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JACOBI NUMERICAL METHOD FOR SOLVING 3D CONTINUATION PROBLEM FOR WAVE EQUATION

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Abstract: In this paper we consider an explicit finite difference scheme to solve an ill-posed continuation problem for the 3D wave equation with the data given on the part of the boundary. We reduce the problem to a system of linear algebraic equations and implement the numerical solution using an iterative solver and discuss an efficient solution to a dense system of linear equations. We use the Jacobi iteration method for solving the linear system to improve computational efficiency and the results of convergence of the proposed method. Numerical experiments are presented.

Keywords: continuation problem, ill-posed problem, 3D wave equation, numerical analysis, regularization, finite difference scheme.

Bakanov, G., Chandragiri, S., Shishlenin, M.A., Jacobi numerical method for solving 3D continuation problem for wave equation.

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1 Introduction

Three-dimensional Inverse and ill-posed acoustic wave problems have been studied theoretically in the time domain with several methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Inverse problems for hyperbolic equations were solved in a class of analytic functions with respect to some variables [12].

Using the conjugate gradient method, an optimization approach was proposed for a three-dimensional inverse acoustic problem to reconstruct density and velocity by minimizing an objective functional, and the uniqueness of the solution was proved [13].

The Cauchy problem was rewritten as an operator equation on the boundary using the Dirichlet-to-Neumann map [14].

The singular values of the operator of continuation problems for PFE were investigated and numerical methods with comparative analysis were presented [15, 17, 18]. In [16, 36] the continuation and coefficient problems for electrodynamics were investigated and numerical approaches were proposed.

The numerical method for a Cauchy problem for a Helmholtz-type equation was proposed using a wavelet regularization method [19].

The inverse source problem for the wave equation was investigated, which, after discretization, was reduced to a system of algebraic equations with a poorly conditioned matrix [21]. Later, this method was applied to the heat equation with data given on a part of the boundary [20, 22].

The solution of the partial differential equations was provided using the finite difference method and the Jacobi iterative solution method theory [23].

Taking into account a priori information in numerical solution methods was presented [24]. For solving multidimensional inverse problems, the optimization issues of numerical methods are relevant [25].

In addition, for three-dimensional problems in the time domain, it usually results in a series of large-scale sparse linear systems that need an iteration at every time step. In this paper, we extend the formulation of the Jacobi iteration method to solve the linear system to improve computational efficiency and prove the convergence properties to the proposed finite difference method. MATLAB code is developed for this approach to implement a numerical solution and discuss an efficient solution of the dense system of linear equations with an iterative solver. Numerical experiments are conducted, and we compare the exact solution and the numerical solution at different time periods. The conclusion of this study finds that the Jacobi iteration method is accurate and in a reasonable amount of time converges to the exact solution.

The paper is organized as follows. In Section 2, the ill-posed continuation problem for the 3D acoustic wave equation is formulated and the iterative method proposed. In Section 3, we present the finite-difference approximations. Section 4 shows numerical experiments that were performed to test the applicability of the proposed method. Section 5 presents the discussion and conclusions.

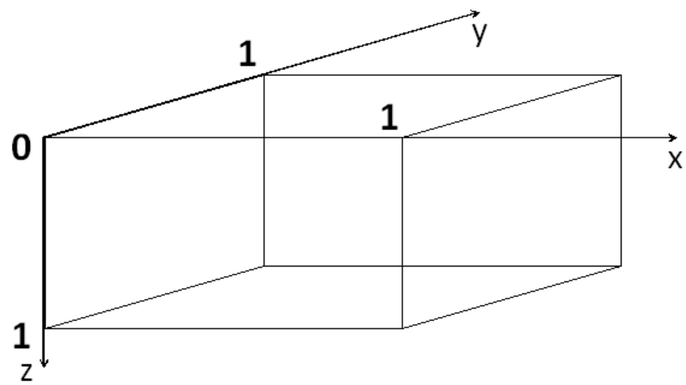

FIG. 1. Domain Ω. Variable z means the depth, variables x and y are horizontal ones

2 Statement of the problem

2.1. Continuation problem. Let us consider the 3D ill-posed [26] continuation problem for the wave equation in the domain $(x, y, z) \in \Omega = \{(0,1)\times(0,1)\times(0,1)\}$, $t \in (0, T)$:

$$v_{tt} = \Delta v - u(x,y,z)v + p(x,y,z,t), \qquad (x,y,z) \in \Omega, t \in (0,T) \tag{1}$$

$$v(x,y,z,0) = a_1(x,y,z), \qquad v_t(x,y,z,0) = a_2(x,y,z), \tag{2}$$

$$v(0,y,z,t) = b_1(y,z,t),\ v(1,y,z,t) = b_2(y,z,t), \tag{3}$$

$$v(x,0,z,t) = c_1(x,z,t),\ v(x,1,z,t) = c_2(x,z,t), \tag{4}$$

$$\frac{\partial v}{\partial z}(x,y,0,t) = g(x,y,t), \tag{5}$$

$$v(x,y,0,t) = f(x,y,t), \tag{6}$$

Here $u(x, y, z)$ is a potential, $p(x, y, z, t)$ is a source function and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The continuation problem is to find the function $v(x, y, z, t)$ using known functions u, p, a_1, a_2, b_1, b_2, c_1, c_2, f and g.

2.2. Inverse problem. Let us reduce the ill-posed continuation problem to the inverse problem. Firstly we formulate the following direct (well-posed) problem (DP): (1), (2), (3), (4), (5) and the boundary condition

$$v(x,y,1,t) = q(x,y,t). \tag{7}$$

The direct problem is to find v for given u, p, a_1, a_2, b_1, b_2, c_1, c_2, g, and q.

The reverse problem is to find the function $q(x, y, t)$ from (1), (2), (3), (4), (5) and (7) by additional known information about the direct problem solution (6).

The inverse problem (1), (2), (3), (4), (5) and (7) is equivalent to the continuation problem (1)—(6) in the following sense. If we solve the continuation problem, then we find the solution $v(x, y, z, t)$ and therefore we can find an unknown boundary condition $q(x, y, t) = v(x, y, 1, t)$, i.e. the solution of the inverse problem. Vice versa if we solve the inverse problem and find the solution of the inverse problem $q(x, y, t)$, we can set $v(x, y, 1, t) = q(x, y, t)$ and solve the direct problem (1), (2), (3), (4), (5), (7) and find $v(x, y, z, t)$ — the solution of the continuation problem.

The continuation problem has been investigated in many works [27, 28, 29]. According to Hadamard [3, 30] a solution of the Cauchy problem for an acoustic wave equation is ill-posed and exists only if smoothness conditions or strong compatibility are imposed on the initial data.

2.3. Operator form of the Inverse problem. The inverse problem (7) can be reduced to the system of algebraic equations:

$$Aq = f. \tag{8}$$

Here, A is the matrix, q is an unknown vector, and f is a data vector.

Equation (8) is

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{bmatrix}$$

2.4. Iterative Methods. Iterative methods are applied to solve the system of linear algebraic equations that arise from the approximation of partial differential equations which results in large and sparse coefficient matrices using the finite difference method. In the iterative method, to find the unknown vector q of $Aq = f$ the process starts with an initial approximation and then successive approximations will be improved by an iterative process.

$$q^{(n+1)} = Rq^{(n)} + C, \quad n = 0, 1, 2 \dots \tag{9}$$

since $q^{(n)}$ and $q^{(n+1)}$ are the n^{th} and $(n+1)^{\text{th}}$ approximations for the solution of the linear system of equations.

R: is called the non-singular iteration matrix depending on A

C: is called the constant column vector.

Given $q^{(0)}$, classical methods generate a sequence $q^{(n)}$ that converges to the solution $A^{-1}f$, where $q^{(n+1)}$ is calculated from $q^{(n)}$ by iterating (9). The iterative method strategy generates a sequence of approximate solution vectors $q^{(0)}, q^{(1)}, q^{(2)}, ..., q^{(n)}$ for the system $Aq = f$. The process can be stopped when

$$\|q^{(n+1)} - q^{(n)}\|_\infty < \varepsilon \tag{10}$$

in the limiting case, when $n \to \infty, q^{(n)}$ converges to the exact solution $q = A^{-1}f$. From (10) we find that the exact solution, q, is a stationary point, that is, in the equation set if $q^{(n)}$ is equal to the exact solution, then $q^{(n+1)}$ will also be equal to the exact solution.

2.5. Classical Iterative Methods. To construct the classical iterative methods, we use the principle of the matrix A, which can be written as the sum of other matrices (see [31]). The matrix A can be divided in several ways; one of them is the creation of the Jacobi iterative method. Classic iterative methods generally have quite a low convergence rate.

The matrix A is split and divided into two other matrices M and K such that $M + K = A$. Here, M is a diagonal matrix with the same entries as A has on the main diagonal, then K has zeros on the diagonal, the rest of the entries in A are equal to the off-diagonal entries. We apply this technique to the set of linear algebraic equations, we get

$$Aq = f.$$

$$(M + K)q = f,$$

here, M is preconditioning matrix and to solve for q, it is taken to be invertible, we get

$$q = Rq + C, \tag{11}$$

where $R = -M^{-1}K$ and $C = M^{-1}f$. We denote R as an iteration matrix.

We write (11) in the component form, which gives the following expression

$$q_i = -\frac{1}{a_{ii}} \left(\sum_{\substack{j=1 \\ j \neq i}}^{k} a_{ij} q_j - f_i \right), \ i = 1, 2, \dots, k. \tag{12}$$

We denote $\rho(A)$ is the spectral radius of matrix A, we write

$$\rho(A) = max\{|\lambda|\},$$

where the maximum is taken overall eigenvalues λ of A.

Note that iteration $q^{(n+1)} = Rq^{(n)} + C$ converges to the solution $Aq = f$ for all initial guesses $q^{(0)}$ and for all f iff $\rho(R) < 1$.

The splitting of the method discussed in this section share the following notation

$$A = M + K = D + L + U, \tag{13}$$

where D is the diagonal of A, L is the strictly lower triangular part of A, U is the strictly upper triangular part of A.

2.5.1. Jacobi Iterative Method. All the entries in the approximation will be updated based on the values in the previous approximation in this method. The splitting of the coefficient matrix A for Jacobi iteration method is $M = D, K = L + U$ and its iteration gives

$$q^{(n+1)} = R_J q^{(n)} + C_J, \tag{14}$$

where $R_J = -M^{-1}K = -D^{-1}(L+U),\ C_J = M^{-1}f = D^{-1}f$.

The component form of equation (14) is (see [32])

$$q_i^{(n+1)} = -\frac{1}{a_{ii}}\left(\sum_{\substack{j=1\\ j\neq i}}^{k} a_{ij}q_j^{(n)} - f_i\right),\ i=1,2,\dots,k \ and\ n\geq 0, \tag{15}$$

where the initial guess $q^{(0)} = (q_1^{(0)}, q_2^{(0)}, q_3^{(0)}, \dots, q_k^{(0)})$ can be chosen arbitrarily for the solution.

Always with the zero initial vector $q^{(0)} = (0,0,0,\dots,0)$ we start the approximation. The Jacobi iteration obviously consists of starting with an initial approximation $q^{(0)}$, and repeatedly applying the Jacobi update, creating a sequence $q^{(0)}, q^{(1)}, q^{(2)}, \dots$ that converges to the exact solution. The control that makes sense to apply to the iteration checks the residual, that is, having computed the n^{th} iterate $q^{(n)}$, we define the residual $\varepsilon^{(n)}$ defined by

$$\varepsilon^{(n)} = Aq^{(n)} - f,$$

and control error using Root Mean Square normalization (RMS norm) $\frac{\|\varepsilon^{(n)}\|}{\sqrt{k}}$. However, if we do not know the real solution, monitoring the residual is the proper way to control and terminate an iteration.

Note that the Jacobi iteration method converges if A is diagonally dominant [31]. Thus, if the given system of linear algebraic equations is strictly diagonally dominant by rows, then the Jacobi iteration method converges.

Jacobi Iterative Method **INPUT:** the entries a_{ij} of the matrix A; the entries f_i of f; the entries $q_i^{(0)}$ of initial guess $q^{(0)}$; the number of equations k; tolerance Tol; maximum number of iterations N.

OUTPUT: the approximate solution $q_1, q_2, \dots, q_k$ or a message that the number of iterations was exceeded.

Step 1 set $n = 1$.

Step 2 While $(n \leq N)$ repeat steps 3-6.

Step 3 For $i = 1, \dots, k$

set $q_i = -\frac{1}{a_{ii}}\left(\sum_{\substack{j=1\\ j\neq i}}^{k} a_{ij}q_j^{(0)} - f_i\right)$.

Step 4 If $\|q - q^{(0)}\| < Tol$ then OUTPUT $(q_1, q_2, \dots, q_k)$; STOP.

Step 5 Set $n = n + 1$.

Step 6 For $i = 1, \dots, k$ set $q_i^{(0)} = q_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded'); STOP.

Remark 1. *Note that R_J is known as the Jacobi iteration matrix, respectively.*

2.6. Convergence. The number of iterations required to find an approximate solution for an iteration method that is within a certain range of the exact solution defines the convergence rate. In finding the spectral radius of the iteration matrix, or an upper bound for it, the convergence rate is largely dependent. In the previous section, we discussed an iteration method which is on the form

$$q^{(n+1)} = Rq^{(n)} + \vec{f}. \tag{16}$$

The matrix R is the iteration matrix and $\vec{f}$ is a vector. The iteration matrix for the Jacobi iterative methods is $-D^{-1}(L+U)$.

An iterative method will converge when the spectral radius of the iterative matrix is less than one for the stability. The following inequality is upheld for any matrix norm (see [32]),

$$\rho(R) \leq \|R\|,$$

where $\rho(R)$ is the spectral radius of the matrix R.

Lemma 1. [33] *If A is irreducible and weakly row diagonally dominant, then the Jacobi method converges, and $\rho(R_J) < 1$.*

3 Finite Difference Method

We divide a three-dimensional region into smaller regions with increments in the x, y and z directions with time t given as Δx, Δy and Δz and Δt is the time interval for time discretization, as shown in the figure mentioned above. Every nodal point is designed by a numbering scheme i, j, k and n where i defines x increment and j defines y increment, k defines z increment, and n defines t increment as shown in Fig.1.

Let us consider the finite difference approximation with the second order approximation:

$$\begin{aligned} v_{i,j,k}^{(n+1)} &= 2v_{i,j,k}^{(n)} - v_{i,j,k}^{(n-1)} + \\ &+ (\Delta t)^2 \left(\frac{v_{i+1,j,k}^{(n)} - 2v_{i,j,k}^{(n)} + v_{i-1,j,k}^{(n)}}{(\Delta x)^2} + \frac{v_{i,j+1,k}^{(n)} - 2v_{i,j,k}^{(n)} + v_{i,j-1,k}^{(n)}}{(\Delta y)^2} + \right. \\ &\left. + \frac{v_{i,j,k+1}^{(n)} - 2v_{i,j,k}^{(n)} + v_{i,j,k-1}^{(n)}}{(\Delta z)^2} - u(x_i, y_j, z_k)v(x_i, y_j, z_k, t_n) + p(x_i, y_j, z_k, t_n) \right). \end{aligned} \tag{17}$$

Let $\Delta x = \Delta y = \Delta z = h$. We can write equation (17) as

$$\begin{aligned} v_{i,j,k}^{(n+1)} &= 2v_{i,j,k}^{(n)} - v_{i,j,k}^{(n-1)} + \\ &+ \left(\frac{\Delta t}{h}\right)^2 \left(v_{i+1,j,k}^{(n)} + v_{i-1,j,k}^{(n)} + v_{i,j+1,k}^{(n)} + v_{i,j-1,k}^{(n)} + v_{i,j,k+1}^{(n)} + v_{i,j,k-1}^{(n)} - 6v_{i,j,k}^{(n)} \right) - \\ &- (\Delta t)^2 u(x_i, y_j, z_k)v(x_i, y_j, z_k, t_n) + (\Delta t)^2 p(x_i, y_j, z_k, t_n). \end{aligned} \tag{18}$$

Higher-order approximations with more accuracy for boundary and interior nodes are also attained in the same manner.

3.1. Reducing to Cube Domains. We first discretize BVP (1)-(5), (7) in all three (x, y, z) dimensions with grid points $N_x \times N_y \times N_z$ on a uniform grid, for which we consider a cube domain where $L = W = H$. If $\Delta x \neq \Delta y \neq \Delta z$, then we can separate our region into subintervals $N_x = \frac{L}{\Delta x}, N_y = \frac{W}{\Delta y}$ and $N_z = \frac{H}{\Delta z}$ along the x, y, and z axis with the current time frame t. The goal is to approximate all the solutions, $v_{i,j,k}^{(n)}$ where $0 \leq i \leq N_x, 0 \leq j \leq N_y, 0 \leq k \leq N_z$, and $t > 0$.

As we have seen from equation (18), any point $v_{i,j,k}^{(n)}$ in the region is related to the six points surrounding it. Consider a sketch of a region where $N_x = 3, N_y = 3$ and $N_z = 3$. Here we can view the cross sections of our cube at different z values. Note that many of the values in this region are already defined. From the boundary conditions, it is known that $v_{0,j,k}^{(n)} = b_1(y_j, z_k, t_n)$, $v_{N_x,j,k}^{(n)} = b_2(y_j, z_k, t_n)$, $v_{i,0,k}^{(n)} = c_1(x_i, z_k, t_n)$, $v_{i,N_y,k}^{(n)} = c_2(x_i, z_k, t_n)$, $v_{i,j,0}^{(n)} = f(x_i, y_j, t_n)$ and $v_z(x_i, y_j, 0, t_n) = g(x_i, y_j, t_n)$. The remaining $(N_x - 1)(N_y - 1)(N_z)$ points will be approximated by building a system of linear equations.

Corollary 1. *To solve the problem (1)-(5), (7) we use the forward finite difference with respect to time t in finite difference approximation (18)*

$$v_t(x_i, y_j, z_k, 0) \approx \frac{v_{i,j,k}^{(1)} - v_{i,j,k}^{(0)}}{\Delta t} = a_2(x_i, y_j, z_k).$$

Corollary 2. *To solve the problem (1)-(5), (7) we use the forward finite difference with respect to variable z in finite difference approximation (18)*

$$v_z(x_i, y_j, 0, t_n) \approx \frac{v_{i,j,1}^{(n)} - v_{i,j,0}^{(n)}}{\Delta z} = g(x_i, y_j, t_n).$$

We can turn the system of linear equation into corresponding matrices and vectors as

$$A\vec{v} = \vec{f} - h^2\vec{p}, \tag{19}$$

where $\vec{v}$ is the vector of approximate solutions at each point, A is the coefficient matrix of these solutions, $\vec{f}$ is the vector of initial and boundary conditions at these points, and $\vec{p}$ is the source function. Although this equation is the same as the two-dimensional case, the coefficient matrix A and the boundary condition vector $\vec{f}$ will have different patterns.

3.2. Direct or Iterative Solution. We can obtain the solution of the above system of equations from the Gaussian elimination method (Direct) for a small system of unknowns $(N_x \times N_y \times N_z)$. Iterative methods give better results for a large system of unknowns.

Let us apply the Jacobi iterative method.

If we apply equation (14) or (15) to solve the system of finite difference equations for 3D acoustic wave equation, we obtain the Jacobi iteration formula (see [23])

$$v_{i,j,k}^{(n+1)} = 2v_{i,j,k}^{(n)} - v_{i,j,k}^{(n-1)} + \\ + \left(\frac{\Delta t}{h}\right)^2 \Big(v_{i+1,j,k}^{(n)} + v_{i-1,j,k}^{(n)} + v_{i,j+1,k}^{(n)} + v_{i,j-1,k}^{(n)} + v_{i,j,k+1}^{(n)} + v_{i,j,k-1}^{(n)} - 6v_{i,j,k}^{(n)}\Big) - \\ - (\Delta t)^2 u(x_i, y_j, z_k) v(x_i, y_j, z_k, t_n) + (\Delta t)^2 p(x_i, y_j, z_k, t_n). \quad (20)$$

In equation (20) the superscript n represents an iterative index. At $n = 0$ the initial iterative guess can be set to produce $v^0_{i,j,k,t}$, the next iteration $(n+1)$ can be found for every grid point (i, j, k, t) across all grid points in the horizontal rows and according to iteration it improves successfully. The difference between the vectors of the next iteration v^{n+1} and the previous iteration v^n is calculated when the iteration is complete for all points on the interior grid. Once the predefined condition (tolerance) which sets for the iteration to converge is met, the iteration terminates and the solution to (20) is v^{n+1}, otherwise the iterations continue. i.e.

$$\left|v^{n+1} - v^n\right| < tolerance.$$

The performance of a matrix vector product $A \cdot q$ plays an important role when an iteration method is used to solve the equation (8). However, this is a matrix-free method and the matrix A is never generated or stored in practice. To produce the action of A on q a Matlab code can be written using the finite difference algorithm.

MATLAB program is developed for the Jacobi iteration method using Dirichlet and Neumann boundary conditions that are applied at the boundary of the domain. We can examine the results of our discretization and iterative approximations for the sample problem with a larger grid size in different time frames. Our Jacobi iterations will use an RMS residual tolerance of 10^{-4}, for the values of $N_x = 100, N_y = 100$ and $N_z = 100$ in different time frames, and the time interval $\Delta t = 0.00405$ used for the Jacobi iterative method for stability in test problem 1.

We compare the analytical solution to the continuation problem with the solution to the discretized problem computed with the Jacobi iteration technique.

4 Numerical Experiments

4.1. Test problem 1: a known analytical solution. Let us set

$$p(x, y, z, t) = (u - 4)\cos(t) sinh(x) sinh(y) sinh(z),$$

$$a_1(x, y, z) = sinh(x) sinh(y) sinh(z), \qquad a_2(x, y, z) = 0,$$

$$b_1(y, z, t) = 0, \qquad b_2(y, z, t) = \cos(t) sinh(1) sinh(y) sinh(z),$$

$$c_1(x, z, t) = 0, \qquad c_2(x, z, t) = \cos(t) sinh(x) sinh(1) sinh(z),$$

$$f(x,y,t) = 0,$$

$$g(x,y,t) = \cos(t) sinh(x) sinh(y),$$

$$q(x,y,t) = \cos(t) sinh(x) sinh(y) sinh(1),$$

$$u(x,y,z) = sinh(x) sinh(y) sinh(z).$$

The aforementioned functions are chosen accordingly the problem such that the problem (1)-(5), (7) has a known analytical solution

$$v(x,y,z,t) = cos(t) sinh(x) sinh(y) sinh(z).$$

4.2. Results. From Table 1 we check the performance of the Jacobi method numerically in three dimensions in solving the acoustic wave equation for the values of $N_x = 100, N_y = 100$ and $N_z = 100$ in different time frames t.

3D - Jacobi Method :					
Grid size	t	$\lVert v - v_{exact} \rVert$	$\lVert q - q_{exact} \rVert$	Iterations	Run time (s)
100 × 100 × 100	0.05	9.9990e-05	1.4358e-05	10706	407.771762
100 × 100 × 100	0.5	9.9982e-05	1.4194e-05	12728	476.778365
100 × 100 × 100	1	9.9999e-05	1.4136e-05	15226	584.356846
100 × 100 × 100	1.5	9.9964e-05	1.4117e-05	16967	612.186274
100 × 100 × 100	2.5	9.9980e-05	1.4111e-05	18853	717.650728
100 × 100 × 100	3.5	9.9998e-05	1.4113e-05	19077	729.772418
100 × 100 × 100	4.25	9.9994e-05	1.4115e-05	18196	710.487008
100 × 100 × 100	5.5	1.0000e-04	1.4152e-05	14241	548.987611
100 × 100 × 100	6	9.9980e-05	1.4268e-05	11522	448.695514

TABLE 1. Numerical results obtained for the test problem 1 using Jacobi iterative method.

The top panel of Table 1 shows the results of the 3D Jacobi iterative method on this problem. The first column represents the mesh size, the second column shows the current time t, the third column represents the error of the numerical method, the fourth column shows the inverse problem error, and the fifth column shows the number of iterations taken by the Jacobi iterative method until convergence to the chosen tolerance of 10^{-4} in the relative residual. The last column is the wall clock time for each run.

From Table 2 we can check the L_1, L_2 and L_∞ norms of the Jacobi method to approximate the acoustic wave equation numerically in three dimensions for the values of $N_x = 100, N_y = 100$ and $N_z = 100$ at different time frames. Here, the L_2 norm represents the spectral radius of the matrix A.

3D - Jacobi Method :				
Grid size	t	L_1norm	L_2norm	L_∞norm
100 × 100 × 100	0.05	1.02e-02	1.66e-02	8.74e-02
100 × 100 × 100	0.5	8.91e-03	1.46e-02	7.69e-02
100 × 100 × 100	1	5.44e-03	8.95e-03	4.74e-02
100 × 100 × 100	1.5	5.42e-04	9.83e-04	5.89e-03
100 × 100 × 100	2.5	8.78e-03	1.42e-02	7.25e-02
100 × 100 × 100	3.5	1.02e-02	1.66e-02	8.48e-02
100 × 100 × 100	4.25	4.95e-03	7.95e-03	4.04e-02
100 × 100 × 100	5.5	7.18e-03	1.18e-02	6.21e-02
100 × 100 × 100	6	9.76e-03	1.60e-02	8.41e-02

TABLE 2. L_1, L_2 and L_∞ norms obtained for Test problem 1 using Jacobi iterative method.

4.3. Error graphs. In Figures 2 and Figure 3 the error difference between the analytical and numerical solutions is plotted with two visualizations of this result. Figure 2(**a**) depicts the error difference between the analytical and numerical solutions at time $t = 0.05$ for the size of the grid $100 \times 100 \times 100$ and Figure 2(**b**) shows the same result, but at time $t = 1$. Figure 3(**a**) depicts the error difference between the analytical and numerical solutions at time $t = 2.5$ for the size of the grid $100 \times 100 \times 100$ and Figure 3(**b**) shows the same result, but at time $t = 4.25$.

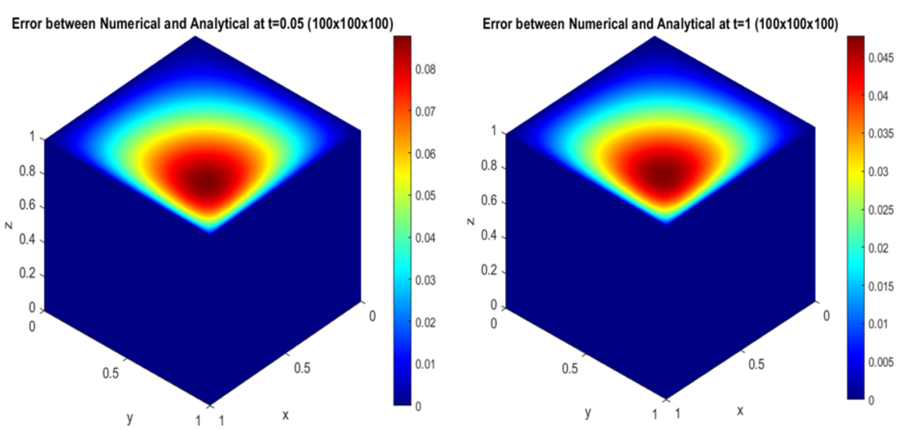

FIG. 2. Error between analytical and numerical solution($100 \times 100 \times 100$): (**a**) at $t = 0.05$ (**b**) at $t = 1$.

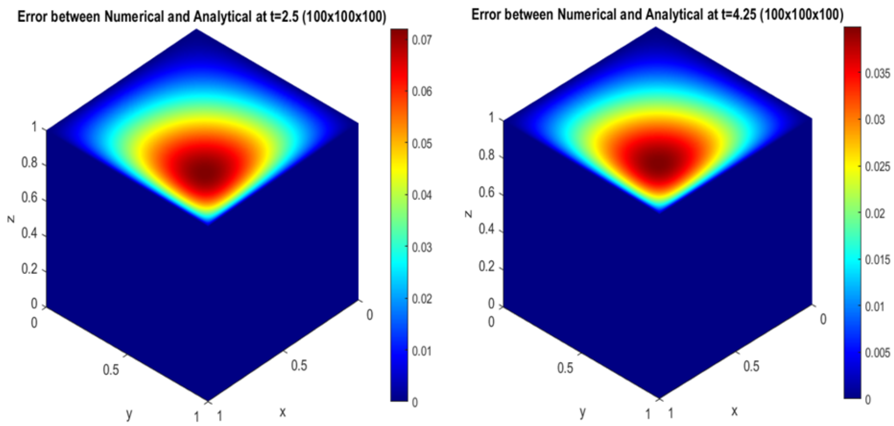

FIG. 3. Error between analytical and numerical solution$(100 \times 100 \times 100)$: (**a**) at $t = 2.5$ (**b**) at $t = 4.25$.

From Table 3, the maximum error difference we can check between the analytical and numerical solutions using the Jacobi method for the acoustic wave equation in three dimensions for the higher values of $N_x = 100, N_y = 100$ and $N_z = 100$ in different time frames t.

3D - Jacobi Method :		
Grid size	t	Max error
$100 \times 100 \times 100$	0.05	0.08742306
$100 \times 100 \times 100$	0.5	0.07688044
$100 \times 100 \times 100$	1	0.04736541
$100 \times 100 \times 100$	1.5	0.00589268
$100 \times 100 \times 100$	2.5	0.07245479
$100 \times 100 \times 100$	3.5	0.08476152
$100 \times 100 \times 100$	4.25	0.04036343
$100 \times 100 \times 100$	5.5	0.06212928
$100 \times 100 \times 100$	6	0.08407013

TABLE 3. Maximum error between Analytical and Numerical solution obtained for Test problem 1 using the Jacobi iterative method.

5 Discussion and Conclusions

In this paper, the 3D-acoustic wave equation has been approximated numerically using a finite difference method through the Jacobi iterative technique. We compared our numerical results with the known analytical solution through numerical experiments and checked the stability.

From Table 1 and Table 2 we check the performance of the Jacobi iterative scheme under the finite difference method numerically in three dimensions to solve the acoustic wave equation at different time periods for test problem 1. We observe that the Jacobi iterative method is accurate and in a reasonable amount of time this method converges to the exact solution for the higher values of N_x, N_y, and N_z.

From Table 3 we checked the difference in error between the analytical and numerical solutions at different time periods for the Jacobi iterative scheme for the Test problem 1. We saw slightly less error difference at time $t = 1.5$ compared to the other time periods that we checked in the problem between the analytical and numerical solutions.

In this work, we presented the Jacobi iterative scheme and numerical results using Matlab software and saw the efficiency of the proposed method. Based on our numerical results, even for complex geometric domains, the FDM is rather simple to implement. Let us note that this approach can be applied to the most ill-posed problems [34, 35] which are reduced to the system of linear algebraic equations. The results obtained clearly showed that the Jacobi iterative scheme is accurate and also in a reasonable amount of time this method converged to the exact solution.

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