S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 22, No. 1, pp. 479–499 (2025)
https://doi.org/10.33048/semi.2025.22.032

УДК 510.64
MSC 03B20, 03B70

ON MODAL PRESENTATION OF EXPLOSIVE AND PARACONSISTENT EQUILIBRIUM LOGIC

S.P. ODINTSOV

Communicated by S.V. Sudoplatov

Abstract: Fariñas del Cerro, Herzig and Su proved that the non-monotonic consequence relation determined by Answer Set Semantics (ASP) for logic programs with negation-as-failure can be embedded into a monotonic modal logic via a variation of Gödel-Tarski Translation. This article generalizes the mentioned result to ASP for logic programs with two kinds of negation: negation-as-failure and strong negation and to PAS, the paraconsistent version of ASP admitting answer sets that are inconsistent w.r.t. the strong negation.

Keywords: logic programs, negation-as-failure, strong negation, equilibrium logic, deductive base, temporal logic, equilibrium modal theory.

The stable model (answer set) semantics for logic programs with negation-as-failure $\neg$ suggested by M. Gelfond and V. Lifschitz [7] gives rise to a separate paradigm in the setting of Logic Programming, so called Answer Set Programming (ASP). An important fact was established by D. Pearce [19], who proved that the intermediate logic HT of "here-and-there", which is known also as the Gödel-Smetanich logic and can be determined by a

Odintsov S.P., On modal presentation of explosive and paraconsistent equilibrium logic.

This work was supported by the Russian Science Foundation under grant no. 23-11-00104, https://rscf.ru/en/project/23-11-00104.

Received March 25, 2025, Published May, 31, 2025.

Kripke frame with two worlds, can serve as a tool for reasoning about answer sets. The main property involved is that answer sets can be viewed as a certain kind of minimal HT-models, which are called equilibrium models. The same holds for logic programs in the extended language, which includes not only the negation-as-failure $\neg$, but also the strong negation $\sim$ (which was originally introduced in Logic Programming under the name of classical negation [8]). In this case [19] answer sets are in one-to one correspondence with equilibrium $\mathbf{N_5}$-models, where $\mathbf{N_5}$ can be considered as HT enriched with the strong negation. More exactly, $\mathbf{N_5}$ is a finite-valued extension of the explosive Nelson logic **N3**, which can be determined via a 5-element algebra. On the other hand, $\mathbf{N_5}$ is the least conservative extension of HT in the lattice of **N3**-extensions. The logic **N3** is based on the concept of constructible falsity, which was introduced into logic by D. Nelson [13] via his system of constructive arithmetic with strong negation. The propositional fragment of Nelson's arithmetic, which is denoted now as **N3**, was subsequently axiomatised by N. Vorob'ev [24, 25].

A second key property relating non-classical logics with ASP was established in [10]: programs are strongly equivalent wrt answer set semantics if and only if they are equivalent viewed as propositional theories in HT (in $\mathbf{N}_5$ if $\sim$ occurs in the language). Here, two programs Π_1 and Π_2 are called strongly equivalent if for any program Π, $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same answer sets. This shows that HT and $\mathbf{N_5}$ can be used for program transformation and optimisation.

Paraconsistent version of answer set semantics (PAS) admits answer sets that are inconsistent w.r.t. the stong negation. PAS was studied as a logic programming semantics by C. Sakama and K. Inoue [21]. Later, the work [1] has made some progress towards a logical, declarative style of characterization for PAS. However, [1] does not axiomatize or otherwise syntactically characterize the underlying (monotonic) logic of PAS. In [16], it was proved that semantical frames for the substructural logics used in [1] can be reduced to a simpler Routley frames [20] with additional falsity constant. This provides a description of paraconsistent answer sets as a special kind of minimal Routley models. It is proved in [16] that these Routley models determine an extension of the paraconsistent Nelson logic $\mathbf{N4}^{\perp}$[14]. This extension was denoted $\mathbf{N_9}$ due to the reason that it can be determined via a 9-element algebra. Again, $\mathbf{N_9}$ is the least conservative extension of HT in the class of $\mathbf{N4}^{\perp}$-extensions. Finally, the strong equivalence theorem was proved [16]: two programs are strongly equivalent iff they are equivalent as $\mathbf{N_9}$-theories.

The next step towards the declarative treatment of ASP was done by L. Fariñas del Cerro, A. Herzig and E. Su [5]. They proved that the nonmonotonic consequence relation determined by Answer Set Semantics (ASP) for logic programs with negation-as-failure can be embedded into a monotonic modal logic **MEM** via a variation of Gödel-Tarski Translation [9]. This article generalizes the mentioned result to the ordinary and paraconsistent versions of ASP for logic programs with two kinds of negation: negation-as-failure and strong negation. To this end we need the possibility to embed the constructive logic with strong negation into a suitable modal logic.

Belnapian version of normal modal logic **BS4** [17] relates to **S4** in exacly the same way as the logic $\mathbf{N4}^{\perp}$ relates to intuitionistic logic. Its semantics can be obtained from that of **S4** via replacement of two-valued valuations by four valued ones. In each of possible worlds a formula may have one of four truth values $True$, $False$, $Neither$, $Both$ of Belnap-Dunn matrix **BD4** [2], which provide a semantics for First Degree Entailment **FDE** [4]. In [17] it was proved that $\mathbf{N4}^{\perp}$ is faithfully embedded into the logic **BS4** via the translation T_B, a natural modification of Gödel-Tarski translation. This result shows that modal companions of Nelson's logic extensions defined via T_B belong to the lattice of **BS4**-extensions (see [23] for details). So logics based on **BS4**-extensions looks suitable for the goals of this article.

In our reasoning we will essentailly follow the line depicted in [5], but we make one serious modification. Following [6] we understand a logic as a structural Tarskian consequence relation defined over some propositional language. According to this definition **MEM** of [5] is rather a theory than a logic because it is not closed under substitutions. In our work we try maximally distinguish 'logical' and 'theoretical' parts of construction. First we define a kind of Belnapian temporal logic $\mathbf{BSK_{t2}}$ and proof that the Belnapian version T_B of Gödel-Tarski translation faithfully embeds the logic $\mathbf{N_9}$ (the deductive base of PAS) into $\mathbf{BSK_{t2}}$, and the same holds for $\mathbf{N_5}$ and the explosive version of $\mathbf{BSK_{t2}}$. Further, we define theories over $\mathbf{BSK_{t2}}$ and its explosive extension and prove that equilibrium entailments over $\mathbf{N_9}$ and $\mathbf{N_5}$ can be embedded into this theories.

The paper is structured as follows. Section 1 contains neccessary information on contsructive logics, Belnapian modal logics and semantics of logic programs with negations. In Section 2 we define a special temporal logic $\mathbf{BSK_t}$ such that future and past modalities of $\mathbf{BSK_t}$ are defined via accessibility relations that are not mutually inverse. We introduce also the logic $\mathbf{BSK_{t2}}$ that extends $\mathbf{BSK_t}$ imposing further restrictions on both accessibility relations. Section 3 investigates the $\blacksquare$-free fragment of $\mathbf{BSK_{t2}}$ ($\blacksquare$ stands for the necessity in the past). We prove that this fragment of $\mathbf{BSK_{t2}}$ is a modal companion of the deductive base of the equilibrium entailment. Finally, in Section 4 we embed the equilibrium entailment into $\mathbf{BSK_{t2}}$.

1 Preliminaries

As usual by a *propositional language* $\mathcal{L}$ we mean a finite tuple of logical connectives and constants. The set $\mathrm{Form}_{\mathcal{L}}$ of $\mathcal{L}$-formulas is constructed in a usual way from the fixed countable set Prop of propositional variables and the constants of $\mathcal{L}$ with the help of $\mathcal{L}$-connectives. The languages we consider will include the implication connective $\to$. We will define logics in different propositional languages via Hilbert style deductive systems. And we assume that every deductive system under consideration includes the standard axioms of intuitionistic logic **Int** in the list of its axioms and the rules of *modus ponens* (MP) and of substitution (SUB)

$$\text{(MP)}\ \ \frac{\varphi, \varphi \to \psi}{\psi}, \quad \text{SUB}\ \ \frac{\varphi(p_1, \dots, p_n)}{\varphi(\psi_1, \dots, \psi_n)}$$

in the set of its inference rules. With every logic L defined in the language $\mathcal{L}$ we associate the inference relation $\vdash_L$. For a subset $\Gamma \cup \{\varphi\} \subseteq \mathrm{Form}_{\mathcal{L}}$, $\Gamma \vdash_L \varphi$ means that φ can be obtained from the elements of Γ and the theorems of L with the help of (MP). Recall that a theorem of L is a formula, which can be inferred from the axioms of L with the help of all inference rules, not only (MP). We write $\varphi \in L$ instead of 'φ is a theorem of L'.

For a logic L in the language $\mathcal{L}$, we denote by $\mathcal{E}L$ the family of all axiomatic extensions of L in the same language.

If L_i is a logic in the language $\mathcal{L}_i$, $i = 1, 2$, $L_1 \cup L_2$ denotes a logic in the language $\mathcal{L}_1 \cup \mathcal{L}_2$ defined by the union of axioms of L_1 and L_2 and the union of rules of these logics.

A proper subset $\Gamma \subseteq \mathrm{Form}_{\mathcal{L}}$ is said to be a *prime L-theory* if (i) Γ contains all L-theorems; (ii) Γ is closed under (MP) ($\varphi,\ \varphi \to \psi \in \Gamma$ implies $\psi \in \Gamma$); (iii) Γ satisfies the disjunction property ($\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$). Notice that the axioms of **Int** and (MP) allow to prove in a standard way the Extension Lemma for every logic L considered in the article.

Lemma 1. *Let $\Gamma \nvdash_L \varphi$. Then there exists a prime L-theory Σ such that $\Gamma \subseteq \Sigma$ and $\Sigma \nvdash_L \varphi$.*

1.1. Constructive logics with strong negation. The paraconsistent version $\mathbf{N4}^{\perp}$ [14] of Nelson's constructive logic with strong negation is defined in the propositional language $\mathcal{L}^{\sim}$ including the absurdity constant $\perp$ and logical connectives $\wedge, \vee, \to, \sim$, standing respectively for conjunction, disjunction, weak implication and strong negation. The set $\mathrm{Lit}^{\sim}$ of *literals* is defined as $\mathrm{Prop} \cup \{\sim p \mid p \in \mathrm{Prop}\}$. Arbitrary $\mathbf{S} \subseteq \mathrm{Lit}^{\sim}$ can be represented as $\mathbf{S} = (\mathbf{S}^+, \mathbf{S}^-)$, where

$$\mathbf{S}^+ = \mathbf{S} \cap \mathrm{Prop} \ \text{ and } \ \mathbf{S}^- = \{p \mid \sim p \in \mathbf{S}\}$$

We say that $\mathbf{S}$ is *consistent* if $\mathbf{S}^+ \cap \mathbf{S}^- = \varnothing$.

The Hilbert style deductive system for $\mathbf{N4}^{\perp}$ has (SUB) and (MP) as its only inference rules. The axioms include the standard list of axioms of intuitionistic logic in the language $\{\wedge, \vee, \to, \perp\}$:

I1. $p \to (q \to p)$ **I2.** $(p \wedge q) \to p$
I3. $(p \to (q \to \gamma)) \to ((p \to q) \to (p \to \gamma))$ **I4.** $(p \wedge q) \to q$
I5. $(p \to q) \to ((p \to \gamma) \to (p \to (q \wedge \gamma)))$ **I6.** $p \to (p \vee q)$
I7. $(p \to \gamma) \to ((q \to \gamma) \to ((p \vee q) \to \gamma))$ **I8.** $q \to (p \vee q)$
I9. $\perp \to p$

plus the following strong negation axioms (where $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$):

N1. $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$ **N2.** $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
N3. $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ **N4.** $\sim\sim p \leftrightarrow p$
N5. $\sim\bot$

The explosive logic $\mathbf{N3}^{\bot}$ is obtained via adding $(p \wedge \sim p) \to q$ to the list of $\mathbf{N4}^{\bot}$-axioms, symbolically $\mathbf{N3}^{\bot} = \mathbf{N4}^{\bot} + \{(p \wedge \sim p) \to q\}$. Notice that intuitionistic logic **Int** coincides with the $\sim$-free fragment of both logics, $\mathbf{N4}^{\bot}$ and $\mathbf{N3}^{\bot}$.

Kripke style semantics for Nelson's Logics is defined as follows. We say that a pair $\mathcal{W} = \langle W, \leq \rangle$ is a *frame* if $\leq$ is a preorder on W, i.e., a reflexive and transitive relation. An $\mathbf{N4}^{\bot}$-*model* (4-*model* over frame $\mathcal{W}$) is a tuple $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$, where $\mathcal{W}$ is a frame and valuations $v^+, v^- : \mathrm{Prop} \to \langle W, \leq \rangle^+$ are such that for $x, y \in W$, $p \in \mathrm{Prop}$, and $\epsilon \in \{+, -\}$ we have

$$(x \in v^{\epsilon}(p) \text{ and } x \leq y) \text{ implies } y \in v^{\epsilon}(p). \tag{1}$$

In other words, both $v^+(p)$ and $v^-(p)$ are *cones* w.r.t. $\leq$.

An $\mathbf{N4}^{\bot}$-model $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is said to be an $\mathbf{N3}^{\bot}$-*model* (3-*model* over $\mathcal{W}$) if

$$v^+(p) \cap v^-(p) \;=\; \varnothing \;\text{ for all }\; p \in \mathrm{Prop}. \tag{2}$$

Now we define two different relations $\models^+$ and $\models^-$ for verification and falsification of formulas in worlds of the model. Naturally, we use v^+ and v^- to define verification and falsification of propositional variables:

$$\mathcal{M}, x \models^+ p \;\Leftrightarrow\; x \in v^+(p)\,; \quad \mathcal{M}, x \models^- p \;\Leftrightarrow\; x \in v^-(p)$$

Verification and falsification of complex formulas are defined as follows:

$$\begin{array}{lll}
\mathcal{M}, x \models^+ \alpha \wedge \beta & \text{iff} & \mathcal{M}, x \models^+ \alpha \text{ and } \mathcal{M}, x \models^+ \beta \\
\mathcal{M}, x \models^- \alpha \wedge \beta & \text{iff} & \mathcal{M}, x \models^- \alpha \text{ or } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \models^+ \alpha \vee \beta & \text{iff} & \mathcal{M}, x \models^+ \alpha \text{ or } \mathcal{M}, x \models^+ \beta \\
\mathcal{M}, x \models^- \alpha \vee \beta & \text{iff} & \mathcal{M}, x \models^- \alpha \text{ and } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \models^+ \alpha \to \beta & \text{iff} & \forall y \geq x \; (\mathcal{M}, y \not\models^+ \alpha \text{ or } \mathcal{M}, y \models^+ \beta) \\
\mathcal{M}, x \models^- \alpha \to \beta & \text{iff} & \mathcal{M}, x \models^+ \alpha \text{ and } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \not\models^+ \bot & \text{and} & \mathcal{M}, x \models^- \bot \\
\mathcal{M}, x \models^+ \sim \alpha & \text{iff} & \mathcal{M}, x \models^- \alpha \\
\mathcal{M}, x \models^- \sim \alpha & \text{iff} & \mathcal{M}, x \models^+ \alpha
\end{array}$$

The persistence condition (1) can be generalized to arbitrary formulas, i.e., for every $\varphi \in \mathrm{Form}_{\mathcal{L}^{\sim}}$ and $\epsilon \in \{+, -\}$ we have

$$(\mathcal{M}, x \models^{\epsilon} \varphi \text{ and } x \leq y) \text{ implies } \mathcal{M}, y \models^{\epsilon} \varphi. \tag{3}$$

If $\mathcal{M}$ is an $\mathbf{N3}^{\bot}$-model, the consistency condition (2) also can be generalized to arbitrary formulas, i.e., for any φ and x we have

$$\mathcal{M}, x \not\models^+ \varphi \;\text{ or }\; \mathcal{M}, x \not\models^- \varphi. \tag{4}$$

We say that φ is *true* on $\mathcal{M}$ and write $\mathcal{M} \models \varphi$ if $\mathcal{M}, x \models^+ \varphi$ for all $x \in W$. We write $\mathcal{W} \models_4 \varphi$ if φ is true on every 4-model over $\mathcal{W}$, and $\mathcal{W} \models_3 \varphi$ if $\mathcal{M} \models \varphi$ for every 3-model $\mathcal{M}$ over $\mathcal{W}$. For $\Gamma \subseteq \mathrm{Form}_{\mathcal{L}^{\sim}}$ and a world x of $\mathcal{M}$, we write $\mathcal{M} \models \Gamma$ ($\mathcal{M}, x \models^{\epsilon} \Gamma$, $\epsilon \in \{+,-\}$) if $\mathcal{M} \models \varphi$ ($\mathcal{M}, x \models^{\epsilon} \varphi$) for all $\varphi \in \Gamma$. If $\mathcal{M} \models \Gamma$, we say that $\mathcal{M}$ is a *model* of Γ. In a similar way, we write $\mathcal{W} \models_{\epsilon} \Gamma$ and say that $\mathcal{W}$ is an ϵ-*model* of Γ, where $\epsilon \in \{3,4\}$, if $\mathcal{W} \models_{\epsilon} \varphi$ for all $\varphi \in \Gamma$. Finally, we write $\Gamma \models_{\mathcal{M}} \varphi$ if for every world x of $\mathcal{M}$ we have $\mathcal{M}, x \models \varphi$, whenever $\mathcal{M}, x \models \Gamma$.

For $L \in \mathcal{E}\mathbf{N4}^{\perp}$ and a class of frames $\mathcal{K}$ we say that

- L is *weakly* 3-*complete* (4-*complete*) w.r.t. $\mathcal{K}$ if for every $\varphi \in \mathrm{Form}_{\mathcal{L}^{\sim}}$

$\varphi \in L$ iff $\mathcal{M} \models \varphi$ for every 3-model (4-model) $\mathcal{M}$ over $\mathcal{W} \in \mathcal{K}$.

- L is *strongly* 3-*complete* (4-*complete*) w.r.t. $\mathcal{K}$ if for every $\Gamma \subseteq \mathrm{Form}_{\mathcal{L}^{\sim}}$ and $\varphi \in \mathrm{Form}_{\mathcal{L}^{\sim}}$ we have

$\Gamma \vdash_L \varphi$ iff $\Gamma \models_{\mathcal{M}} \varphi$ for every 3-model (4-model) $\mathcal{M}$ over $\mathcal{W} \in \mathcal{K}$.

Obviously, the strong 4- or 3-completeness implies the weak 4- or 3-completeness w.r.t. the same class of frames.

The following characterization of $\mathbf{N4}^{\perp}$ [11] and $\mathbf{N3}^{\perp}$ [22] is well known:

- $\mathbf{N4}^{\perp}$ is strongly 4-complete w.r.t. the class of all frames;
- $\mathbf{N3}^{\perp}$ is strongly 3-complete w.r.t. the class of all frames.

Recall that $HT = \mathbf{Int} + \{p \vee (p \to q) \vee \neg q\}$, where $\neg q$ abbreviates $q \to \perp$, is the greatest extension of $\mathbf{Int}$ different from the classical logic $\mathbf{CL}$. HT is known as Gödel-Smetanich logic or "hear-and-there" logic. We are interested in $\mathbf{N4}^{\perp}$- and $\mathbf{N3}^{\perp}$-extensions via the same axiom:

$$\mathbf{N_9} = \mathbf{N4}^{\perp} + \{p \vee (p \to q) \vee \neg q\} \quad \text{and} \quad \mathbf{N_5} = \mathbf{N3}^{\perp} + \{p \vee (p \to q) \vee \neg q\}.$$

Both logics $\mathbf{N_9}$ and $\mathbf{N_5}$ are determined by the same two-element partially ordered frame $\mathcal{W}^{HT} = \langle W^{HT}, \leq \rangle$, where $W^{HT} = \{h, t\}$ and $h \leq t$. More exactly, we have:

- $\mathbf{N_9}$ is strongly 4-complete w.r.t. the class $\{\mathcal{W}^{HT}\}$;
- $\mathbf{N_5}$ is strongly 3-complete w.r.t. the class $\{\mathcal{W}^{HT}\}$.

The choice of notation $\mathbf{N_9}$ and $\mathbf{N_5}$ is conditioned by the facts that $\mathbf{N_9}$ can be determined by a 9-element algebra [16], and $\mathbf{N_5}$ by a 5-element algebra [19].

Since we have only two worlds, an $\mathbf{N_9}$-model $\mathcal{M}$ (over $\mathcal{W}^{HT}$) is completely determined by sets of literals verified in the worlds h and t, so it can be identified with a pair $\langle \mathbf{H}, \mathbf{T} \rangle$, where $\mathbf{H}, \mathbf{T} \subseteq \mathrm{Lit}^{\sim}$ and

$$\mathbf{H}^{+} = \{p \mid \mathcal{M}, h \models^{+} p\}, \quad \mathbf{H}^{-} = \{p \mid \mathcal{M}, h \models^{-} p\},$$

$$\mathbf{T}^{+} = \{p \mid \mathcal{M}, t \models^{+} p\}, \quad \mathbf{T}^{-} = \{p \mid \mathcal{M}, t \models^{-} p\}.$$

In view of (1) we have $\mathbf{H} \subseteq \mathbf{T}$. If $\mathcal{M}$ is an $\mathbf{N_5}$-model, the pair $\langle \mathbf{H}, \mathbf{T} \rangle$ satisfies additionally the condition that $\mathbf{H}$ and $\mathbf{T}$ are consistent.

Further, we put

$$\mathbf{B4} = \mathbf{N4}^{\perp} + \{p \vee \neg p\} \quad \text{and} \quad \mathbf{B3} = \mathbf{N3}^{\perp} + \{p \vee \neg p\}.$$

These logics can be considered as expansions of four- and three-valued Belnap-Dunn logics (see [15] and [12] for details) via connectives $\to$ and $\bot$. They are characterized by a one-element frame $\mathcal{W}^T = \langle \{t\}, \leq \rangle$:

- **B4** is strongly 4-complete w.r.t. the class $\{\mathcal{W}^T\}$;
- **B3** is strongly 3-complete w.r.t. the class $\{\mathcal{W}^T\}$.

Naturally, every **B4**-model $\mathcal{M}$ (over $\mathcal{W}^T$) can be identified with the set $\mathbf{T}$ of literals verified at t, i.e.

$$\mathbf{T}^+ = \{p \mid \mathcal{M}, h \models^+ p\}, \quad \mathbf{T}^- = \{p \mid \mathcal{M}, h \models^- p\},$$

If $\mathcal{M}$ is a **B3**-model, then $\mathbf{T}$ must be consistent.

1.2. Belnapian modal logics. The Belnapian versions **BK** and **BS4** of normal modal logics **K** and, respectively, **S4** were defined in [17]. We define **BK** in the language $\mathcal{L}^\Box = \mathcal{L}^\sim \cup \{\Box\}$ as it was done in [18]. The possibility operator is defined as $\Diamond\varphi := \sim\Box\sim\varphi$. We also need the following abbreviations: $\neg\varphi := \varphi \to \bot$, $\varphi \Leftrightarrow \psi := (\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi)$. The list of axioms of **BK** includes the following groups of axioms:

I. The axioms of classical logic in the language $\{\wedge, \vee, \to, \bot\}$.

II. The strong negation axioms of $\mathbf{N4}^\bot$ plus

$$\neg\sim\Box p \leftrightarrow \Box\neg\sim p$$

III. The modal axiom of **K**: $\Box(p \to q) \to (\Box p \to \Box q)$

The list of inference rules includes (SUB), (MP) and the normalization rule ($\mathrm{NR}_\Box$):

$$\frac{\varphi}{\Box\varphi}$$

The following formulas are **BK**-theorems:

$$\neg\Box p \leftrightarrow \Diamond\neg p, \quad \neg\Diamond p \leftrightarrow \Box\neg p, \quad \Diamond(p \wedge q) \to (\Diamond p \wedge \Diamond q) \tag{5}$$

Logic **BS4** is an extension of **BK** obtained via adding the modal axioms of **S4**, i.e.,

$$\mathbf{BS4} = \mathbf{BK} + \{\Box p \to p,\ \Box p \to \Box\Box p\}.$$

The explosive extensions of **BK** and **BS4** are defined as follows:

$$\mathbf{B3K} = \mathbf{BK} + \{(p \wedge \sim p) \to q\}, \quad \mathbf{B3S4} = \mathbf{BS4} + \{(p \wedge \sim p) \to q\}$$

To define Kripke style semantics for **BK** we use the same frames as for **K**. Namely, we say that a pair $\mathcal{W} = \langle W, R \rangle$ is an **K**-*frame* if R is a binary relation on W. A **BK**-*model* (4-*model* over $\mathcal{W}$) is a tuple $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$, where $v^+, v^- : \mathrm{Prop} \to 2^W$.

An **S4**-*frame* $\mathcal{W} = \langle W, R \rangle$ is a **K**-frame, where R is preorder. A **BS4**-*model* is a **BK**-model over an **S4**-frame.

A **B3K**-*model* $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ (3-*model* over $\mathcal{W}$) is a **BK**-model satisfying the consistency condition (2). A **B3S4**-*model* $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is a **B3K**-model over an **S4**-frame.

The verification $\models^+$ and falsification $\models^-$ relations between worlds and formulas are defined in exactly the same way as for $\mathbf{N4}^\perp$-models in case of propositional variables, constant $\perp$, and connectives $\vee$, $\wedge$, $\sim$. For $\to$ and $\Box$ we have:

$$\begin{array}{lll}
\mathcal{M}, x \models^+ \alpha \to \beta & \text{iff} & \mathcal{M}, x \not\models^+ \alpha \text{ or } \mathcal{M}, x \models^+ \beta \\
\mathcal{M}, x \models^- \alpha \to \beta & \text{iff} & \mathcal{M}, x \models^+ \alpha \text{ and } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \models^+ \Box\alpha & \text{iff} & \forall y (xRy \text{ implies } \mathcal{M}, y \models^+ \alpha) \\
\mathcal{M}, x \models^- \Box\alpha & \text{iff} & \exists y (xRy \text{ and } \mathcal{M}, y \models^- \alpha)
\end{array}$$

It is easy to see that for possibility operator we have then:

$$\begin{array}{lll}
\mathcal{M}, x \models^+ \Diamond\alpha & \text{iff} & \exists y (xRy \text{ and } \mathcal{M}, y \models^+ \alpha) \\
\mathcal{M}, x \models^- \Diamond\alpha & \text{iff} & \forall y (xRy \text{ implies } \mathcal{M}, y \models^- \alpha)
\end{array}$$

Again for a $\mathbf{B3K}$-model $\mathcal{M}$ the consistency condition (2) can be generalized to arbitrary formulas, i.e. $\mathcal{M}, x \not\models^+ \varphi$ or $\mathcal{M}, x \not\models^- \varphi$ for any φ and x.

The truth of a formulas in a $\mathbf{BK}$-model is defined via the verification relation, i.e., $\mathcal{M} \models \varphi$ means that $\mathcal{M}, x \models^+ \varphi$ for all $x \in W$. For a $\mathbf{K}$-frame $\mathcal{W}$ we write $\mathcal{W} \models_4 \varphi$ if $\mathcal{M} \models \varphi$ for every 4-model $\mathcal{M}$ over $\mathcal{W}$, and $\mathcal{W} \models_3 \varphi$ if $\mathcal{M} \models \varphi$ for every 3-model $\mathcal{M}$ over $\mathcal{W}$. For $\Gamma \cup \{\varphi\} \subseteq \mathrm{Form}_{\mathcal{L}^\Box}$, the relations $\mathcal{M} \models \Gamma$, $\Gamma \models_{\mathcal{M}} \varphi$, $\mathcal{W} \models_4 \Gamma$, and $\mathcal{W} \models_3 \Gamma$ are defined in an obvious way. For $L \in \mathcal{E}\mathbf{BK}$ and a class of $\mathbf{BK}$-frames $\mathcal{K}$, the sense of expressions 'L is weakly 4-*complete* (3-*complete*) w.r.t. the class $\mathcal{K}$' and 'L is strongly 4-*complete* (3-*complete*) w.r.t. the class $\mathcal{K}$' is defined in exactly the same way as for $\mathbf{N4}^\perp$-extensions.

If $\mathcal{W} = \langle W, R \rangle$ is an $\mathbf{S4}$-frame, $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is a 4-model over $\mathcal{W}$, and $K \subseteq W$ is a cone w.r.t. R ($x \in K$ and xRy imply $y \in K$), then

$$\mathcal{W}^K := \langle K, R \cap K^2 \rangle, \quad \mathcal{M}^K = \langle \mathcal{W}^K, v^+_K, v^-_K \rangle,$$

where $v^+_K(p) = v^+(p) \cap K$ and $v^-_K(p) = v^-(p) \cap K$. For any $x \in K$, $\varphi \in \mathrm{Form}_{\mathcal{L}^\Box}$, and $\epsilon \in \{+, -\}$ we have

$$\mathcal{M}, x \models^\epsilon \varphi \text{ iff } \mathcal{M}^K, x \models^\epsilon \varphi \tag{6}$$

In [17] the following results were proved:

- $\mathbf{BK}$ is strongly 4-complete w.r.t. the class of all $\mathbf{K}$-frames;
- $\mathbf{B3K}$ is strongly 3-complete w.r.t. the class of all $\mathbf{K}$-frames;
- $\mathbf{BS4}$ is strongly 4-complete w.r.t. the class of all $\mathbf{S4}$-frames;
- $\mathbf{B3S4}$ is strongly 3-complete w.r.t. the class of all $\mathbf{S4}$-frames.

Moreover, it was proved in [17] that $\mathbf{N4}^\perp$ and $\mathbf{N3}^\perp$ are faithfully embedded into $\mathbf{BS4}$ and, respectively, into $\mathbf{B3S4}$ via an analog T_B of the Gödel-Tarski translation that embeds $\mathbf{Int}$ into $\mathbf{S4}$. The translation $T_B : \mathrm{Form}_{\mathcal{L}^\sim} \to \mathrm{Form}_{\mathcal{L}^\Box}$ is defined as follows:

$$
\begin{aligned}
T_B p &= \Box p \\
T_B(\varphi \vee \psi) &= T_B\varphi \vee T_B\psi \\
T_B(\varphi \wedge \psi) &= T_B\varphi \wedge T_B\psi \\
T_B(\varphi \to \psi) &= \Box(T_B\varphi \to T_B\psi) \\
T_B\bot &= \bot
\end{aligned}
\qquad
\begin{aligned}
T_B{\sim}p &= \Box{\sim}p \\
T_B{\sim}(\varphi \vee \psi) &= T_B{\sim}\varphi \wedge T_B{\sim}\psi \\
T_B{\sim}(\varphi \wedge \psi) &= T_B{\sim}\varphi \vee T_B{\sim}\psi \\
T_B{\sim}(\varphi \to \psi) &= T_B\varphi \wedge T_B{\sim}\psi \\
T_B{\sim}{\sim}\varphi &= T_B\varphi \\
T_B{\sim}\bot &= {\sim}\bot
\end{aligned}
$$

A logic $M \in \mathcal{E}\mathbf{BS4}$ is said to be a *modal companion* of $L \in \mathcal{E}\mathbf{N4}^{\bot}$ if T_B faithfully embeds L into M, i.e.

$$\varphi \in L \ \text{ iff } \ T_B\varphi \in M$$

for all $\varphi \in \mathrm{Form}_{\mathcal{L}^{\sim}}$. According to this definition **BS4** is a modal companion of $\mathbf{N4}^{\bot}$, and **B3S4** is a modal companion of $\mathbf{N3}^{\bot}$.

Let $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ be a **BS4**-model. Define the new valuations

$$v'^+(p) = \{w \in W \mid \mathcal{M}, w \models^+ \Box p\} \ \text{ and } \ v'^-(p) = \{w \in W \mid \mathcal{M}, w \models^+ \Box{\sim}p\}.$$

It is obvious that $\mathcal{M}' = \langle W, R, v'^+, v'^- \rangle$ is an $\mathbf{N4}^{\bot}$-model too.

Lemma 2. [17] *Let $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ be a* **BS4***-model, $x \in W$, and $\varphi \in \mathrm{Form}_{\mathcal{L}^{\sim}}$. Then*

$$\mathcal{M}', x \models^+ \varphi \ \Leftrightarrow \ \mathcal{M}, x \models^+ T_B\varphi.$$

This simple fact allows to prove (see [17]) that **BS4** is a modal companion of $\mathbf{N4}^{\bot}$, and **B3S4** is a modal companion of $\mathbf{N3}^{\bot}$.

1.3. Logic programming preliminaries. .

By a *logic program* Π we mean a set of rules of the form

$$(r) \quad \alpha_1 \vee \ldots \vee \alpha_k \leftarrow \beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\beta_{n+1} \wedge \ldots \wedge \neg\beta_{n+m},$$

where $\alpha_i, \beta_j \in \mathrm{Lit}^{\sim}$. We say that logic program Π is *normal* if $k = 1$ for all rules in Π, and that Π is *positive* (w.r.t. $\neg$) if $m = 0$ for all rules in Π.

Thus, the programs under consideration may contain two kinds of negation: the default negation, or negation-as-failure, denoted as $\neg$ (usually written as 'not') and the strong or explicit negation [8] that may occur in α_i and β_j. In what follows we will identify a rule of the form (r) with a formula

$$(\beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\beta_{n+1} \wedge \ldots \wedge \neg\beta_{n+m}) \ \to \ (\alpha_1 \vee \ldots \vee \alpha_k) \in \mathrm{Form}_{\mathcal{L}^{\sim}},$$

where $\neg\beta_j$ is understood as $\beta_j \to \bot$. A set $\mathbf{H} \subseteq \mathrm{Lit}^{\sim}$ is a model of a logic program Π if $\mathbf{H}$ is a **B4**-model of the set of all formulas corresponding to the rules of Π.

Now we recall the definition of stable models. Notice that originally Gelfond and Lifschitz [7] defined stable models for positive normal programs.

Let Π be a logic program and $\mathbf{T} \subseteq \mathrm{Lit}^{\sim}$. The *Gelfond-Lifschitz reduct* (*GL-reduct*) of Π w.r.t. $\mathbf{T}$ is a positive program obtained from Π in two steps. First, we exclude from Π all rules containing $\neg\beta_i$ for $\beta_i \in \mathbf{T}$. Second, we delete all conjunctive terms of the form $\neg\beta_i$ from the rest of rules.

We say that $\mathbf{T} \subseteq \mathrm{Lit}^{\sim}$ is a *stable model* of Π if $\mathbf{T} \models \Pi^{\mathbf{T}}$, and $\mathbf{H} \models \Pi^{\mathbf{T}}$ for $\mathbf{H} \subseteq \mathbf{T}$ implies $\mathbf{H} = \mathbf{T}$. In other words, $\mathbf{T}$ is a minimal w.r.t. inclusion $\mathbf{B4}$-model of $\Pi^{\mathbf{T}}$.

Now we define the relation $\trianglelefteq$ among $\mathbf{N_9}$-models as follows. Let $\langle \mathbf{H}_1, \mathbf{T}_1\rangle$ and $\langle \mathbf{H}_2, \mathbf{T}_2\rangle$ be $\mathbf{N_9}$-models. We set

$$\langle \mathbf{H}_1, \mathbf{T}_1\rangle \trianglelefteq \langle \mathbf{H}_2, \mathbf{T}_2\rangle \ \text{ iff } \ \mathbf{T}_1 = \mathbf{T}_2 \ \text{ and } \ \mathbf{H}_1 \subseteq \mathbf{H}_2.$$

An $\mathbf{N_9}$-model of the form $\langle \mathbf{T}, \mathbf{T}\rangle$ is called *total*.

For an arbitrary subset $\Gamma \subseteq \mathrm{Form}_{\mathcal{L}^{\sim}}$, a total model $\langle \mathbf{T}, \mathbf{T}\rangle$ is said to be an *equilibrium model of* Γ if $\langle \mathbf{T}, \mathbf{T}\rangle \models \Gamma$ and there is no $\mathbf{H} \subseteq \mathrm{Lit}^{\sim}$ such that $\mathbf{H} \neq \mathbf{T}$ and $\langle \mathbf{H}, \mathbf{T}\rangle \models \Gamma$. In other words an equilibrium model of Γ is a total model of Γ, which is $\trianglelefteq$-minimal in the class of $\mathbf{N_9}$-models of Γ.

For logic programs, there is a close connection between stable and equilibrium models.

Theorem 1. [16] *For a logic program* Π*, a set* $\mathbf{T} \subseteq \mathrm{Lit}^{\sim}$ *is a stable model of* Π *iff* $\langle \mathbf{T}, \mathbf{T}\rangle$ *is an equilibrium model of* Π*.*

Originally [16] this statement was proved via the reduction to the results of [1], a short direct proof can be found in [12].

In what follows, $\mathcal{E}l_9(\Gamma)$ denotes the set of all equilibrium models of Γ, and $\mathcal{E}l_5(\Gamma)$ denotes the set of all consistent equilibrium models of Γ, i.e. the set of those equilibrium models of Γ that are $\mathbf{N_5}$-models. We define equilibrium consequence relations as follows:

$$\Gamma \mid\!\sim^{9}_{el} \varphi \ \text{ iff } \ \langle \mathbf{H}, \mathbf{T}\rangle \models \varphi \ \text{ for every} \langle \mathbf{H}, \mathbf{T}\rangle \in \mathcal{E}l_9(\Gamma).$$

$$\Gamma \mid\!\sim^{5}_{el} \varphi \ \text{ iff } \ \langle \mathbf{H}, \mathbf{T}\rangle \models \varphi \ \text{ for every} \langle \mathbf{H}, \mathbf{T}\rangle \in \mathcal{E}l_5(\Gamma).$$

2 Special temporal logic

Similar to [5] we define a special temporal logic, where the future ($\Box$, $\Diamond$) and past ($\blacksquare$, $\blacklozenge$) modalities are defined via accessibility relations that are not mutually inverse, but however are closely connected. Prior to do it we recall the definition of a fusion of modal logics, and of the temporary version of $\mathbf{BS4}$ defined in [17].

Let $\mathcal{L}^{\blacksquare} := \mathcal{L}^{\sim} \cup \{\blacksquare\}$, $\mathcal{L}^{t} := \mathcal{L}^{\Box} \cup \{\blacksquare\}$, and $\blacklozenge\varphi := {\sim}\blacksquare{\sim}\varphi$.

For $L \in \mathcal{E}\mathbf{BK}$, we denote by $L_{\blacksquare}$ the logic in the language $\mathcal{L}^{\blacksquare}$ defined via the same axioms and rules as L but with $\Box$ replaced by $\blacksquare$. Clearly, $L_{\blacksquare} \in \mathcal{E}\mathbf{BK}_{\blacksquare}$.

For L^1, $L^2 \in \mathcal{E}\mathbf{BK}$, we put $L^1 * L^2 := L^1 \cup L^2_{\blacksquare}$. We say that $L^1 * L^2$ is a *fusion* of logics L^1 and L^2.

The temporal version $\mathbf{BS4_t}$ of $\mathbf{BS4}$ was defined in [17] as

$$\mathbf{BS4_t} = \mathbf{BS4} * \mathbf{BS4} + \{p \to \Box\blacklozenge p,\ p \to \blacksquare\Diamond p\}.$$

Frames and models are defined for $\mathbf{BS4}_t$ in the same way as for **BS4**. For connectives of $\mathcal{L}^{\Box}$, the verification and falsification also are defined as for **S4**-frames. For $\blacksquare$ we have:

$$\begin{array}{lll} \mathcal{M}, x \vDash^+ \blacksquare\alpha & \text{iff} & \forall y (yRx \text{ implies } \mathcal{M}, y \vDash^+ \alpha) \\ \mathcal{M}, x \vDash^- \blacksquare\alpha & \text{iff} & \exists y (yRx \text{ and } \mathcal{M}, y \vDash^- \alpha) \end{array}$$

As a consequence for $\blacklozenge$ we have:

$$\begin{array}{lll} \mathcal{M}, x \vDash^+ \blacklozenge\alpha & \text{iff} & \exists y (yRx \text{ and } \mathcal{M}, y \vDash^+ \alpha) \\ \mathcal{M}, x \vDash^- \blacklozenge\alpha & \text{iff} & \forall y (yRx \text{ implies } \mathcal{M}, y \vDash^- \alpha) \end{array}$$

All related notions are modified for the language $\mathcal{L}^t$ in an obvious way. Naturally, $\mathbf{BS4_t}$ is strongly 4-complete w.r.t. the class of all **S4**-frames, and $\mathbf{B3S4_t} = \mathbf{B3S4_t} + \{(p \wedge {\sim}p) \to q\}$ is strongly 3-complete w.r.t. the class of all **S4**-frames.

Further, let us consider the fusion

$$\mathbf{BSK} = \mathbf{BS4} * \mathbf{BK}$$

and its explosive extension $\mathbf{B3SK} = \mathbf{BSK} + \{(p \wedge {\sim}p) \to q\}$.

A **BSK**-frame is a tuple $\mathcal{W} = \langle W, R, S\rangle$, where R is a preoder on W and $S \subseteq W^2$. A **BSK**-model $\mathcal{M} = \langle \mathcal{W}, v^+, v^-\rangle$ is a **BSK**-frame $\mathcal{W}$ augmented with two valuations $v^+, v^- : \mathrm{Prop} \to 2^W$. One can combine in an obvious way the completeness results for **BS4**, **BK** and their explosive extensions from [17] to obtain the following

Theorem 2. *Logic* **BSK** *(***B3SK***) is strongly 4-complete (3-complete) w.r.t. the class of* **BSK***-frames.*

We denote by id_W the diagonal of W, i.e., $\mathrm{id}_W = \{(a, a) \mid a \in W\}$.

Proposition 1. *Let $\mathcal{W} = \langle W, R, S\rangle$ be a* **BSK***-frame and $\epsilon \in \{3, 4\}$. The following equivalences hold:*

1) $\mathcal{W} \models_\epsilon p \to \blacksquare\Diamond p$ *iff* $S \subseteq R^{-1}$;
2) $\mathcal{W} \models_\epsilon p \to \Box(p \vee \blacklozenge p)$ *iff* $R \subseteq S^{-1} \cup \mathrm{id}_W$.

Proof. We fix some **BSK**-frame $\mathcal{W} = \langle W, R, S\rangle$.

1) This is one of standard axioms of temporal logic, and since $\sim$ does not occur in this formula the three- or four-valued case should not differ from the ordinary one. However we provide this proof to be self contained. Let $S \subseteq R^{-1}$, and let $\mathcal{M} = \langle \mathcal{W}, v^+, v^-\rangle$ be a model over $\mathcal{W}$. Assume that $\mathcal{M}, x \models^+ p$ and choose some y with xSy. We have then yRx by $S \subseteq R^{-1}$, and so $\mathcal{M}, y \models^+ \Diamond p$. Since y is an arbitrary S-successor of x, we have $\mathcal{M}, x \models^+ \blacksquare\Diamond p$. We have thus proved that $S \subseteq R^{-1}$ implies $\mathcal{W} \models_4 p \to \blacksquare\Diamond p$ and, in particular, $\mathcal{W} \models_3 p \to \blacksquare\Diamond p$.

To prove the inverse implication we assume that $S \not\subseteq R^{-1}$ and $x, y \in W$ are such that xSy but $\neg(yRx)$. Let $\mathcal{M} = \langle \mathcal{W}, v^+, v^-\rangle$ be such that $v^+(p) = \{x\}$ and $v^-(q) = \varnothing$ for all $q \in \mathrm{Prop}$. Obviously, $\mathcal{M}$ is a **B3SK**-model. In this case we have $\mathcal{M}, x \models^+ p$ and $\mathcal{M}, y \not\models^+ \Diamond p$, whence $\mathcal{M}, x \not\models^+ \blacksquare\Diamond p$. Thus, $S \not\subseteq R^{-1}$ implies $\mathcal{W} \not\models_3 p \to \blacksquare\Diamond p$, moreover, $\mathcal{W} \not\models_4 p \to \blacksquare\Diamond p$.

2) First we assume that $R \subseteq S^{-1} \cup \mathrm{id}_W$. Let $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ be a model over $\mathcal{W}$ and $x, y \in W$ be such that $\mathcal{M}, x \models^+ p$ and xRy. If $x = y$, then $\mathcal{M}, y \models^+ p \vee \blacklozenge p$. If $x \neq y$, then ySx, and we again have $\mathcal{M}, y \models^+ p \vee \blacklozenge p$. Consequently, $\mathcal{M}, x \models^+ \Box(p \vee \blacklozenge p)$. We proved thus $\mathcal{W} \models_4 p \to \Box(p \vee \blacklozenge p)$, moreover, $\mathcal{W} \models_3 p \to \Box(p \vee \blacklozenge p)$.

Now we assume that $R \not\subseteq S^{-1} \cup \mathrm{id}_W$ and choose $x, y \in W$ such that xRy, $x \neq y$, and $\neg(ySx)$. As in Item 1 we take a **B3SK**-model $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ such that $v^+(p) = \{x\}$ and $v^-(q) = \varnothing$ for all $q \in \mathrm{Prop}$. We have $\mathcal{M}, x \models^+ p$. At the same time the conditions $x \neq y$ and $\neg(ySx)$ imply $\mathcal{M}, y \not\models^+ p \vee \blacklozenge p$, whence $\mathcal{M}, x \not\models^+ \Box(p \vee \blacklozenge p)$. Thus, $\mathcal{W} \not\models_3 p \to \Box(p \vee \blacklozenge p)$ and $\mathcal{W} \not\models_4 p \to \Box(p \vee \blacklozenge p)$.

$\Box$

We define a weak version of $\mathbf{BS4_t}$ as follows:

$$\mathbf{BSK_t} := \mathbf{BSK} + \{p \to \blacksquare \Diamond p,\ p \to \Box(p \vee \blacklozenge p)\}.$$

We put also $\mathbf{B3SK_t} := \mathbf{BSK_t} + \{(p \wedge {\sim} p) \to q\}$.

To prove the completeness of $\mathbf{BSK_t}$ and of its axiomatic extensions via different classes of frames we will use the canonical model method.

First we notice that every prime L-theory Γ over $L \in \mathcal{E}\mathbf{BSK_t}$ is complete and consistent w.r.t. $\neg$. Indeed, $\varphi \vee \neg\varphi \in \Gamma$ since $\mathbf{BSK_t}$ contains axioms of classical logic in the language $\{\vee, \wedge, \to, \bot\}$. Consequently, the disjunction property of Γ implies $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. In particular, any two prime L-theories are incomparable w.r.t. set-theoretical inclusion $\subseteq$, i.e. for prime L-theories Γ and Δ we have:

$$\Gamma \neq \Delta \ \text{ implies } \ \Gamma \setminus \Delta \neq \varnothing. \tag{7}$$

If $\varphi, \neg\varphi \in \Gamma$, then $(\varphi \wedge \neg\varphi) \to \psi \in \Gamma$ implies that $\Gamma = \mathrm{Form}_{\mathcal{L}^t}$.

For $L \in \mathcal{E}\mathbf{BSK_t}$, the *canonical L-frame* is defined as $\mathcal{W}^L = \langle W^L, R^L, S^L \rangle$, where

- W^L is the set of all prime L-theories;
- $\Gamma R^L \Delta$ iff $\Gamma_\Box \subseteq \Delta$, where $\Gamma_\Box = \{\varphi \mid \Box\varphi \in \Gamma\}$;
- $\Gamma S^L \Delta$ iff $\Gamma_\blacksquare \subseteq \Delta$, where $\Gamma_\blacksquare = \{\varphi \mid \blacksquare\varphi \in \Gamma\}$;

The *canonical L-model* has the form $\mathcal{M}^L = \langle \mathcal{W}, v_L^+, v_L^- \rangle$, where

$$v_L^+(p) = \{\Gamma \in W^L \mid p \in \Gamma\} \text{ and } v_L^-(p) = \{\Gamma \in W^L \mid \sim p \in \Gamma\}.$$

The abbreviations $\Diamond\varphi := {\sim}\Box{\sim}\varphi$ and $\blacklozenge\varphi := {\sim}\Diamond{\sim}\varphi$ easily imply that

$$\Gamma R^L \Delta \text{ iff } \Delta^\Diamond \subseteq \Gamma; \quad \Gamma S^L \Delta \text{ iff } \Delta^\blacklozenge \subseteq \Gamma,$$

where $\Delta^\Diamond = \{\Diamond\varphi \mid \varphi \in \Delta\}$ and $\Delta^\blacklozenge = \{\blacklozenge\varphi \mid \varphi \in \Delta\}$. Further, by induction on the structure of formulas one can easily prove the canonical model lemma:

Lemma 3. *Let $L \in \mathcal{E}\mathbf{BSK_t}$. For every prime L-theory Γ and formula φ, the following equivalences hold:*

$$\mathcal{M}^L, \Gamma \models^+ \varphi \ \text{ iff } \ \varphi \in \Gamma; \quad \mathcal{M}^L, \Gamma \models^- \varphi \ \text{ iff } \ \sim\varphi \in \Gamma.$$

Theorem 3. *Logic* $\mathbf{BSK_t}$ ($\mathbf{B3SK_t}$) *is strongly 4-complete (3-complete) w.r.t. the class of* $\mathbf{BSK}$*-frames* $\langle W, R, S\rangle$ *such that*

$$R = S^{-1} \cup \mathrm{id}_W. \tag{8}$$

Proof. We omit the routine correctness proof and check the completeness. Let $\Gamma \nvdash_{\mathbf{BSK_t}} \varphi$. By Lemma 1 there is a prime $\mathbf{BSK_t}$-theory Δ with $\Gamma \subseteq \Delta$ and $\varphi \notin \Delta$. According to Lemma 3, we have

$$\mathcal{M}^{\mathbf{BSK_t}}, \Delta \models^+ \psi \text{ for all } \psi \in \Gamma \text{ and } \mathcal{M}^{\mathbf{BK}}, \Delta \not\models^+ \varphi.$$

It remains to check that the canonical $\mathbf{BSK_t}$-frame $\mathcal{W}^{\mathbf{BSK_t}}$ is a $\mathbf{BSK}$-frame, i.e., that the relation $R^{\mathbf{BSK_t}}$ is reflexive and transitive, and that

$$R^{\mathbf{BSK_t}} = (S^{\mathbf{BSK_t}})^{-1} \cup \mathrm{id}_{W^{\mathbf{BSK_t}}}. \tag{9}$$

As well as in case of normal modal logics (see [3, Theorem 5.16]) we can we can check that the axiom $\Box p \to p$ implies that $R^{\mathbf{BSK_t}}$ is reflexive, and that $\Box p \to \Box\Box p$ implies that $R^{\mathbf{BSK_t}}$ is transitive.

Let us check (9). For brevity we will omit the upper index $(\cdot)^{\mathbf{BSK_t}}$. Assume that $R \not\subseteq S^{-1} \cup \mathrm{id}_W$. In this case there are $\Gamma, \Delta \in W$ such that $\Gamma R \Delta$, $\Gamma \neq \Delta$, and $\neg(\Delta S \Gamma)$. The latter is equivalent to $\Delta^{\blacksquare} \not\subseteq \Gamma$. Let $\blacksquare\varphi \in \Delta$ and $\varphi \notin \Gamma$. The completeness of Γ implies $\neg\varphi \in \Gamma$. By (7) and $\Gamma \neq \Delta$ there is $\psi \in \Gamma \setminus \Delta$, so $\neg\varphi \wedge \psi \in \Gamma$. The axiom $p \to \Box(p \vee \blacklozenge p)$ implies

$$(\neg\varphi \wedge \psi) \to \Box((\neg\varphi \wedge \psi) \vee \blacklozenge(\neg\varphi \wedge \psi)) \in \Gamma.$$

By (MP) and $\Gamma R \Delta$ we obtain $(\neg\varphi \wedge \psi) \vee \blacklozenge(\neg\varphi \wedge \psi) \in \Delta$. Since $\psi \notin \Delta$, we have $\neg\varphi \wedge \psi \notin \Delta$, so $\blacklozenge(\neg\varphi \wedge \psi) \in \Delta$. From $\blacklozenge(\neg\varphi \wedge \psi) \to (\blacklozenge\neg\varphi \wedge \blacklozenge\psi) \in \mathbf{BSK_t}$, we conclude $\blacklozenge\neg\varphi \in \Delta$. By (5) $\neg\blacksquare\varphi \in \Delta$, which contradicts to the $\neg$-consistency of Δ. We have thus proved $R \subseteq S^{-1} \cup \mathrm{id}_W$.

Now we prove that $S \subseteq R^{-1}$. Let $\Gamma, \Delta \in W$ be such that $\Gamma S \Delta$, i.e., $\Delta^{\blacklozenge} \subseteq \Gamma$. If $\Box\varphi \in \Delta$, then $\blacklozenge\Box\varphi \in \Gamma$. By axiom $p \to \blacksquare\Diamond p$ we have $\neg\varphi \to \blacksquare\Diamond\neg\varphi \in \Gamma$. By (5) $\blacksquare\Diamond\neg\varphi \leftrightarrow \neg\blacklozenge\Box\varphi \in \Gamma$. Consequently, $\neg\varphi \to \neg\blacklozenge\Box\varphi \in \Gamma$, whence $\blacklozenge\Box\varphi \to \varphi \in \Gamma$. Finally, $\varphi \in \Gamma$. We have thus proved that $\Gamma S \Delta$ implies $\Delta R \Gamma$, which completes the proof of 4-completeness for $\mathbf{BSK_t}$.

In case of $\mathbf{B3SK_t}$ we only have to check that the canonical $\mathbf{B3SK_t}$-model is a 3-model, i.e., that $\{p, \sim p\} \subseteq \Gamma$ does not hold for any $p \in \mathrm{Prop}$ and $\Gamma \in W^{\mathbf{B3SK_t}}$. This fact readily follows from the $\mathbf{B3SK_t}$-axiom $(p \wedge \sim p) \to q$. $\Box$

Now it is natural to say that $\mathcal{W} = \langle W, R, S\rangle$ is a $\mathbf{BSK_t}$-*frame*, if $\mathcal{W}$ is a $\mathbf{BSK}$-frame and $R = S^{-1} \cup \mathrm{id}_W$.

We consider some further conditions on the accessibility relations of a $\mathbf{BSK_t}$-frame $\langle W, R, S\rangle$:

(1^R) $\forall x, y, z \in W((xRy \;\&\; xRz \;\&\; x \neq y \;\&\; x \neq z) \Rightarrow y = z)$;
(2^S) $\forall x, y(xSy \;\Rightarrow\; ySy)$;
(3^S) $\forall x, y, z \in W((xSy \;\&\; ySz \;\Rightarrow\; y = z)$.

We will need also $\mathcal{L}^t$-formulas:

$$\mathbf{alt}_2 : \; \Box p \vee \Box(p \to q) \vee \Box((p \wedge q) \to r), \quad \blacksquare \cdot \mathbf{T}_{\blacksquare} : \; \blacksquare(\blacksquare p \to p).$$

Proposition 2. *Let $\mathcal{W} = \langle W, R, S\rangle$ be a $\mathbf{BSK_t}$-frame and $\epsilon \in \{3, 4\}$. Then the following equivalences hold:*

(1) $\mathcal{W} \models_\epsilon \mathbf{alt}_2$ *iff* R *satisfies* (1^R)*;*

(2) $\mathcal{W} \models_\epsilon \blacksquare \cdot \mathbf{T}_\blacksquare$ *iff* S *satisfies* (2^S)*;*

(3) $\mathcal{W} \models_\epsilon \blacksquare(p \to \blacksquare p)$ *iff* S *satisfies* (3^S)*.*

Proof. (1) It is clear that the validity of a formula which does not contain $\sim$ and $\blacksquare$ on a frame $\langle W, R, S\rangle$ is equivalent to the validity of this formulas on a frame $\langle W, R\rangle$ for normal modal logics. It is also known (see,e.g. [3, Prop. 3.45]) that the validity of $\mathbf{alt}_2$ is equivalent to the condition that every world has at most 2 different R-successors. In view of reflexivity of R in $\mathbf{BSK_t}$-frames we obtain that the validity of $\mathbf{alt}_2$ in a $\mathbf{BSK_t}$-frame $\langle W, R, S\rangle$ is equivalent to the condition that every world has at most one proper R-successor, i.e., to (1^R).

(2) Again, it is known [3, Prop. 3.30] that the validity of $\blacksquare p \to p$ on $\langle W, R, S\rangle$ is equivalent to the reflexivity of S. The additional $\blacksquare$ in front of this formula restricts this condition to worlds that are S-successors. So the validity of $\blacksquare \cdot \mathbf{T}_\blacksquare$ is equivalent to (2^S).

(3) Assume that $\mathcal{W} = \langle W, R, S\rangle$ satisfies (3^S), i.e., every S-successor has no proper S-successors, and that $\mathcal{M}$ is a 4-model over $\mathcal{W}$. Check that $\mathcal{M} \models \blacksquare(p \to \blacksquare p)$. Let $x, y \in W$ and xSy. If $\mathcal{M}, y \models^+ p$ and ySz, then $y = z$ by (3^S) and $\mathcal{M}, z \models^+ p$. So $\mathcal{M}, y \models^+ p \to \blacksquare p$ and $\mathcal{M}, x \models^+ \blacksquare(p \to \blacksquare p)$.

Assume that $\mathcal{W} = \langle W, R, S\rangle$ is such that xSy, ySz, and $y \neq z$. Consider a 3-model cM over $\mathcal{W}$ such that $v^+(p) = \{y\}$ and $v^-(y) = \varnothing$. Then $\mathcal{M}, y \models^+ p$, $\mathcal{M}, y \not\models^+ \blacksquare p$, and so $\mathcal{M}, x \not\models^+ \blacksquare(p \to \blacksquare p)$. $\square$

Now we put

$$\mathbf{BSK_{t2}} := \mathbf{BSK_t} + \{\mathbf{alt}_2,\ \blacksquare(p \leftrightarrow \blacksquare p)\},\ \mathbf{B3SK_{t2}} := \mathbf{BSK_{t2}} + \{(p \wedge {\sim} p) \to q\}.$$

A $\mathbf{BSK_t}$-frame $\mathcal{W} = \langle W, R, S\rangle$ is said to be a $\mathbf{BSK_{t2}}$*-frame* if it satisfies the conditions (1^R), (2^S), (3^S).

Theorem 4. *Logic* $\mathbf{BSK_{t2}}$ $(\mathbf{B3SK_{t2}})$ *is strongly 4-complete (3-complete) w.r.t. the class of* $\mathbf{BSK_{t2}}$*-frames.*

Proof. Since $\blacksquare(p \leftrightarrow \blacksquare p)$ is equivalent to the conjunction of $\blacksquare \cdot \mathbf{T}_\blacksquare$ and $\blacksquare(p \to \blacksquare p)$, the correctness part follows from Proposition 2. For the completeness part, it would be enough to check that the canonical $\mathbf{BSK_{t2}}$-frame, which we denote for brevity as $\mathcal{W}_2 = \langle W_2, R_2, S_2\rangle$ is a $\mathbf{BSK_{t2}}$-frame. That $\mathcal{W}_2$ is a $\mathbf{BSK_t}$-frame can be checked as in the proof of Theorem 3. The axiom $\mathbf{alt}_2$ implies that every $\Gamma \in W_2$ has at most two R_2-successors. This fact and the reflexivity of R_2 imply the condition (1^R).

Now we check that $\Gamma S_2 \Delta$ implies $\Delta S_2 \Delta$ for all $\Gamma, \Delta \in W_2$. Let $\Gamma S_2 \Delta$, i.e., $\Gamma_\blacksquare \subseteq \Delta$. Assume that $\blacksquare\varphi \in \Delta$. Since $\blacksquare(\blacksquare\varphi \to \varphi)$ is a theorem of $\mathbf{BSK_{t2}}$ we have $\blacksquare(\blacksquare\varphi \to \varphi) \in \Gamma$, whence $\blacksquare\varphi \to \varphi \in \Delta$. Applying (MP) we obtain $\varphi \in \Delta$. So $\Delta_\blacksquare \subseteq \Delta$, and the condition (2^S) holds for $\mathcal{W}_2$.

Finally, we assume that $\Gamma, \Delta, \Sigma \in W_2$ are such that $\Gamma_{\blacksquare} \subseteq \Delta$ and $\Delta_{\blacksquare} \subseteq \Sigma$. We have to check that $\Delta = \Sigma$. Let $\varphi \in \Delta$. Since $\blacksquare(\varphi \to \blacksquare\varphi) \in \Gamma$, we have $\varphi \to \blacksquare\varphi \in \Delta$. By (MP) $\blacksquare\varphi \in \Delta$, and so $\varphi \in \Sigma$. We proved $\Delta \subseteq \Sigma$. By (7) we conclude $\Delta = \Sigma$. Thus, the condition (3^S) holds for the canonical $\mathbf{BSK}_{t2}$-frame too. $\square$

3 $\blacksquare$-free fragments of $\mathbf{BSK_{t2}}$ and $\mathbf{B3SK_{t2}}$

We define

$$\mathbf{BS4_2} = \mathbf{BS4} + \{\mathbf{alt}_2\}, \quad \mathbf{B3S4_2} = \mathbf{BS4_2} + \{(p \wedge {\sim}p) \to q\}$$

Proposition 3.

(1) *Logic* $\mathbf{BS4_2}$ ($\mathbf{B3S4_2}$) *is strongly 4-complete (3-complete) w.r.t. the class of* $\mathbf{BS4}$*-frames satisfying* (1^R).

(2) *Logic* $\mathbf{BS4_2}$ ($\mathbf{B3S4_2}$) *is strongly 4-complete (3-complete) w.r.t. the class* $\{\mathcal{W}^{HT}\}$.

Proof. (1) As above we use [3, Theorem 5.16] to check that the canonical frames of logics $\mathbf{BS4_2}$ and $\mathbf{B3S4_2}$ are $\mathbf{BS4}$-frames satisfying (1^R). According to this theorem the axiom $\Box p \to p$ implies the reflexivity of canonical frames. In a similar way, $\Box p \to \Box\Box p$ and $\mathbf{alt}_2$ imply the transitivity and the condition (1^R) respectively.

(2) We consider the case of $\mathbf{BS4_2}$. It is clear that $\mathcal{W}^{HT}$ is a $\mathbf{BS4}$-frame satisfying (1^R), so $\Gamma \vdash_{\mathbf{BS4_2}} \varphi$ implies $\Gamma \models_{\mathcal{M}} \varphi$ for every 4-model $\mathcal{M}$ over $\mathcal{W}^{HT}$.

Let $\Gamma \nvdash_{\mathbf{BS4_2}} \varphi$. Then $\mathcal{M}, x \models^+ \Gamma$ and $\mathcal{M}, x \not\models^+ \varphi$ for some 4-model $\mathcal{M}$ over a $\mathbf{BS4}$-frame $\mathcal{W} = \langle W, R\rangle$ satisfying (1^R). By (6) we have

$$\mathcal{M}^{x\uparrow}, x \models^+ \Gamma \text{ and } \mathcal{M}^{x\uparrow}, x \not\models^+ \varphi,$$

where $x\uparrow = \{y \in W \mid xRy\}$. The condition (1^R) implies that $|x\uparrow| \leq 2$, i.e., $\mathcal{W}^{x\uparrow}$ is isomorphic to $\mathcal{W}^{HT}$ or $\mathcal{W}^T$. Since $\mathcal{W}^T$ can be identified with the upper world of $\mathcal{W}^{HT}$, we obtain $\mathcal{M}', x \models^+ \Gamma$ and $\mathcal{M}', x \not\models^+ \varphi$ for some 4-model $\mathcal{M}'$ over $\mathcal{W}^{HT}$ and $x \in \{h, t\}$. $\square$

Corollary 1. *Logic* $\mathbf{BS4_{t2}}$ ($\mathbf{B3S4_{t2}}$) *is a conservative extension of* $\mathbf{BS4_2}$ ($\mathbf{BS4_2}$).

Proof. This statement readily follows from the observation that $\langle W, R, S\rangle$ is a $\mathbf{BSK_{t2}}$-frame iff $R = S^{-1} \cup \mathrm{id}_W$ and $\langle W, R\rangle$ is a $\mathbf{BS4}$-frame satisfying (1^R). $\square$

Theorem 5. *The logic* $\mathbf{BS4_2}$ ($\mathbf{B3S4_2}$) *is a modal companion of* $\mathbf{N_9}$ ($\mathbf{N_5}$), *i.e., for every* $\varphi \in \mathrm{Form}_{\mathcal{L}^\sim}$ *the following two equivalences hold:*

$$\varphi \in \mathbf{N_9} \quad \textit{iff} \quad T_B\varphi \in \mathbf{BS4_2},$$
$$\varphi \in \mathbf{N_5} \quad \textit{iff} \quad T_B\varphi \in \mathbf{B3S4_2}.$$

Proof. We consider the case of $\mathbf{N_9}$. If $T_B\varphi \notin \mathbf{BS4_2}$, then by Item 2 of Proposition 3 $\mathcal{M}, x \not\models^+ T_B\varphi$ for some 4-model $\mathcal{M}$ over $\mathcal{W}^{HT}$ and $x \in \{h, t\}$. By Lemma 2 we have $\mathcal{M}', x \not\models \varphi$. Moreover, $\mathcal{M}'$ is obviously an $\mathbf{N_9}$-model. Consequently, $\varphi \notin \mathbf{N_9}$.

If $\varphi \notin \mathbf{N_9}$. Then $\mathcal{M}, x \not\models^+ \varphi$ for a suitable $\mathbf{N_9}$-model and $x \in \{h, t\}$. Obviously, $\mathcal{M}$ is a $\mathbf{BS4}$-model over $\mathcal{W}^{HT}$. Since $v^+(p)$ and $v^-(p)$ are cones for any $p \in \mathrm{Prop}$, we have

$$x \in v^+(p) \;\Leftrightarrow\; \mathcal{M}, x \models^+ \Box p \;\text{ and }\; w \in v^-(p) \;\Leftrightarrow\; \mathcal{M}, w \models^+ \Box{\sim}p,$$

which implies $\mathcal{M}' = \mathcal{M}$. We have then $\mathcal{M}, w \not\models^+ T_B\varphi$ by Lemma 2, and so $T_B\varphi \notin \mathbf{BS4}$.

$\Box$

This theorem together with Proposition 1 yields

Corollary 2. *The translation T_B faithfully embeds the logic $\mathbf{N_9}$ ($\mathbf{N_5}$) into the special temporal logic $\mathbf{BSK_{t2}}$ ($\mathbf{B3SK_{t2}}$).*

4 Equilibrium theory over $\mathbf{BS4_{t2}}$

We define a theory $\mathbf{BKE}$ as the least set of formulas closed under the rules of $\mathbf{BS4_{t2}}$, containing the axioms of $\mathbf{BS4_{t2}}$ as well as the following formulas:

$$\Diamond\Big(\bigwedge_{\alpha\in A} \Box\alpha \wedge \bigwedge_{\beta\in B} \Box\beta \wedge \bigwedge_{\gamma\in C} \Box\neg\gamma\Big) \to \Diamond\blacklozenge\Big(\bigwedge_{\alpha\in A\cup C} \neg\alpha \wedge \bigwedge_{\beta\in B} \beta\Big), \tag{10}$$

where A, B, C are disjoint finite subsets of $\mathrm{Lit}^{\sim}$, and $A \neq \varnothing$.

It is clear that $\mathbf{BKE}$ contains all theorems of $\mathbf{BS4_{t2}}$ and is closed under the rules ($\mathrm{NR}_\Box$) and ($\mathrm{NR}_\blacksquare$), but it need not be closed under (SUB). Substituting to the axioms (10) we obtain formulas which are not of the form (10). So $\mathbf{BKE}$ is namely a theory over $\mathbf{BS4_{t2}}$, not an axiomatic extension.

The theory $\mathbf{B3KE}$ is defined in exactly the same way but over $\mathbf{B3S4_{t2}}$.

Now we describe models of theories $\mathbf{BKE}$ and $\mathbf{B3KE}$.

First, for a model $\mathcal{M} = \langle W, R, v^+, v^-\rangle$ and $x \in W$ we denote by At_x the set of all literals true at x:

$$At_x = \{\alpha \mid \alpha \in \mathrm{Lit}^{\sim} \text{ and } \mathcal{M}, x \models^+ \alpha\}.$$

We say that $\mathcal{M} = \langle W, R, S, v^+, v^-\rangle$ is a $\mathbf{BKE}$-*model* if $\langle W, R, S\rangle$ is a $\mathbf{BS4_{t2}}$-frame, i.e., R and S are related by (8), R is a preoder on W satisfying (1^R), S satisfies (2^S), (3^S), and for every $x \in W$ the following two conditions are satisfied:

(1^E) if x is an *isolated point*, i.e., there is no $y \in W$ such that $y \neq x$, and yRx or xRy, then $At_x = \varnothing$.

(2^E) if x is a proper R-successor of some $y \in W$ and $At_x \neq \varnothing$, then for every proper subset $B \subsetneq At_x$ there is $z \in W$ such that xSz and $At_z = B$.

$\mathbf{B3KE}$-*models* are $\mathbf{BKE}$-models satisfying the consistency condition (2).

Theorem 6. *Let $\varphi \in \mathrm{Form}_{\mathcal{L}^t}$. The following equivalences are true:*

(1) $\varphi \in \mathbf{BKE}$ *iff* $\mathcal{M} \models \varphi$ *for every* **BKE**-*model;*

(2) $\varphi \in \mathbf{B3KE}$ *iff* $\mathcal{M} \models \varphi$ *for every* **B3KE**-*model;*

Proof. (1) First we prove that $\varphi \in \mathbf{BKE}$ implies the truth of φ on all **BKE**-models. Since **BKE**-models are based on $\mathbf{BS4_{t2}}$-frames, Theorem 4 implies that it would be enough to check the truth of axioms (10) on **BKE**-models. Let $\mathcal{M} = \langle W, R, S, v^+, v^- \rangle$ be a **BKE**-model and $x \in W$. Assume that

$$\mathcal{M}, x \models^+ \Diamond(\bigwedge_{\alpha \in A} \Box\alpha \wedge \bigwedge_{\beta \in B} \Box\beta \wedge \bigwedge_{\gamma \in C} \Box\neg\gamma),$$

where A, B, C are disjoint finite subsets of $\mathrm{Lit}^{\sim}$, and $A \neq \varnothing$. Then there is $y \in W$ such that xRy and

$$\mathcal{M}, y \models^+ \bigwedge_{\alpha \in A} \Box\alpha \wedge \bigwedge_{\beta \in B} \Box\beta \wedge \bigwedge_{\gamma \in C} \Box\neg\gamma.$$

Reflexivity of R implies $A \cup B \subseteq At_y$. Since $\mathcal{M}, y \models^+ \neg\alpha$ is equivalent to $\mathcal{M}, y \not\models^+ \alpha$, we have $C \cap At_y = \varnothing$. If x has no proper R-successors, then $x = y$. In this case $A \neq \varnothing$ and (1^E) imply that there is z such that $x \neq z$ and zRx. By (2^E) there is an S-successor u of x such that $At_u = B$. So we have $\mathcal{M}, x \models^+ \blacklozenge(\bigwedge_{\alpha \in A \cup C} \neg\alpha \wedge \bigwedge_{\beta \in B} \beta)$ and

$$\mathcal{M}, x \models^+ \Diamond\blacklozenge(\bigwedge_{\alpha \in A \cup C} \neg\alpha \wedge \bigwedge_{\beta \in B} \beta). \tag{11}$$

Otherwise we can assume that y is a proper R-successor of x. From (2^E) we again obtain that $\blacklozenge(\bigwedge_{\alpha \in A \cup C} \neg\alpha \wedge \bigwedge_{\beta \in B} \beta)$ is verified at y, and (11) as a consequence.

To prove the inverse implication we define the canonical frame $\mathcal{W}^E$ for the theory **BKE** in exactly the same way as it was done earlier for axiomatic extensions of $\mathbf{BSK_t}$. We put $\mathcal{W}^E = \langle W^E, R^E, S^E \rangle$, where W^E is the set of all prime $\mathbf{BSK_{t2}}$-theories containing **BKE**,

$$\Gamma R^E \Delta \text{ iff } \Gamma_\Box \subseteq \Delta, \quad \Gamma S^E \Delta \text{ iff } \Gamma_\blacksquare \subseteq \Delta.$$

If $\varphi \in \mathrm{Form}_{\mathcal{L}^t}$, we denote the set of its propositional variables as $\mathrm{var}(\varphi)$ and put

$$\mathrm{Lit}_\varphi = \{p, \sim p \mid p \in \mathrm{var}(\varphi)\}.$$

For $\varphi \notin \mathbf{BKE}$ we define a kind of canonical model $\mathcal{M}^E_\varphi = \langle \mathcal{W}^E, v^+_E, v^-_E \rangle$ putting

$$v^+_\varphi(p) = \{\Gamma \in W^L \mid p \in \Gamma \cap \mathrm{var}(\varphi)\}, \quad v^-_\varphi(p) = \{\Gamma \in W^L \mid \sim p \in \Gamma \cap \mathrm{Lit}_\varphi\}.$$

The analog of canonical model lemma (Lemma 3) can also be proved for $\mathcal{M}^E_\varphi$ by induction on the structure of formulas. For every $\psi \in \mathrm{Form}_{\mathcal{L}^t}$ and $\Gamma \in W^E$ we have

$$\mathcal{M}^E_\varphi, \Gamma \models^+ \psi \text{ iff } \psi \in \Gamma; \quad \mathcal{M}^E_\varphi, \Gamma \models^- \psi \text{ iff } \sim\psi \in \Gamma. \tag{12}$$

That $\mathcal{W}^E$ is a $\mathbf{BSK_{t2}}$-frame can be proved in exactly the same way as in Theorem 4. Let us check that $\mathcal{M}^E_\varphi$ is a **BKE**-model.

We take some $\Gamma \in W^E$ that has no proper R^E-successors. If $\Gamma \cap \mathrm{Lit}_\varphi \neq \varnothing$, we choose finite disjoint A, B, $C \subseteq \mathrm{Lit}_\varphi$ such that

$$A \cup B = \Gamma \cap \mathrm{Lit}_\varphi, \quad A \neq \varnothing, \quad C = \mathrm{Lit}_\varphi \setminus \Gamma.$$

Then $\mathcal{M}^E_\varphi, \Gamma \models^+ \bigwedge_{\alpha\in A} \alpha \wedge \bigwedge_{\beta\in B} \beta \wedge \bigwedge_{\gamma\in C} \neg\gamma$. Since R^E is reflexive and Γ has no other R^E-successors, we have

$$\mathcal{M}^E_\varphi, \Gamma \models^+ \bigwedge_{\alpha\in A} \Box\alpha \wedge \bigwedge_{\beta\in B} \Box\beta \wedge \bigwedge_{\gamma\in C} \Box\neg\gamma.$$

The reflexivity of R^E yields $\mathcal{M}^E_\varphi, \Gamma \models^+ \Diamond(\bigwedge_{\alpha\in A} \Box\alpha \wedge \bigwedge_{\beta\in B} \Box\beta \wedge \bigwedge_{\gamma\in C} \Box\neg\gamma)$. Finally, by (12) we obtain $\Diamond(\bigwedge_{\alpha\in A} \Box\alpha \wedge \bigwedge_{\beta\in B} \Box\beta \wedge \bigwedge_{\gamma\in C} \Box\neg\gamma) \in \Gamma$. Since Γ contains all **BKE**-axioms, we have $\Diamond\blacklozenge(\bigwedge_{\alpha\in A\cup C} \neg\alpha \wedge \bigwedge_{\beta\in B} \beta) \in \Gamma$, which is equivalent by (12) to

$$\mathcal{M}^E_\varphi, \Gamma \models^+ \Diamond\blacklozenge(\bigwedge_{\alpha\in A\cup C} \neg\alpha \wedge \bigwedge_{\beta\in B} \beta). \tag{13}$$

The lack of proper R^E-successors implies $\mathcal{M}^E_\varphi, \Gamma \models^+ \blacklozenge(\bigwedge_{\alpha\in A\cup C} \neg\alpha \wedge \bigwedge_{\beta\in B} \beta)$. This fact together with $A \cup B \cup C = \mathrm{Lit}_\varphi$ implies in turn that Γ has an S^E-successor Δ with $At_\Delta = B$. Since $At_\Gamma \neq At_\Delta$, Δ is a proper S^E-successor or, equivalently, R^E-predecessor of Γ. We have thus proved that $At_\Gamma \neq \varnothing$ implies that Γ is not an isolated point. So (1^E) holds for $\mathcal{M}^E_\varphi$.

Let $\Gamma, \Delta \in W^E$ be such that $\Gamma R^E \Delta$, $\Gamma \neq \Delta$, $\Delta \cap \mathrm{Lit}_\varphi \neq \varnothing$, and $B \subsetneq \Delta \cap \mathrm{Lit}_\varphi$. In this case Δ has no proper R^E-successors, and arguing as above we obtain that Δ has an S^E-successor Σ with $At_\Sigma = B$. Thus, $\mathcal{M}^E_\varphi$ satisfies (2^E) too.

We have proved that $\mathcal{M}^E_\varphi$ is a **BKE**-model. Since $\varphi \notin \mathbf{BKE}$, by extension lemma there is $\Gamma \in W^E$ such that $\varphi \notin \Gamma$, whence $\mathcal{M}^E_\varphi, \Gamma \not\models^+ \varphi$.

(2) That **BKE**-axioms are true on **B3KE**-models readily follows from the previous item. It is obvious too that a canonical model

$$\mathcal{M}^{3E}_\varphi = \langle W^{3E}, R^{3E}, S^{3E}, v^+_{3E}, v^-_{3E}\rangle$$

defined as above, but with W^{3E} consisting of all prime theories extending **B3KE**, is a 3-model.

$\square$

Theorem 7. *Let* $\varphi, \psi \in \mathrm{Form}_{\mathcal{L}^\sim}$. *Then*

$$\varphi \mid\!\sim^9_{el} \psi \quad \textit{iff} \quad (T_B\varphi \wedge \blacksquare\neg T_B\varphi) \to T_B\psi \in \mathbf{BKE}.$$

$$\varphi \mid\!\sim^5_{el} \psi \quad \textit{iff} \quad (T_B\varphi \wedge \blacksquare\neg T_B\varphi) \to T_B\psi \in \mathbf{B3KE}.$$

Proof. We consider only the case of paraconsistent equilibrium consequence. Assume that $\varphi \not\mid\!\sim^9_{el} \psi$, i.e., there is an equilibrium model $\langle \mathbf{T}, \mathbf{T}\rangle$ of φ such that $\langle \mathbf{T}, \mathbf{T} \rangle \not\models \psi$. Recall that an equilibrium model of φ is a total $\mathbf{N_9}$-model $\langle \mathbf{T}, \mathbf{T} \rangle$ such that

$$\langle \mathbf{T}, \mathbf{T} \rangle \models \varphi \text{ and } \langle \mathbf{H}, \mathbf{T} \rangle \not\models \varphi \text{ for all } \mathbf{H} \subsetneq \mathbf{T}.$$

Let us consider a **BKE**-model $\mathcal{M}^{\mathbf{T}} = \langle W^{\mathbf{T}}, R^{\mathbf{T}}, S^{\mathbf{T}}, v^+_{\mathbf{T}}, v^-_{\mathbf{T}} \rangle$, where

- $W^{\mathbf{T}} = \{\mathbf{H} \mid \mathbf{H} \subseteq \mathbf{T}\}$;
- $R^{\mathbf{T}} = \{(\mathbf{H}, \mathbf{H}) \mid \mathbf{H} \in W^{\mathbf{T}}\} \cup \{(\mathbf{H}, \mathbf{T}) \mid \mathbf{H} \subseteq \mathbf{T}\}$;
- $S^{\mathbf{T}} = \{(\mathbf{H}, \mathbf{H}) \mid \mathbf{H} \in W^{\mathbf{T}} \setminus \{\mathbf{T}\}\} \cup \{(\mathbf{T}, \mathbf{H}) \mid \mathbf{H} \subsetneq \mathbf{T}\}$;
- $v^+_{\mathbf{T}}(p) = \{\mathbf{H} \mid p \in \mathbf{H}\}$, $v^-_{\mathbf{T}}(p) = \{\mathbf{H} \mid {\sim} p \in \mathbf{H}\}$, $p \in \mathrm{Prop}$.

It is clear that $R^{\mathbf{T}}$ is a partial order of depth 2 with the greatest element $\mathbf{T}$, $S^{\mathbf{T}}$ satisfies (8) and has the only irreflexive point $\mathbf{T}$. So $\langle W^{\mathbf{T}}, R^{\mathbf{T}}, S^{\mathbf{T}} \rangle$ is a $\mathbf{BSK_{t2}}$-frame. The valuations $v^+_{\mathbf{T}}$, $v^-_{\mathbf{T}}$ are defined so that $At_{\mathbf{H}} = \mathbf{H}$ for all $\mathbf{H} \in W^{\mathbf{T}}$.

Further, for every $\mathbf{H} \in W^{\mathbf{T}}$ with $\mathbf{H} \neq \mathbf{T}$, the generated submodel $\mathcal{M}^{\mathbf{H}\uparrow} := (\mathcal{M}^{\mathbf{T}})^{\mathbf{H}\uparrow}$ is of the form $\langle \mathbf{H}, \mathbf{T} \rangle$. Moreover, since $\mathbf{H} \subseteq \mathbf{T}$, we can consider $\langle \mathbf{H}, \mathbf{T} \rangle$ as an $\mathbf{N_9}$-model, i.e. $(\mathcal{M}^{\mathbf{H}\uparrow})' = \mathcal{M}^{\mathbf{H}\uparrow}$. If $\mathbf{H} = \mathbf{T}$, then $\mathcal{M}^{\mathbf{T}\uparrow} = \mathbf{T}$ and it can be identified with the total model $\langle \mathbf{T}, \mathbf{T} \rangle$. By Lemma 2 and (6) we have for every $\chi \in \mathrm{Form}_{\mathcal{L}^\sim}$ and $\mathbf{H} \in W^{\mathbf{T}}$,

$$\mathcal{M}^{\mathbf{T}}, \mathbf{H} \models^+ T_B \chi \text{ iff } \langle \mathbf{H}, \mathbf{T} \rangle \models \chi. \tag{14}$$

From this observation we obtain

$$\mathcal{M}^{\mathbf{T}}, \mathbf{T} \models^+ T_B \varphi \text{ and } \mathcal{M}^{\mathbf{T}}, \mathbf{H} \models^+ \neg T_B \varphi \text{ for } \mathbf{H} \subsetneq \mathbf{T}.$$

We also have $\mathcal{M}^{\mathbf{T}}, \mathbf{T} \models^+ \blacksquare \neg T_B \varphi$, since $\{\mathbf{H} \mid \mathbf{H} \subsetneq \mathbf{T}\}$ is the set of all $S^{\mathbf{T}}$-successors of $\mathbf{T}$. From $\langle \mathbf{T}, \mathbf{T} \rangle \not\models \psi$ and (14) we have $\mathcal{M}^{\mathbf{T}}, \mathbf{T} \not\models^+ T_B \psi$, whence

$$\mathcal{M}^{\mathbf{T}}, \mathbf{T} \not\models^+ (T_B \varphi \wedge \blacksquare \neg T_B \varphi) \to T_B \psi.$$

We have thus proved the right-to-left implication.

Let us assume now that $(T_B \varphi \wedge \blacksquare \neg T_B \varphi) \to T_B \psi \notin \mathbf{BKE}$, i.e., that there are a **BKE**-model $\mathcal{M} = \langle W, R, S, v^+, v^- \rangle$ and $x \in W$ such that

$$\mathcal{M}, x \not\models^+ (T_B \varphi \wedge \blacksquare \neg T_B \varphi) \to T_B \psi.$$

Then we have

$$\mathcal{M}, x \models^+ T_B \varphi, \quad \mathcal{M}, y \not\models^+ T_B \varphi \text{ whenever } xSy, \quad \mathcal{M}, x \not\models^+ T_B \psi.$$

If there is y such that xRy and $x \neq y$, then ySx by (8), and xSx by (2^S). So we have simultaneously $\mathcal{M}, x \models^+ T_B \varphi$ and $\mathcal{M}, x \not\models^+ T_B \varphi$. Consequently, x has no proper R-successors. Assume that $At_x = \varnothing$. By Lemma 2 and (6) we have $\varnothing \models \varphi$ and $\varnothing \not\models \psi$. Thus, $\langle \varnothing, \varnothing \rangle$ is an equilibrium model of φ that refutes $\varphi \mathrel{\mid\!\sim^9_{el}} \psi$.

In case $At_x \neq \varnothing$ by (1^E) we obtain that x is not an isolated point. Since x has no proper R-successors there is y such that $y \neq x$ and yRx. Take some $\mathbf{H} \subsetneq At_x$. By (2^E) there is z such that $At_z = \mathbf{H}$ and xSz. Applying again Lemma 2 and (6) we obtain $\langle \mathbf{H}, At_x \rangle \not\models \varphi$. Thus, $\langle At_x, At_x \rangle$ is an equilibrium model of φ that refutes ψ. So, $\varphi \mathrel{\not\mid\!\sim^9_{el}} \psi$. $\square$

References

[1] J. Alcântara, C. Demásio, L.M. Pereira, *A declarative characterisation of disjunctive paraconsistent answer sets*, in R. López de Mántaras, L. Saitta (eds.), Proc. of ECAI 2004, IOS Press, 2004, 951-957.

[2] N. Belnap, *A useful four-valued logic*, In J.M. Dunn, G. Epstein (eds), *Modern uses of multiple-valued logic*, 5th int. Symp., Bloomington 1975, D. Reidel, Dordrecht-Boston, 1977, 5–37. Zbl 0417.03009

[3] A. Chagrov, M. Zakhariaschev, *Modal logic*, Clarendon Press, Oxford, 1997. Zbl 0871.03007

[4] J.M. Dunn, *Intuitive semantics for first-degree entailments and 'coupled trees'*, Philos. Stud., **29**:3 (1976), 149–168. Zbl 1435.03043

[5] L. Fariñas del Cerro, A. Herzig, E.I. Su, *Capturing equilibrium models in modal logic*, J. Appl. Log., **12**:2 (2014), 192–207. Zbl 1328.03018

[6] J. Font, *Abstract algebraic logic. An introductory textbook*, College Publications, London, 2016. Zbl 1375.03001

[7] M. Gelfond, V. Lifschitz, *The stable model semantics for logic programming*, Proc. of ICLP'88, The MIT Press, 1988, 1070–1080.

[8] M. Gelfond, V. Lifschitz, *Classical negation in logic programs and disjunctive databases*, New Gener. Comput., **9**:3-4 (1991), 365–385. Zbl 0735.68012

[9] K. Gödel, *Eine Interpretation des intuitionistischen Aussagenkalküls*, Erg. Math. Kolloqu., **4** (1933), 39–40. Zbl 0007.19303

[10] V. Lifschitz, D. Pearce, A. Valverde, *Strongly equivalent logic programs*, ACM Trans. Comput. Log., **2**:4 (2001), 526–541. Zbl 1365.68149

[11] E.G.K. López-Escobar, *Refutability and elementary number theory*, Indagationes Mathematicae, (Proceedings), **75**:4 (1972), 362–374. Zbl 0262.02027

[12] N.V. Mayatskiy, S.P. Odintsov, *On deductive bases for paraconsistent answer set semantics*, J. Appl. Non-Class. Log., **23**:1-2 (2013), 131–146. Zbl 1400.68050

[13] D. Nelson, *Constructible falsity*, J. Symb. Log., **14** (1949), 16–26. Zbl 0033.24304

[14] S.P. Odintsov, *The class of extensions of Nelson's paraconsistent logic*, Stud. Log., **80**:2-3 (2005), 291–320. Zbl 1097.03019

[15] S.P. Odintsov *Constructive negations and paraconsistency*, Trends in Logic–Studia Logica Library, **26**, Springer, Dordrecht, 2008. Zbl 1161.03014

[16] S. Odintsov, D. Pearce, *Routley semantics for answer sets*, in Baral, Chitta (ed.) et al., *Logic programming and nonmonotonic reasoning, LPNMR 2005*, Lecture Notes in Computer Science, **3662**, Springer, Berlin, Heidelberg, 2005, 343–355. Zbl 1152.68416

[17] S.P. Odintsov, H. Wansing, *Modal logics with Belnapian truth values*, J. Appl. Non-Class. Log., **20**:3 (2010), 279–301. Zbl 1242.03050

[18] S.P. Odintsov, H. Wansing, *Disentangling FDE-based paraconsistent modal logics*, Stud. Log., **105**:6 (2017), 1221–1254. Zbl 1417.03160

[19] D. Pearce, *A new logical characterization of stable models and answer sets*, In Dix, Jürgen (ed.) et al., *Non-monotonic extensions of logic programming*, (NMELP 1996), Lect. Notes Comput. Sci., **1216**, Springer, Berlin, Heidelberg, 1997, 57–70. Zbl 1547.68090

[20] R. Routley. *Semantical analyses of propositional systems of Fitch and Nelson*, Stud. Log., **33**:3 (1974), 283–298. Zbl 0356.02022

[21] C. Sakama, K. Inoue, *Paraconsistent stable semantics for extended disjunctive programs*, J. Log. Comput., **5**:3 (1995), 265–285. Zbl 0827.68070

[22] R.H. Thomason, *A semantical study of constructible falsity*, Z. Math. Logik Grundlagen Math., **15** (1969), 247–257. Zbl 0181.00901

[23] A.G. Vishneva, S.P. Odintsov *Modal companions for the special extensions of Nelson's constructive logic*, Matematicheskie Zametki, **117**:3 (2025), 344–364.

[24] N.N. Vorob'ev, *A constructive calculus of propositions with strong negation*, Doklady Akad. Nauk SSSR (N.S.), **85** (1952), 465–468. MR0049836

[25] N.N. Vorob'ev, *The decision problem in the constructive calculus of propositions with strong negation*, Doklady Akad. Nauk SSSR (N.S.), **85** (1952), 689–692. MR0049837

Sergei Pavlovich Odintsov
Steklov Mathematical Institute of Russian Academy of Sciences,
8 Gubkina St.,
119991, Moskva, Russia
Email address: odintsov.sergey2013@yandex.ru