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RINGS WITH THE 2-ΔU PROPERTY

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Abstract: Rings in which the square of each unit lies in $1+\Delta(R)$ are said to be 2-ΔU *rings*, where $J(R) \subseteq \Delta(R) =: \{r \in R \mid r + U(R) \subseteq U(R)\}$. The set $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by units of R and is detailed studied in [21]. The class of 2-ΔU rings consists several rings including UJ-rings, 2-UJ rings and ΔU-rings, respectively, and we observe that ΔU-rings are UUC in terms of [2]. Furthermore, the structure of 2-ΔU rings is examined under various algebraic conditions. Moreover, the 2-ΔU property is explored under some extended constructions.

The established by us achievements substantially improved on the existing in the literature relevant results.

Keywords: $\Delta(R)$, ΔU ring, 2-ΔU ring, Matrix ring.

1 Introduction and Basic Concepts

In the current paper, let R denote an associative not necessarily commutative ring with identity element. Typically, for such a ring R, the sets $U(R)$, $Nil(R)$, $C(R)$, $Id(R)$ and $J(R)$ represent the set of invertible elements in R, the set of nilpotent elements in R, the set of central elements in R, the set of idempotent elements in R and the Jacobson radical of R, respectively.

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Additionally, the ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are, respectively, denoted by $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$. Traditionally, a ring is termed *abelian* if each idempotent element is central, meaning that $Id(R) \subseteq C(R)$.

The key instrument of the present study is the set $\Delta(R)$ which was handled by Lam in [20, Exercise 4.24] and recently investigated by Leroy-Matczuk in [21]. As pointed out by the authors in [21, Theorem 3 and 6], the subring $\Delta(R)$ is the largest Jacobson radical of R that is closed with respect to multiplication by all units (quasi-invertible elements) of R. Also, $J(R) \subseteq \Delta(R)$, and $\Delta(R) = J(T)$, where T is the subring of R generated by units of R, and the equality $\Delta(R) = J(R)$ holds if, and only if, $\Delta(R)$ is an ideal of R.

It is well known that $1 + J(R) \subseteq U(R)$. A ring R is said to be an *UJ-ring* if the reverse inclusion holds, i.e., $U(R) = 1 + J(R)$ (see [7] and [14]). Imitating [6], a ring R is said to be *2-UJ* if, for each $u \in U(R)$, $u^2 = 1 + j$, where $j \in J(R)$. These rings are a common generalization of UJ rings. The authors showed there that for 2-UJ rings the notions of being semi-regular, exchange and clean rings are all equivalent.

In the other vein, recall that a ring R is called an *UU-ring* if $U(R) = 1 + Nil(R)$ (see, e.g., [9]). As a natural expansion of UU rings, Sheibani and Chen introduced in [25] the so-called *2-UU rings* – a ring R is called 2-UU if the square of every unit is the sum of 1_R and a nilpotent. They showed that R is strongly 2-nil-clean if, and only if, R is an exchange 2-UU ring.

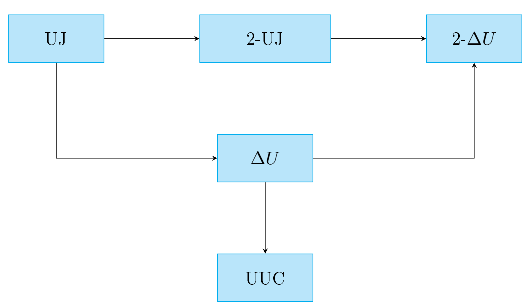
Let us also recollect certain classical concepts, needed for our successful presentation: a ring R is known to be *Boolean* if every element of R is idempotent. Also, as a more general setting, a ring R is said to be *regular* (resp., *unit-regular*) in the sense of von Neumann if, for every $a \in R$, there is an $x \in R$ (resp., $x \in U(R)$) such that $axa = a$ and, in addition, R is said to be *strongly regular* if, for each $a \in R$, $a \in a^2R$. Recall that a ring R is *exchange* if, for each $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$, and a ring R is *clean* if every element of R is a sum of an idempotent and an unit (cf. [23]). Notice that every clean ring is exchange, but the converse is manifestly *not* true in general; however, it is true in the abelian case (see, for more details, [23, Proposition 1.8]). Likewise, a ring R is called *semi-regular*, provided $R/J(R)$ is regular and idempotents lift modulo $J(R)$. Note that semi-regular rings are exchange, but the opposite is generally *not* valid (see [23]).

Further, according to Chen ([3]), an element of a ring is called *J-clean*, provided that it can be written as the sum of an idempotent and an element from its Jacobson radical. Accordingly, a ring is termed *J-clean* in the case when each of its elements is J-clean or, equivalently, $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ (which is also called *semi-boolean* in the language of [24]). It was shown in [14] that a ring R is J-clean if, and only if, R is a clean UJ ring. Later on, in 2019, Karabacak et al. introduced new rings that are a non-trivial generalization of UJ rings; in fact, they named these rings ΔU (see [11]) and R is said to be a ΔU *ring* if $1+\Delta(R)=U(R)$. Besides, due to Karabacak et al. ([11]), a ring R is called Δ-*clean*, provided every element of R is a sum of an idempotent and an element from the $\Delta(R)$. Thus, Δ-clean rings are clean, but the reciprocality is *not* fulfilled in all generality. They also showed in [11] that a ring R is Δ-clean if, and only if, R is a clean ΔU ring. Some other interesting results close to this material can be found in [15] as well.

As a proper expansion of some of the above concepts, we introduce the new class of *2-ΔU rings* as follows: a ring R is called 2-ΔU if the square of each unit is a sum of an idempotent and an element from the $\Delta(R)$ (or, in an equivalent form, for each $u \in U(R)$, $u^2 = 1+r$, where $r \in \Delta(R)$). Clearly, all ΔU rings, and hence the unit uniquely clean rings from [2] as well as the rings with only two units are 2-ΔU. Also, 2-UJ rings and hence UJ rings are 2-ΔU, but the converse does *not* hold in general. Our motivating tool is to give a satisfactory description of these 2-ΔU rings by comparing their crucial properties with these of 2-UU and 2-UJ rings, respectively, as well as to find some new exotic properties of 2-ΔU rings that are not too characteristically or frequently seen in the existing literature.

We are now planning to give a brief program of our main material established in the sequel: In Section 2, we achieve to exhibit some major properties and characterizations of ΔU rings in various different aspects (see, for instance, Propositions 1, 2 and 3 and Theorem 1). In Section 3, we establish some fundamental characterizing properties of 2-ΔU rings that are mainly stated and proved in Theorems 9, 2, 3 and 4 and the other statements associated with them. In Section 4, we give some extensions of 2-ΔU rings; for instance, polynomial extensions, matrix extensions, trivial extensions and Morita contexts. We close our work in the final Section 5 with challenging questions, namely Problems 1, 2, 3 and 4.

Now, we have the following diagram which violates the relationships between the defined above sorts of rings:



2 ΔU rings

In this section, we reinvestigate some major properties of ΔU rings which were *not* found in [11], as well as we give a few close relations between ΔU rings and some related type of rings.

Definition 1 ([11]). *A ring R is called ΔU if $1+\Delta(R)=U(R)$.*

Mimicking Calugareanu and Zhou (see [2]), a ring R is called UUC if every unit is uniquely clean.

We start with a series of preliminaries.

Proposition 1. *Let R be a ΔU ring. Then, the following three points hold:*

(1) $U(R)+U(R)\subseteq\Delta(R)$.
(2) *R is a UUC ring.*
(3) $(U(R)+U(R))\cap Id(R)=\{0\}$.

Proof. (1) Choose $x\in U(R)+U(R)$. So, $x=u_1+u_2$, where $u_1,u_2\in U(R)=1+\Delta(R)$ yielding $x=1+r_1+1+r_2=2+(r_1+r_2)$, where $r_1,r_2\in\Delta(R)$. On the other hand, we know that $2\in\Delta(R)$ by [11, Proposition 2.4]. But, $\Delta(R)$ is a subring of R and thus $x\in\Delta(R)$, as required.

(2) Assume that $u=e+v$, where $u,v\in U(R)$ and $e\in Id(R)$. It suffices to show that $e=0$. To this target, as R is ΔU, we may write $u=1+r$ and $v=1+r'$, where $r,r'\in\Delta(R)$. Hence, $e=0$ in view [21, Proposition 15], as needed.

(3) This is clear combining (i) and (ii).

□

Proposition 2. *Let R be a ΔU ring and set $\bar{R}:=R/J(R)$. The following two items hold:*

(1) *For any $u_1,u_2\in U(R)$, $u_1+u_2\neq 1$.*
(2) *For any $\bar{u_1},\bar{u_2}\in U(\bar{R})$, $\bar{u_1}+\bar{u_2}\neq\bar{1}$.*

Proof. (1) This is immediate utilizing Proposition 1 and [2, Example 2.2].

(2) Since R is ΔU, one sees that $\bar{R}$ is ΔU and hence it is UUC. Now, the result follows directly from [2, Example 2.2].

□

Remember for completeness of the exposition that a ring R is called *semi-potent* if every one-sided ideal *not* contained in $J(R)$ contains a non-zero idempotent. Moreover, a semi-potent ring R is called *potent*, provided all idempotents lift modulo $J(R)$.

Proposition 3. *Let R be a potent ΔU ring and set $\bar{R}:=R/J(R)$. Then, we have:*

(1) *For any $\bar{e}=\bar{e}^2\in\bar{R}$ and any $\bar{u_1},\bar{u_2}\in U(\bar{e}\bar{R}\bar{e})$, $\bar{u_1}+\bar{u_2}\neq\bar{e}$.*
(2) *There does not exist $\bar{e}=\bar{e}^2\in\bar{R}$ such that $\bar{e}\bar{R}\bar{e}\cong M_2(S)$ for some ring S.*

Proof. (1) Given $\bar{e},\bar{u_1},\bar{u_2}$ as in (i), we can assume that $e^2=e\in R$, because idempotents lift modulo $J(R)$. Thus, $\bar{e}\bar{R}\bar{e}\cong eRe/J(eRe)$. Finally, since eRe is ΔU in virtue of [11, Proposition 2.6], (i) follows automatically from Proposition 2(i).

(2) Note that, in a 2×2 matrix ring, it is always valid that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in U(M_2(S)) + U(M_2(S)).$$

Hence, there exist $\bar{u_1}, \bar{u_2} \in U(\bar{e}\bar{R}\bar{e})$ such that $\bar{u_1} + \bar{u_2} = \bar{e}$. This, however, is a contradiction with (i), as expected.

□

A ring R is called *reduced* if it contains no non-zero nilpotent elements, that is, $Nil(R) = (0)$.

We now come to the following criterion.

Theorem 1. *Let R be a semi-potent ring. Then, the following statements are equivalent:*

(1) *R is a ΔU ring.*
(2) *$R/J(R)$ is Boolean.*
(3) *R is a UJ ring.*
(4) *$R/J(R)$ is a UU ring.*

Proof. $(i) \Rightarrow (ii)$. Since R is semi-potent, $R/J(R)$ is too semi-potent (and, indeed, even potent). Also, $R/J(R)$ is ΔU. So, without loss of generality, it can be assumed that $J(R) = (0)$. Thus, using [11, Theorem 4.4], R is reduced and hence abelian.

Now, assume that there exists $x \in R$ such that $x - x^2 \neq 0$ in R. Since R is a semi-potent ring, there exists $e = e^2 \in R$ such that $e \in (x - x^2)R$. So, write $e = (x - x^2)y$ for some $y \in R$. Since e is central, it must be that

$$[er(1-e)]^2 = 0 = [(1-e)re]^2,$$

whence we have

$$er(1-e) = 0 = e(1-e)re.$$

We now can write:

$$e = ex.e(1-e).ey,$$

so that both $ex, e(1-x) \in U(eRe)$. But, we know that eRe is a ΔU ring. However, $ex + e(1-x) = e$, which contradicts Proposition 2(i). Therefore, R is a Boolean ring, as desired.

$(ii) \Rightarrow (iii)$. Let us assume $u \in U(R)$. Then, $\bar{u} \in U(\bar{R})$, where $\bar{R} = R/J(R))$. But, since $\bar{R}$ is a Boolean ring, we obtain $\bar{u} = \bar{1}$ which implies $u - 1 \in J(R)$, as required.

$(iii) \Rightarrow (i)$. This is evident, because we know that always $J(R) \subseteq \Delta(R)$.

$(ii) \Rightarrow (iv)$. This is obvious, so we omit the necessary arguments.

$(iv) \Rightarrow (ii)$. Knowing that $R/J(R)$ is semi-potent, one derives that it is also strongly nil-clean applying [12, Theorem 2.25]. Hence, $R/J(R)$ is an exchange UU ring consulting with [9, Theorem 4.3]. In conclusion, it is Boolean (see [9, Theorem 4.1]), as wanted. □

As five consequences, we deduce:

Corollary 1. *A regular ring R is ΔU if, and only if, R is UJ if, and only if, R is UU if, and only if, R is Boolean.*

Proof. Since R is regular, $J(R) = (0)$ and R is semi-potent. So, the result follows from Theorem 1. □

Corollary 2. *Let R be a potent ring. Then, the following are equivalent:*

(1) *R is a ΔU ring.*
(2) *$R/J(R)$ is a ΔU ring.*
(3) *$R/J(R)$ is a Boolean ring.*
(4) *R is a UJ ring.*
(5) *$R/J(R)$ is a UJ ring.*
(6) *$R/J(R)$ is a UU ring.*

Proof. We know that every potent ring is semi-potent. Then, issues (i), (iii), (iv) and (vi) are equivalent invoking Theorem 1. On the other side, issues (i) and (ii) are equivalent employing [11, Proposition 2.4]. Finally, issues (iv) and (v) are tantamount in conjunction with [14, Proposition 1.3]. □

Corollary 3. *Let R be an Artinian ring. Then, the following are equivalent:*

(1) *R is a ΔU ring.*
(2) *R is a UJ ring.*
(3) *R is a UU ring.*

Proof. We know that every Artinian ring is always clean. Also, since R is Artinian, we have $J(R) \subseteq Nil(R)$. □

Corollary 4. *Let R be a finite ring. Then, the following conditions are equivalent:*

(1) *R is a ΔU ring.*
(2) *R is a UJ ring.*
(3) *R is a UU ring.*

Proof. In fact, any finite ring is known to be Artinian. □

Corollary 5. *For a ring R, the following two conditions are equivalent:*

(1) *R is a potent ΔU ring.*
(2) *R is a J-clean ring.*

Proof. $(ii) \Rightarrow (i)$. This is apparent by a combination of [14, Theorem 3.2] and Corollary 2.

$(i) \Rightarrow (ii)$. Thanks to Corollary 2, we infer that $R/J(R)$ is Boolean. Therefore, for each $a \in R$, we receive $a - a^2 \in J(R)$. Besides, since R is a potent ring, there exists an idempotent $e \in R$ such that $a - e \in J(R)$. Thus, R is a J-clean ring, as promised. □

Let $Nil_*(R)$ denote the *prime* radical (or, in other terms, the *lower* nil-radical) of a ring R, i.e., the intersection of all prime ideals of R. We know that $Nil_*(R)$ is a nil-ideal of R. It is also long known that a ring R is called 2-*primal* if its lower nil-radical $Nil_*(R)$ consists precisely of all the nilpotent elements of R. For instance, it is well known that reduced rings and commutative rings are both 2-primal.

For an endomorphism α of a ring R, R is called α-*compatible* if, for any $a, b \in R$,

$$ab = 0 \iff a\alpha(b) = 0,$$

and in this case α is clearly injective.

Let R be a ring and $\alpha : R \to R$ a ring endomorphism; then, $R[x;\alpha]$ denotes the *skew polynomial ring* over R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[x] = R[x; 1_R]$ is the ordinary *polynomial ring* over R.

Our next result of interest is the following.

Proposition 4. *Let R be a 2-primal ring and let α be an endomorphism of R. If R is α-compatible, then*

$$\Delta(R[x;\alpha]) = \Delta(R) + J(R[x;\alpha]).$$

Proof. Suppose first that R is a reduced ring. As R is α-compatible, [4, Corollary 2.12] applies to get that $U(R[x;\alpha]) = U(R)$. Also, it is easy to see that $\Delta(R) \subseteq \Delta(R[x;\alpha])$. We claim that this is exactly an equality. In fact, let $r + r_0 \in \Delta(R[x;\alpha])$, where $r \in R[x;\alpha]x$ and $r_0 \in R$. Then, for any $u \in U(R)$, $r + r_0 + u \in U(R)$. This shows that $r = 0$ and $r_0 + u \in U(R)$. Thus, we conclude that $\Delta(R[x;\alpha]) \subseteq \Delta(R)$ and hence $\Delta(R[x;\alpha]) = \Delta(R)$, as claimed.

Now assume that R is 2-primal. Obviously,

$$Nil_*(R[x;\alpha]) = Nil_*(R)[x;\alpha] \subseteq J(R[x;\alpha])$$

consulting with [4, Lemma 2.2]. As R is 2-primal, $R/Nil_*(R)$ is reduced, and so we arrive at

$$J(R[x;\alpha]) = \mathrm{Nil}_*(R[x;\alpha]) = \mathrm{Nil}_*(R)[x;\alpha].$$

By the first part of the proof applied to $R/Nil_*(R)$ and referring to [21, Proposition 6(3)], we deduce that

$$\Delta(R) + Nil_*(R)[x;\alpha] = \Delta(\frac{R}{Nil_*(R)}[x;\alpha]) = \Delta(\frac{R[x;\alpha]}{J(R[x;\alpha])}) = \frac{\Delta(R[x;\alpha])}{J(R[x;\alpha])}.$$

Now, summarizing all the above, we conclude the pursued equality. □

It is principally known that, for any two elements $a, b \in R$, $1 - ab$ is a unit if, and only if, $1 - ba$ is a unit. This result is attributed to as *Jacobson's lemma* for units. There are several analogous results in the literature as well.

We now have the validity of the following.

Corollary 6. *Let R be a ΔU ring and let $a, b \in R$. Then, $1 - ab \in \Delta(R)$ if, and only if, $1 - ba \in \Delta(R)$.*

Proof. Assuming that $1 - ab \in \Delta(R)$, we can write $ab \in U(R) = 1 + \Delta(R)$. Therefore, [11, Proposition 2.4] is applicable to get that $a \in U(R)$. Thus,

$$1 - ba = a^{-1}(1 - ab)a \in \Delta(R),$$

because $\Delta(R)$ is closed with respect to multiplication by all units (see [21, Theorem 3]). The converse implication is similar, concluding the proof. □

3 2-ΔU rings

In this section, we introduce the concept of 2-ΔU rings and investigate their elementary properties.

We now give our main definition.

Definition 2. *A ring R is called 2-ΔU if the square of each unit is a sum of an idempotent and an element from the $\Delta(R)$ (equivalently, for each $u \in U(R)$, $u^2 = 1 + r$, where $r \in \Delta(R)$).*

two more constructions clarify the given definition a bit more.

Example 1. *Unambiguously, 2-UJ rings are 2-ΔU. But, the converse is definitely not true in general. For example, consider the ring $R = \mathbb{F}_2\langle x, y\rangle/\langle x^2\rangle$. Thus, one calculates that $J(R) = (0)$, $\Delta(R) = \mathbb{F}_2 x + xRx$ and $U(R) = 1 + \mathbb{F}_2 x + xRx$. Therefore, R is ΔU by* [11, Example 2.2]*, and hence it is 2-ΔU. But, it is readily seen that R is not 2-UJ, as asserted.*

Example 2. *The ring $\mathbb{Z}_3$ is 2-ΔU, but is not ΔU.*

We are now in a position to explore some critical properties of the newly defined notion.

Proposition 5. *A direct product $\prod_{i \in I} R_i$ of rings is 2-ΔU if, and only if, each direct component R_i is 2-ΔU.*

Proof. As $\Delta(\prod_{i \in I} R_i) = \prod_{i \in I} \Delta(R_i)$ and $U(\prod_{i \in I} R_i) = \prod_{i \in I} U(R_i)$, the result follows without any difficult. □

Proposition 6. *Let R be a 2-ΔU ring. If T is a factor ring of R such that all units of T lift to units of R, then T is 2-ΔU.*

Proof. Suppose that $f : R \to T$ is a ring epimorphism. Choosing $v \in U(T)$, there exists $u \in U(R)$ such that $v = f(u)$ and $u^2 = 1 + r \in 1 + \Delta(R)$. Thus,

$$v^2 = (f(u))^2 = f(u^2) = f(1 + r) = f(1) + f(r) = 1 + f(r) \in 1 + \Delta(T),$$

as required. □

Example 3. *A division ring R is 2-ΔU if, and only if, either $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*

Proof. Since R is a division ring, one has that $\Delta(R) = 0$, and the result follows from [6, Example 2.1]. □

Remark 1. *The condition "all units of T lift to units of R"in Proposition 6 is necessary and cannot be ignored. Indeed, the ring $\mathbb{Z}_7$ is a factor ring of the 2-ΔU ring $\mathbb{Z}$. But, $\mathbb{Z}_7$ is not 2-ΔU by Example 3. Note that not all of units of $\mathbb{Z}_7$ can lift to units of $\mathbb{Z}$.*

Proposition 7. *Let R be a 2-ΔU ring. For a unital subring S of R, if $S \cap \Delta(R) \subseteq \Delta(S)$, then S is a 2-ΔU ring. In particular, the center of R is a 2-ΔU ring.*

Proof. Let $v \in U(S) \subseteq U(R)$. Since R is 2-ΔU, we have $v^2 - 1 \in \Delta(R) \cap S \subseteq \Delta(S)$. So, S is a 2-ΔU ring. Now, the rest follows from [21, Corollary 8]. □

Proposition 8. *Let R be a 2-ΔU ring and $2 \in \Delta(R)$. Then, the following two relations are fulfilled:*

(1) $(U(R))^2 + (U(R))^2 \subseteq \Delta(R)$.
(2) $[(U(R))^2 + (U(R))^2] \cap Id(R) = \{0\}$.

Proof. (1) Let $t \in (U(R))^2 + (U(R))^2$, so $t = u^2 + v^2$, where $u, v \in U(R)$. Since R is 2-ΔU, one may write that $t = 1 + r + 1 + s$, where $r, s \in \Delta(R)$. So, we have $t = 2 + (r + s)$. But, $2 \in \Delta(R)$ and $\Delta(R)$ is a subring of R, so that $t \in \Delta(R)$ follows.
(2) It follows immediately from (i) and [21, Proposition 15].

□

Proposition 9. *Let $I \subseteq J(R)$ be an ideal of a ring R. Then, R is 2-ΔU if, and only if, so is R/I.*

Proof. Let R be a 2-ΔU ring and $u + I \in U(R/I)$. Thus, $u \in U(R)$ and hence $u^2 = 1 + r$, where $r \in \Delta(R)$. Therefore,

$$(u + I)^2 = u^2 + I = (1 + I) + (r + I),$$

where $r + I \in \Delta(R)/I = \Delta(R/I)$ in view of [21, Proposition 6].

Conversely, let R/I be a 2-ΔU ring and $u \in U(R)$. Thus, $u + I \in U(R/I)$ whence $(u + I)^2 = (1 + I) + (r + I)$, where $r + I \in \Delta(R/I)$. This means that $u^2 + I = (1 + r) + I$. So,

$$u^2 - (1 + r) \in I \subseteq J(R) \subseteq \Delta(R).$$

Consequently, $u^2 = 1 + r'$, where $r' \in \Delta(R)$. Hence, R is a 2-ΔU ring. □

Corollary 7. *A ring R is 2-ΔU if, and only if, $R/J(R)$ is 2-ΔU.*

Proposition 10. *Let R be a 2-ΔU ring and e an idempotent of R. Then, eRe is too a 2-ΔU ring.*

Proof. Letting $u \in U(eRe)$, we have $u + (1 - e) \in U(R)$. Under validity of the hypothesis,

$$(u + (1 - e))^2 = u^2 + (1 - e) = 1 + r \in 1 + \Delta(R).$$

Thus, $u^2 - e \in \Delta(R)$.

Now, we need to show that $u^2 - e \in \Delta(eRe)$. To that end, let v be an arbitrary unit of eRe. One inspects that $v + 1 - e \in U(R)$. Note also that $u^2 - e \in \Delta(R)$ gives that $u^2 - e + v + 1 - e \in U(R)$ under presence of the definition of $\Delta(R)$. Taking $u^2 - e + v + 1 - e = t \in U(R)$, one can check that $et = te = ete = u^2 - e + v$, and so $ete \in U(eRe)$. This shows that $u^2 - e + U(eRe) \subseteq U(eRe)$, so that $u^2 - e \in \Delta(eRe)$ and $u^2 \in e + \Delta(eRe)$ implying eRe is a 2-ΔU ring, as asked for. □

Proposition 11. *For any ring $R \neq 0$ and any integer $n \geq 2$, the ring $M_n(R)$ is not a 2-ΔU ring.*

Proof. Since it is well known that $M_2(R)$ is isomorphic to a corner ring of $M_n(R)$ whenever $n \geq 2$, it suffices to show that $M_2(R)$ is not a 2-ΔU ring in conjunction with Proposition 10. To this purpose, consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in U(M_2(R)).$$

Then, one verifies that

$$A^2 - I_2 = A \notin J(M_2(R)) = \Delta(M_2(R)),$$

as required. □

A set $\{e_{ij} : 1 \leq i, j \leq n\}$ of non-zero elements of R is said to be a system of n^2 matrix units if $e_{ij}e_{st} = \delta_{js}e_{it}$, where $\delta_{jj} = 1$ and $\delta_{js} = 0$ for $j \neq s$. In this case, $e := \sum_{i=1}^{n} e_{ii}$ is an idempotent of R and $eRe \cong M_n(S)$, where

$$S = \{r \in eRe : re_{ij} = e_{ij}r, \;\; \text{for all} \;\; i, j = 1, 2, ..., n\}.$$

Recall that a ring R is said to be *Dedekind finite* if $ab = 1$ ensures $ba = 1$ for any $a, b \in R$. In other words, all one-sided inverses in such a ring are necessarily two-sided.

Proposition 12. *Every 2-ΔU ring is Dedekind finite.*

Proof. If we assume to the contrary that R is *not* a Dedekind finite ring, then there exist elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Assuming $e_{ij} = a^i(1 - ba)b^j$ and $e = \sum_{i=1}^{n} e_{ii}$, there exists a non-zero ring S such that $eRe \cong M_n(S)$. However, according to Proposition 10, eRe is a 2-ΔU ring, whence $M_n(S)$ must also be a 2-ΔU ring, thus contradicting Proposition 11. □

Example 4. *A local ring R is 2-ΔU if, and only if, either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.*

Proof. Assume one of the possibilities $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$. We, however, know that $R/J(R)$ is a division ring, so $R/J(R)$ is 2-ΔU viewing Example 3. Thus, R is 2-ΔU in accordance with Corollary 7.

Conversely, letting R be 2-ΔU, we directly check that $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$ with the aid of Example 3. □

As an obvious consequence, we derive:

Corollary 8. *(i) A semi-simple ring R is 2-ΔU if, and only if, $R \cong \bigoplus_{i=1}^{n} R_i$, where $R_i \cong \mathbb{Z}_2$ or $R_i \cong \mathbb{Z}_3$ for every index i.*

(ii) A semi-local ring R is 2-ΔU if, and only if, $R/J(R) \cong \bigoplus_{i=1}^{m} R_i$, where $R_i \cong \mathbb{Z}_2$ or $R_i \cong \mathbb{Z}_3$ for every index i.

Example 5. *The ring $\mathbb{Z}_m$ is 2-ΔU if, and only if, $m = 2^k 3^l$ for some positive integers k and l.*

Lemma 1. *Let R be a 2-ΔU ring. If $J(R) = (0)$ and every non-zero right ideal of R contains a non-zero idempotent, then R is reduced.*

Proof. Suppose that the contradiction R is not reduced holds. Then, there exists a non-zero element $a \in R$ such that $a^2 = 0$. With [22, Theorem 2.1] at hand, there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some non-trivial ring T. Now, Proposition 10 tells us that eRe is a 2-ΔU ring, and hence $M_2(T)$ is a 2-ΔU ring. This, however, contradicts Proposition 11. $\square$

A ring R is called *π-regular* if, for each $a \in R$, $a^n \in a^n R a^n$ for some integer $n \geq 1$. Regular rings are always π-regular. Also, a ring R is said to be *strongly π-regular*, provided that, for any $a \in R$, there exists $n \geq 1$ such that $a^n \in a^{n+1}R$.

We are now ready to attack the following pivotal result.

Theorem 2. *Let R be a ring. Then, the following three assertions are equivalent:*

(1) *R is a regular 2-ΔU ring.*
(2) *R is a π-regular reduced 2-ΔU ring.*
(3) *R has the identity $x^3 = x$ (i.e., R is a tripotent ring).*

Proof. $(i) \Rightarrow (ii)$. Since R is regular, $J(R) = (0)$, and every non-zero right ideal contains a non-zero idempotent. In virtue of Lemma 1, R is reduced. Also, every regular ring is π-regular.

$(ii) \Rightarrow (iii)$. Notice that reduced rings are abelian, so R is abelian regular by virtue of [1, Theorem 3], and hence it is strongly regular. Thus, R is unit-regular, so that $\Delta(R) = (0)$ in accordance with [21, Corollary 16]. Therefore, we have $Nil(R) = J(R) = \Delta(R) = (0)$.

On the other hand, one knows that R is strongly π-regular. Choose $x \in R$. The application of [10, Proposition 2.5] insures that there are an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u$ and $ex = xe \in Nil(R) = (0)$. So, we deduce

$$x = x - xe = x(1 - e) = u(1 - e) = (1 - e)u.$$

Since R is a 2-ΔU ring, $u^2 = 1$. It now follows that $x^2 = (1 - e)$. Hence, $x = x(1 - e) = x.x^2 = x^3$.

$(iii) \Rightarrow (i)$. It is not so hard to verify that R is regular. Choosing $u \in U(R)$, we infer $u^3 = u$, that is, $u^2 = 1$, and thus R is a 2-ΔU ring, as asserted. $\square$

A ring R is termed *strongly 2-nil-clean* if every element in R is a sum of two idempotents and a nilpotent that commute (for more account, we refer to [5]).

Our next chief result is as follows.

Theorem 3. *The following four statements are equivalent for a ring R:*

1. *R is a regular 2-ΔU ring.*
2. *R is a strongly regular 2-ΔU ring.*
3. *R is a unit-regular 2-ΔU ring.*
4. *R has the identity $x^3 = x$ (i.e., R is a tripotent ring).*

Proof. $(i) \Rightarrow (ii)$. Observe that Lemma 1 gives R is reduced and hence abelian. Then, R is strongly regular.

$(ii) \Rightarrow (iii)$. This is pretty obvious, so we drop off the arguments.

$(iii) \Rightarrow (iv)$. Choose $x \in R$ and write $x = ue$ for some $u \in U(R)$ and $e \in Id(R)$. We know that every unit-regular ring is regular, so that R is regular 2-ΔU whence R is abelian.

On the other side, [21, Corollary 16] informs us that $\Delta(R) = 0$. Therefore, for any $u \in U(R)$, we have $u^2 = 1$. Then, $x^2 = u^2e^2 = e$. So, R is a 2-Boolean ring. Thus, [5, Corollary 3.4] enables us that R is strongly 2-nil-clean, and hence [5, Theorem 3.3] guarantees that R is tripotent, as formulated.

$(iv) \Rightarrow (i)$. It is quite elementary looking at Theorem 2. □

Proposition 13. *A ring R is ΔU if, and only if,*

1. *$2 \in \Delta(R)$,*
2. *R is a 2-ΔU ring,*
3. *If $x^2 \in \Delta(R)$, then $x \in \Delta(R)$ for every $x \in R$.*

Proof. $\Rightarrow$. As R is a ΔU ring, one has that $-1 = 1 + r$ for some $r \in \Delta(R)$. This insures $-2 \in \Delta(R)$ and so $2 \in \Delta(R)$. But, every ΔU ring is 2-ΔU. The conclusion now follows from [11, Proposition 2.4].

$\Leftarrow$. Let $u \in U(R)$. Then,

$$(u-1)^2 + 2(u-1) = (u-1)(u+1) = u^2 - 1 \in \Delta(R),$$

because R is a 2-ΔU ring. It follows from the facts $2 \in \Delta(R)$ and $\Delta(R)$ is a subring of R that $(u-1)^2 \in \Delta(R)$. So, by (iii) it must be that $u - 1 \in \Delta(R)$ whence R is a ΔU ring, as stated. □

Following Kosan et al. ([18]), a ring R is called *semi-tripotent* if, for each $a \in R$, $a = e + j$, where $e^3 = e$ and $j \in J(R)$ (or, equivalently, $R/J(R)$ satisfies the identity $x^3 = x$ and all idempotents lift modulo $J(R)$).

We now have all the ingredients necessary to prove our next basic result.

Theorem 4. *Let R be a ring. Then, the following three conditions are equivalent:*

1. *R is a semi-regular 2-ΔU ring.*
2. *R is an exchange 2-ΔU ring.*

(3) *R is a semi-tripotent ring.*

Proof. $(i) \Rightarrow (ii)$. Observe that [23, Proposition 1.6] assures that semi-regular rings are exchange.

$(ii) \Rightarrow (iii)$. Monitoring [23, Corollary 2.4], $R/J(R)$ is exchange and idempotents lift modulo $J(R)$. Moreover, Proposition 9 ensures that $R/J(R)$ is 2-ΔU. So, with no loss of generality, it can be assumed that $J(R) = (0)$. Since R is an exchange ring, every non-zero one sided ideal contains a non-zero idempotent. However, Lemma 1 is a guarantor that R is reduced and so abelian. Thus, R is abelian clean. Hence, [17, Proposition 14] employs to write that $R/J(R) \cong M_n(D)$, where $1 \leq n \leq 2$ and D is a division ring. Therefore, [21, Theorem 11] helps us to get that $\Delta(R) = J(R)$ whence $\Delta(R) = (0)$. But, as R is 2-ΔU, we have $v^2 = 1$ for every $v \in U(R)$. Consequently, the conclusion follows now from [6, Theorem 3.3].

$(iii) \Rightarrow (i)$. Adapting [6, Theorem 3.3], R is semi-regular 2-UJ meaning that R is semi-regular 2-ΔU. □

As four valuable corollaries, we yield:

Corollary 9. *Let R be a 2-ΔU ring. Then, the following are equivalent:*

(1) *R is a semi-regular ring.*
(2) *R is an exchange ring.*
(3) *R is a clean ring.*

Proof. By Theorem 4, (i) ⇔ (ii).

$(iii) \Rightarrow (ii)$. This is easy, so we leave the arguments to the interested reader.

$(ii) \Rightarrow (iii)$. If R is exchange 2-ΔU, then R is reduced via Lemma 1, and hence it is abelian. Therefore, R is abelian exchange, so it is clean, ending the implication. □

Corollary 10. *Let R be a ring. The following are equivalent:*

(1) *R is a semi-regular 2-ΔU ring and $J(R)$ is nil.*
(2) *R is an exchange 2-ΔU ring and $J(R)$ is nil.*
(3) *R is a strongly 2-nil-clean ring.*

Proof. $(i) \Rightarrow (ii)$. This can easily be deduced, so the detailed argumentation is leaved.

$(ii) \Rightarrow (iii)$. Since R is exchange 2-ΔU, $\Delta(R) = J(R)$. Then, for any $u \in U(R)$, we have

$$u^2 - 1 \in \Delta(R) = J(R) \subseteq Nil(R).$$

So, R is 2-UU ring. Therefore, R is exchange 2-UU ring, whence it is strongly 2-nil-clean in regard to [25, Theorem 4.1].

$(iii) \Rightarrow (i)$. This follows from [6, Corollary 3.5] and knowing that every 2-UJ ring is 2-ΔU. □

Corollary 11. *Let R be a ring. Then, the following are equivalent:*

(1) *R is a regular* ΔU *ring.*
(2) *R is a* π*-regular reduced* ΔU *ring.*
(3) *R is a Boolean ring.*

Proof. This is an automatic consequence of Theorem 2 and [11, Theorem 4.4]. □

Corollary 12. *Let R be a ring. The following are equivalent:*
(1) *R is a semi-regular* ΔU *ring.*
(2) *R is an exchange* ΔU *ring.*
(3) *R is a clean* ΔU *ring.*

Proof. The result follows from [11, Theorem 4.2 and Corollary 4.7]. □

4 Some extensions of 2-ΔU rings

We say that C is an unital subring of a ring D if $\emptyset \neq C \subseteq D$ and, for any $x, y \in C$, the relations $x - y$, $xy \in C$ and $1_D \in C$ hold. Let D be a ring and C an unital subring of D, and designate by $R[D, C]$ the set

$$\{(d_1, \dots, d_n, c, c, \dots) : d_i \in D, c \in C, 1 \leq i \leq n\}.$$

Then, $R[D, C]$ forms a ring under the usual component-wise addition and multiplication. The ring $R[D, C]$ is called the *tail ring extension*.

We are now attacking the following two preliminary claims giving us some useful necessary and sufficient conditions.

Proposition 14. *$R[D, C]$ is a 2-ΔU ring if, and only if, D and C are 2-ΔU rings.*

Proof. Let $R[D, C]$ be a 2-ΔU ring. Firstly, we prove that D is a 2-ΔU ring. Let $u \in U(D)$. Then $\bar{u} = (u, 1, 1, \dots) \in U(R[D, C])$. By existing hypothesis, we have $(u^2 - 1, 0, 0, \dots) \in \Delta(R[D, C])$, so

$$(u^2 - 1, 0, 0, \dots) + U(R[D, C]) \subseteq U(R[D, C]).$$

Thus, for all $v \in U(D)$,

$$(u^2 - 1 + v, 1, 1, \dots) = (u^2 - 1, 0, 0, \dots) + (v, 1, 1, \dots) \in U(R[D, C]).$$

Hence, $u^2 - 1 + v \in U(D)$ forcing that $u^2 - 1 \in \Delta(D)$.

Now, we show that C is a 2-ΔU ring. To this aim, let $v \in U(C)$. Then, $(1, \dots, 1, 1, v, v, \dots) \in U(R[D, C])$. By assumption,

$$(0, \dots, 0, v^2 - 1, v^2 - 1, \dots) \in \Delta(R[D, C]),$$

and so

$$(0, \dots, 0, v^2 - 1, v^2 - 1, \dots) + U(R[D, C]) \subseteq U(R[D, C]).$$

Thus, for all $u \in U(C)$,

$$(1, 1, \dots, v^2 - 1 + u, v^2 - 1 + u, \dots) \in U(R[D, C]).$$

But then, we have $v^2 - 1 + u \in U(C)$ and hence $v^2 - 1 \in \Delta(C)$, as needed.

For the converse, assume that D and C are 2-ΔU rings. Let $\bar{u} = (u_1, ..., u_n, v, v, \dots) \in U(R[D, C])$, where $u_i \in U(D)$ and $v \in U(C) \subseteq U(D)$. We must show that $\bar{u}^2 - 1 + U(R[D, C]) \subseteq U(R[D, C])$. In fact, for all $\bar{a} = (a_1, \dots, a_m, b, b, \dots) \in U(R[D, C])$ with $a_i \in U(D)$ and $b \in U(C) \subseteq U(D)$, take $z = \max\{m, n\}$, and thus we obtain

$$\bar{u}^2 - 1 + \bar{a} = (u_1^2 - 1 + a_1, \dots, u_z^2 - 1 + a_z, v^2 - 1 + b, v^2 - 1 + b, \dots).$$

Note that $u_i^2 - 1 + a_i \in U(D)$ for all $1 \leq i \leq z$ and $v^2 - 1 + b \in U(C) \subseteq U(D)$. We, thereby, deduce that $\bar{u}^2 - 1 + \bar{a} \in U(R[D, C])$. Thus, $\bar{u}^2 - 1 \in \Delta(R[D, C])$ and $\bar{u}^2 \in 1 + \Delta(R[D, C])$. This shows that $R[D, C]$ is a 2-ΔU ring, as required. □

Let R be a ring and $\alpha : R \to R$ a ring endomorphism. As usual, $R[[x; \alpha]]$ denotes the ring of *skew formal power series* over R; that is, all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x; 1_R]]$ is the ring of *formal power series* over R.

Proposition 15. *A ring R is 2-ΔU if, and only if, so is $R[[x; \alpha]]$.*

Proof. Consider $I = R[[x; \alpha]]x$. Then, I is an ideal of $R[[x; \alpha]]$. A simple check gives that $J(R[[x; \alpha]]) = J(R) + I$, so $I \subseteq J(R[[x; \alpha]])$. Since $R[[x; \alpha]]/I \cong R$, the result follows with the help of Proposition 9. □

Corollary 13. *A ring R is 2-ΔU if, and only if, so is $R[[x]]$.*

Our further main achievement is the following one.

Theorem 5. *Let R be a 2-primal ring and let α be an endomorphism of R such that R is α-compatible. The following two statements are equivalent:*

(1) *$R[x; \alpha]$ is a 2-ΔU ring.*
(2) *R is a 2-ΔU ring.*

Proof. $(ii) \Rightarrow (i)$. Let

$$u(x) = a_0 + a_1 x + \dots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

be in $U(R[x; \alpha])$. So, the usage of [4, Corollary 2.14] allows us to infer that $a_0 \in U(R)$ and $a_i \in Nil(R)$ for each $i \geq 1$. Then, by assumption, $a_0^2 = 1 + r$, where $r \in \Delta(R)$.

In the other vein, we know that

$$J(R[x; \alpha]) = Nil_*(R[x; \alpha]) = Nil_*(R)[x; \alpha] = Nil(R)[x; \alpha].$$

Now, we conclude that

$$\begin{aligned}(u(x))^2 &= a_0^2 + a_0 a_1 x + \dots + a_0 a_n x^n + a_1 x a_0 + \dots = (1 + r) + a_0 a_1 x + \dots \\ &= 1 + (r + a_0 a_1 x + \dots) \in 1 + \Delta(R) + J(R[x; \alpha]).\end{aligned}$$

On the other side, it must be that $\Delta(R) + J(R[x;\alpha]) = \Delta(R[x;\alpha])$ by Proposition 4. Thus, this means that $R[x;\alpha]$ is a 2-ΔU ring, as required.
$(i) \Rightarrow (ii)$. Let $u \in U(R) \subseteq U(R[x;\alpha])$. Then,

$$u^2 \in 1 + \Delta(R[x;\alpha]) = 1 + \Delta(R) + J(R[x;\alpha]).$$

Thus, one detects that $u^2 \in 1 + \Delta(R)$, and hence R is a 2-ΔU ring, as needed. □

The following consequence is now immediate.

Corollary 14. *Let R be a 2-primal ring. Then, the following are equivalent:*

(1) *$R[x]$ is a 2-ΔU ring.*
(2) *R is a 2-ΔU ring.*

Let R be a ring and M a bi-module over R. The trivial extension of R and M is defined as

$$T(R, M) = \{(r, m) : r \in R \text{ and } m \in M\},$$

with addition defined componentwise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

Note that the trivial extension $T(R, M)$ is isomorphic to the subring

$$\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M\right\}$$

of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$, and also $T(R, R) \cong R[x]/\langle x^2 \rangle$. We, likewise, notice that the set of units of the trivial extension $T(R, M)$ is

$$U(T(R, M)) = T(U(R), M).$$

Besides, thanks to [11], we can write

$$\Delta(T(R, M)) = T(\Delta(R), M).$$

We proceed by proving the following.

Proposition 16. *Suppose R is a ring and M is a bi-module over R. Then, the following hold:*

(1) *The trivial extension $T(R, M)$ is a 2-ΔU ring if, and only if, R is a 2-ΔU ring.*
(2) *For $n \geq 2$, the quotient-ring $\dfrac{R[x;\alpha]}{\langle x^n \rangle}$ is a 2-ΔU ring if, and only if, R is a 2-ΔU ring.*
(3) *For $n \geq 2$, the quotient-ring $\dfrac{R[[x;\alpha]]}{\langle x^n \rangle}$ is a 2-ΔU ring if, and only if, R is a 2-ΔU ring.*
(4) *The upper triangular matrix ring $T_n(R)$ is a 2-ΔU if, and only if, R is a 2-ΔU ring.*

Proof. (1) Set $A := T(R, M)$ and consider $I := T(0, M)$. It is not so hard to see that $I \subseteq J(A)$ such that $\dfrac{A}{I} \cong R$. So, the result follows directly from Proposition 9.

(2) Put $A := \dfrac{R[x;\alpha]}{\langle x^n \rangle}$. Considering the ideal $I := \dfrac{\langle x \rangle}{\langle x^n \rangle}$ of A, we routinely obtain that $I \subseteq J(A)$ with $\dfrac{A}{I} \cong R$. So, the wanted result follows automatically from Proposition 9.

(3) Knowing that the isomorphism $\dfrac{R[x;\alpha]}{\langle x^n \rangle} \cong \dfrac{R[[x;\alpha]]}{\langle x^n \rangle}$ holds, point (iii) follows immediately from (ii).

(4) Setting $I := \{(a_{ij}) \in T_n(R) \,|\, a_{ii} = 0\}$, we then have $I \subseteq J(T_n(R))$ and $T_n(R)/I \cong R^n$. Therefore, the desired result follows from Propositions 9 and 5.

□

Corollary 15. *Let R be a ring. Then, the following are equivalent:*

(1) *R is a 2-ΔU ring.*
(2) *For $n \geq 2$, the quotient-ring $\dfrac{R[x]}{\langle x^n \rangle}$ is a 2-ΔU ring.*
(3) *For $n \geq 2$, the quotient-ring $\dfrac{R[[x]]}{\langle x^n \rangle}$ is a 2-ΔU ring.*

Example 6. *The upper triangular ring $T_n(\mathbb{Z}_3)$ for all $n \geq 1$ is 2-ΔU (see Proposition 16(4) and Example 2). But, it is not a ΔU ring as Example 2 and* [11, Corollary 2.9] *show.*

Suppose R is a ring and M is a bi-module over R. Putting

$$DT(R, M) := \{(a, m, b, n)|a, b \in R, m, n \in M\}$$

with addition defined componentwise and multiplication defined by

$$(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) =$$

$$= (a_1a_2, a_1m_2 + m_1a_2, a_1b_2 + b_1a_2, a_1n_2 + m_1b_2 + b_1m_2 + n_1a_2),$$

we then see that $DT(R, M)$ is a ring that is isomorphic to $T(T(R, M), T(R, M))$. Moreover, we have

$$DT(R, M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} |a, b \in R, m, n \in M \right\}.$$

We now establish the following isomorphism as rings: the map $\dfrac{R[x,y]}{\langle x^2, y^2\rangle} \to DT(R,R)$ is defined by

$$a + bx + cy + dxy \mapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

We, thereby, arrive at the following.

Corollary 16. *Let R be a ring and M a bi-module over R. Then, the following statements are equivalent:*

(1) *R is a 2-ΔU ring.*
(2) *$DT(R,M)$ is a 2-ΔU ring.*
(3) *$DT(R,R)$ is a 2-ΔU ring.*
(4) *$\dfrac{R[x,y]}{\langle x^2, y^2\rangle}$ is a 2-ΔU ring.*

Let A, B be two rings and let M, N be an (A,B)-bi-module and a (B,A)-bi-module, respectively. Also, we consider the two bi-linear maps $\phi : M \otimes_B N \to A$ and $\psi : N \otimes_A M \to B$ that apply to the following properties.

$$Id_M \otimes_B \psi = \phi \otimes_A Id_M, Id_N \otimes_A \phi = \psi \otimes_B Id_N.$$

For $m \in M$ and $n \in N$, we define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Now the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes to an associative ring with obvious matrix operations that is called a *Morita context ring*. Denote two-sided ideals $Im\phi$ and $Im\psi$ to MN and NM, respectively, that are called the *trace ideals* of the Morita context ring.

The following assertion holds.

Proposition 17. *Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring such that $MN \subseteq J(A)$ and $NM \subseteq J(B)$. Then, R is a 2-ΔU ring if, and only if, both A and B are 2-ΔU.*

Proof. One observes that [28, Lemma 3.1] can be applied to argue that $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$, and hence $\dfrac{R}{J(R)} \cong \dfrac{A}{J(A)} \times \dfrac{B}{J(B)}$. Thus, the result follows from Corollary 7 and Proposition 5. □

Now, let R, S be two rings and let M be an (R,S)-bi-module such that the operation $(rm)s = r(ms)$ is valid for all $r \in R$, $m \in M$ and $s \in S$. Given such a bi-module M, we can set

$$T(R,S,M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it obviously forms a ring with the usual matrix operations. The so-stated formal matrix $T(R,S,M)$ is called a *formal triangular matrix ring*.

It is worthy of noticing that, if we set $N=0$ in Proposition 17, then we will obtain the following statement.

Corollary 17. *Let R,S be rings and let M be an (R,S)-bi-module. Then, the formal triangular matrix ring $T(R,S,M)$ is a 2-ΔU ring if, and only if, R and S are both 2-ΔU.*

Given a ring R and a central element s of R, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition component-wise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix}=\begin{pmatrix} a_1a_2+sx_1y_2 & a_1x_2+x_1b_2 \\ y_1a_2+b_1y_2 & sy_1x_2+b_1b_2 \end{pmatrix}.$$

This ring is denoted by $K_s(R)$. A *Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A=B=M=N=R$ is called a *generalized matrix ring* over R. It was observed by Krylov in [19] that a ring S is generalized matrix over R if, and only if, $S=K_s(R)$ for some $s\in C(R)$. Here $MN=NM=sR$, so that $MN\subseteq J(A)\iff s\in J(R)$ and $NM\subseteq J(B)\iff s\in J(R)$.

We can now extract the following.

Corollary 18. *Let R be a ring and $s\in C(R)\cap J(R)$. Then, $K_s(R)$ is a 2-ΔU ring if, and only if, R is 2-ΔU.*

Following Tang and Zhou (cf. [27]), for $n\geq 2$ and for $s\in C(R)$, the $n\times n$ *formal matrix ring* over R defined by s, and designated by $M_n(R;s)$, is the set of all $n\times n$ matrices over R with usual addition of matrices and with multiplication defined below:

For (a_{ij}) and (b_{ij}) in $M_n(R;s)$,

$$(a_{ij})(b_{ij})=(c_{ij}),\quad \text{where}\quad (c_{ij})=\sum s^{\delta_{ikj}}a_{ik}b_{kj}.$$

Here, $\delta_{ijk}=1+\delta_{ik}-\delta_{ij}-\delta_{jk}$, where δ_{jk}, δ_{ij}, δ_{ik} are the *Kronecker delta symbols*.

We now manage to prove the following.

Corollary 19. *Let R be a ring and $s\in C(R)\cap J(R)$. Then, $M_n(R;s)$ is a 2-ΔU ring if, and only if, R is 2-ΔU.*

Proof. If $n=1$, then $\mathrm{M}_n(R;s)=R$. So, in this situation, there is nothing to establish. That is why, suppose $n=2$. Using the definition of $\mathrm{M}_n(R;s)$, we have $\mathrm{M}_2(R;s)\cong \mathrm{K}_{s^2}(R)$. Evidently, $s^2\in J(R)\cap C(R)$, so the assertion is true for $n=2$ taking into account Corollary 18.

To proceed by induction, assume now that $n>2$ and that the claim holds for $\mathrm{M}_{n-1}(R;s)$. Set $A:=\mathrm{M}_{n-1}(R;s)$. Then, one inspects that $\mathrm{M}_n(R;s)=$

$\begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a Morita context, where

$$M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1,n} \end{pmatrix} \quad \text{and} \quad N = (M_{n1} \dots M_{n,n-1})$$

with $M_{in} = M_{ni} = R$ for all $i = 1, \dots, n-1$, and

$$\psi : N \otimes M \to N, \quad n \otimes m \mapsto snm$$
$$\phi : M \otimes N \to M, \quad m \otimes n \mapsto smn.$$

Moreover, for $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{n-1,n} \end{pmatrix} \in M$ and $y = (y_{n1} \dots y_{n,n-1}) \in N$, we may write

$$xy = \begin{pmatrix} s^2x_{1n}y_{n1} & sx_{1n}y_{n2} & \dots & sx_{1n}y_{n,n-1} \\ sx_{2n}y_{n1} & s^2x_{2n}y_{n2} & \dots & sx_{2n}y_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ sx_{n-1,n}y_{n1} & sx_{n-1,n}y_{n2} & \dots & s^2x_{n-1,n}y_{n,n-1} \end{pmatrix} \in sA$$

as well as

$$yx = s^2y_{n1}x_{1n} + s^2y_{n2}x_{2n} + \dots + s^2y_{n,n-1}x_{n-1,n} \in s^2R.$$

Since $s \in J(R)$, we observe that $MN \subseteq J(A)$ and $NM \subseteq J(A)$. So, we receive that

$$\frac{\mathrm{M}_n(R; s)}{J(\mathrm{M}_n(R; s))} \cong \frac{A}{J(A)} \times \frac{R}{J(R)}.$$

Finally, the induction hypothesis along with Proposition 17 yield the desired conclusion after all. □

A *Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called *trivial*, if the context products are trivial, i.e., $MN = 0$ and $NM = 0$. We now see that

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context bearing in mind [13].

We, thus, obtain the following.

Corollary 20. *The trivial Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *is a 2-ΔU ring if, and only if, A and B are both 2-ΔU.*

Proof. It is plainly seen that the isomorphisms

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}$$

hold. Then, the rest of the proof follows by a combination of Propositions 16(i) and 5. □

We shall now deal with group rings of 2-ΔU rings as follows.

As usual, for an arbitrary ring R and an arbitrary group G, the symbol RG stands for the *group ring* of G over R. Standardly, $\varepsilon(RG)$ denotes the kernel of the classical *augmentation map* $\varepsilon : RG \to R$, defined by $\varepsilon(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g$, and this ideal is called the *augmentation ideal* of RG.

Besides, a group G is called a *p-group* if every element of G has order which is a power of the prime number p. Moreover, a group G is said to be *locally finite* if every finitely generated subgroup is finite.

We begin our work with two preliminary technicalities.

Lemma 2. [28, Lemma 2]. *Let p be a prime with $p \in J(R)$. If G is a locally finite p-group, then $\varepsilon(RG) \subseteq J(RG)$.*

Proposition 18. *(i) If RG is a 2-ΔU ring, then R is also a 2-ΔU ring.*

(ii) If R is a 2-ΔU ring and G is a locally finite p-group, where p is a prime number such that $p \in J(R)$, then RG is a 2-ΔU ring.

Proof. (i) Assume $u \in U(R)$, then $u \in U(RG)$. Thus, $u^2 = 1 + r$, where $r \in \Delta(RG)$. Since $r = 1 - u^2 \in R$, it suffices to verify that $r \in \Delta(R)$, which is pretty obvious as, for any $v \in U(R) \subseteq U(RG)$, we see that $v - r \in U(RG) \cap R \subseteq U(R)$. Therefore, $r \in \Delta(R)$.

(ii) Obviously, Lemma 2 gives us that $\varepsilon(RG) \subseteq J(RG)$. However, since $RG/\varepsilon(RG) \cong R$, Theorem 9 applies to conclude that RG is a 2-ΔU ring. □

The following reversed implication is somewhat slightly curious.

Proposition 19. *If RG is a 2-ΔU ring with $2 \in \Delta(RG)$, then G is a 2-group.*

Proof. We first consider two claims:

Claim 1: Every element $g \in G$ has a finite order.

Assume the contrary, namely that there exists $g \in G$ with infinite order. Since RG is a 2-ΔU ring, we have $1 - g^2 \in \Delta(RG)$. Given $2 \in \Delta(RG)$, we then can write $1 + g^2 \in \Delta(RG)$, ensuring $1 + g + g^2 \in U(RG)$. Therefore, there exist integers $n < m$ and elements a_i with $a_n \neq 0 \neq a_m$ such that

$$(1 + g + g^2)\sum_{n}^{m} a_i g^i = 1.$$

This, however, leads to a contradiction, and thus every element $g \in G$ must have finite order, as expected.

Claim 2: For any $g \in G$ and $k \in \mathbb{N}$, we have $\sum_{i=0}^{2k} g^i \in U(RG)$.

We will show this only for $k = 1, 2$, because the general claim follows in a way of similarity.

For $k = 1$ and any $g \in G$, we have $1 - g^2 \in \Delta(RG)$. Since $2 \in \Delta(RG)$, we then can write $1 + g^2 \in \Delta(RG)$ and hence $1 + g + g^2 \in U(RG)$.

For $k = 2$ and any $g \in G$, observing that $g, g^2 \in U(RG)$, we get $1 - g^2 \in \Delta(RG)$ and hence $1 + g^2 \in \Delta(RG)$. Thus, $g + g^3 \in \Delta(RG)$. But, $1 - g^4 \in \Delta(RG)$ and, therefore, $1 + g^4 \in \Delta(RG)$.

Furthermore, since $\Delta(RG)$ is closed under addition, it follows that

$$2 + g + g^2 + g^3 + g^4 \in \Delta(RG),$$

assuring that

$$g + g^2 + g^3 + g^4 \in \Delta(RG)$$

and so

$$1 + g + g^2 + g^3 + g^4 \in U(RG).$$

Continuing this process, we can show that $\sum_{i=0}^{2k} g^i \in U(RG)$, as claimed.

If now $g \in G$ has an order p that does not divide 2, then p has to be odd, whence $p - 1 = 2k$. Consequently, by what we have shown above $\sum_{i=0}^{2k} g^i \in U(RG)$, and since $(1 - g)(\sum_{i=0}^{2k} g^i) = 0$, we deduce $1 - g = 0$, which is a contradiction. Thus, G must be a 2-group, as stated. □

5 Open Questions

We finish our work with the following four questions which allude us.

A ring R is called UQ if $U(R) = 1 + QN(R)$ (see [8]).

Problem 1. *Examine those rings R whose for each $u \in U(R)$, $u^2 = 1 + q$ where $q \in QN(R)$ (i.e., 2-UQ rings).*

Problem 2. *Characterize regular (or semi-regular) 2-UQ rings.*

A ring R is called UNJ if $U(R) = 1 + Nil(R) + J(R)$ (see [16]).

Problem 3. *Examine those rings R whose for each $u \in U(R)$, $u^2 = 1 + n + j$, where $n \in Nil(R)$ and $j \in J(R)$ (i.e., 2-UNJ rings).*

Problem 4. *Characterize regular (or semi-regular) 2-UNJ rings.*

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