Ration. Mech. Anal., **28** (1968), 1–39. Zbl 0157.56703

[10] I. Straškraba, A. Zlotnik, *Global properties of solutions to 1D-viscous compressible barotropic fluid equations with density dependent viscosity*, Z. Angew. Math. Phys., **54**:4 (2003), 593–607. Zbl 1040.35064

[11] I. Straškraba, A. Zlotnik, *Global behavior of 1d-viscous compressible barotropic fluid with a free boundary and large data*, J. Math. Fluid Mech., **5**:2 (2003), 119–143. Zbl 1042.76050

[12] A.A. Amosov, A.A. Zlotnik, *Uniqueness and stability of generalized solutions to quasiaveraged equations of one-dimensional motion of a viscous barotropic medium*, Differ. Equations, **31**:7 (1995), 1056–1066. Zbl 0864.76079

[13] A.A. Zlotnik, N.Z. Bao, *Properties and asymptotic behavior of solutions of some problems of one-dimensional motion of a viscous barotropic gas*, Math Notes, **55**:5 (1994), 471–482. Zbl 0828.76070

[14] S. Jiang, *On initial boundary value problems for a viscous heat-conducting, one-dimensional real gas*, J. Differ. Equations, **110**:2 (1994), 157–181. Zbl 0805.35074

[15] S. Yanagi, *Global existence for one-dimensional motion of non-isentropic viscous fluids*, Math. Methods Appl. Sci., **16**:9 (1993), 609–624. Zbl 0780.35082

[16] D. Hoff, *Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data*, J. Differ. Equations, **95**:1 (1992), 33–74. Zbl 0762.35085

[17] V.A. Vaigant, *On the Cauchy problem for the system of equations of a viscous gas*, Din. Splosh. Sredy, **102** (1991), 31–40. Zbl 0805.76065

[18] A.A. Amosov, A.A. Zlotnik, *Solvability «in the large» of a system of equations of the one-dimensional motion of an inhomogeneous viscous heat-conducting gas*, Math. Notes, **52**:2 (1992), 753–763. Zbl 0779.76079

[19] A.A. Zlotnik, *On equations for one-dimensional motion of a viscous barotropic gas in the presence of a body force*, Sib. Math. J., **33**:5 (1992), 798–815. Zbl 0811.76075

[20] D. Hoff, *Discontinuous solutions of the Navier-Stokes equations for compressible flow*, Arch. Ration. Mech. Anal., **114**:1 (1991), 15–46. Zbl 0732.35071

[21] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*, Studies in Mathematics and its Applications, **22**, North-Holland, Amsterdam etc., 1990. Zbl 0696.76001

[22] V.A. Vaigant, *Nonhomogeneous boundary value problems for the equation of a viscous heat-conducting gas*, Din. Splosh. Sredy, **97** (1990), 3–21. Zbl 0743.76063

[23] T. Nagasawa, *On the one-dimensional free boundary problem for the heat-conductive compressible viscous gas*, North-Holland Mathematics Studies, **160** (1989), 83–99. Zbl 0712.35113

[24] A.A. Zlotnik, A.A. Amosov, *Generalized global solutions of the equations of the one-dimensional motion of a viscous barotropic gas*, Sov. Math., Dokl., **37**:2 (1988), 554–558. Zbl 0682.76061

[25] V.A. Vaigant, A.A. Papin, *Solvability of the initial-boundary value problem for equations of barotropic gas with viscosity depending on density*, Din. Splosh. Sredy, **79** (1987), 3–9. Zbl 0684.76085

[26] T. Nagasawa, *On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary*, J. Differ. Equations, **65**:1 (1986), 49–67. Zbl 0598.34021

[27] B. Kawohl, *Global existence of large solutions to initial boundary value problem for a viscous, heat-conducting, one-dimensional real gas*, J. Differ. Equations, **58**:1 (1985), 76–103. Zbl 0579.35052

[28] M. Okada, S. Kawashima, *On the equations of one-dimensional motion of compressible viscous fluids*, J. Math. Kyoto Univ., **23**:1 (1983), 55–71. Zbl 0529.76070

[29] S.Ya. Belov, *On the flow problem for the system of equations of the one-dimensional motion of viscous heat conducting gas*, Din. Splosh. Sredy, **56** (1982), 22–43. Zbl 0562.76071

[30] A.V. Kazhikhov, *Cauchy problem for viscous gas equations*, Sib. Math. J., **23** (1982), 44–49. Zbl 0519.35065

[31] S. Kawashima, T. Nishida, *Global solutions to the initial value problems for the equations of one-dimensional motion of viscous polytropic gases*, J. Math. Kyoto Univ., **21**:4 (1981), 825–837. Zbl 0478.76097

[32] A. Matsumura, T. Nishida, *The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids*, Proc. Japan Acad. Ser. A, Math. Sci., **55**:9 (1979), 337–342. Zbl 0447.76053

[33] V.V. Shelukhin, *Periodic flows of viscous gas*, Din. Splosh. Sredy, **42** (1979), 80–102. MR0602221

[34] A.V. Kazhikhov, *The global solvability of one-dimensional boundary value problems for the equations of a viscous heat-carrying gas*, Din. Splosh. Sredy, **24** (1976), 45–61. MR0459229

[35] A.V. Kazhikhov, *Correctness «in general» of mixed boundary value problems for a model system of equations of a viscous gas*, Din. Splosh. Sredy, **21** (1975), 18–47. MR0478984

[36] A.E. Mamontov, D.A. Prokudin, *Global unique solvability of the initial-boundary value problem for the equations of one-dimensional polytropic flows of viscous compressible multifluids*, J. Math. Fluid Mech., **21**:1 (2019), Paper No. 9. Zbl 1411.76150

[37] A.E. Mamontov, D.A. Prokudin, *Unique solvability of initial-boundary value problem for one-dimensional equations of polytropic flows of multicomponent viscous compressible fluids*, Sib. Èlektron. Mat. Izv., **15** (2018), 631–649. Zbl 1428.35387

[38] A.E. Mamontov, D.A. Prokudin, *Asymptotic behavior of the solution to the initial-boundary value problem for one-dimensional motions of a barotropic compressible viscous multifluid*, Lobachevskii J. Math., **45**:4 (2024), 1463–1471. Zbl 7940773

[39] D.A. Prokudin *On the stabilization of the solution to the initial-boundary value problem for one-dimensional isothermal equations of viscous compressible multicomponent media dynamics*, Mathematics, **11**:14 (2023), Article ID 3065.

[40] A.E. Mamontov, D.A. Prokudin, *Global unique solvability of an initial-boundary value problem for the one-dimensional barotropic equations of binary mixtures of viscous compressible fluids*, J. Appl. Ind. Math., **15**:1 (2021), 50–61. Zbl 1511.35283

[41] D.A. Prokudin, *On the stabilization of solutions to the initial-boundary value problem for the equations of dynamics of viscous compressible multicomponent media*, Sib. Èlektron. Mat. Izv., **18**:2 (2021), 1278–1285. Zbl 1479.35693

[42] D.A. Prokudin, *Unique solvability of initial-boundary value problem for a model system of equations for the polytropic motion of a mixture of viscous compressible fluids*, Sib. Èlektron. Mat. Izv., **14** (2017), 568–585. Zbl 1375.35007

[43] D.A. Prokudin, *Global solvability of the initial boundary value problem for a model system of one-dimensional equations of polytropic flows of viscous compressible fluid mixtures*, J. Phys.: Conf. Ser., **894** (2017), Article ID 012076.

[44] S. Li, *On one-dimensional compressible Navier-Stokes equations for a reacting mixture in unbounded domains*, Z. Angew. Math. Phys., **68**:5 (2017), Paper No. 106. Zbl 1378.35220

[45] D. Bresch, X. Huang, J. Li, *Global weak solutions to one-dimensional non-conservative viscous compressible two-phase system*, Commun. Math. Phys., **309**:3 (2012), 737–755. Zbl 1235.76182

[46] A.A. Papin, *On the uniqueness of the solutions of an initial boundary-value problem for the system of a heat-conducting two-phase mixture*, Math. Notes, **87**:4 (2010), 594–598. Zbl 1197.35247

[47] A.A. Zlotnik, *Weak solutions to the equations of motion of viscous compressible reacting binary mixtures: uniqueness and Lipschitz-continuous dependence on data*, Math. Notes, **75**:2 (2004), 278–283. Zbl 1122.35118

[48] A.A. Zlotnik, *Uniform estimates and stabilization of solutions to equations of one-dimensional motion of a multicomponent barotropic mixture*, Math. Notes, **58**:2 (1995), 885–889. Zbl 0852.76078

[49] A.N. Petrov, *Well-posedness of initial-boundary value problems for one-dimensional equations of the interpenetrating motion of perfect gases*, Din. Splosh. Sredy, **56** (1982), 105–121. MR0706350

[50] A.V. Kazhikov, A.N. Petrov, *Well-posedness of the initial-boundary value problem for a model system of equations of a multicomponent mixture*, Din. Splosh. Sredy, **35** (1978), 61–73. MR0539168

[51] D.A. Zakora, *Spectral properties of the operator in the problem of oscillations in a mixture of viscous compressible fluids*, Differ. Equ., **59**:4 (2023), 473–490. Zbl 1530.35236

[52] D.A. Zakora, *The problem on small motions of a mixture of viscous compressible fluids*, Sib. Èlektron. Mat. Izv., **20**:2 (2023), 1552–1589. Zbl 1543.76095

[53] R.J. DiPerna, P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., **98**:3 (1989), 511–547. Zbl 0696.34049

[54] A. Novotný, I. Straškraba, *Introduction to the mathematical theory of compressible flow*, Oxford University Press, Oxford, 2004. Zbl 1088.35051

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