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PROPERTIES OF MOMENTS OF DENSITY FOR NONLOCAL MEAN FIELD GAME EQUATIONS WITH A QUADRATIC COST

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Abstract: We consider mean field game equations with an underlying jump-diffusion process X_t for the case of a quadratic cost function and show that the expectation and variance of X_t obey second-order ordinary differential equations with coefficients depending on the parameters of the cost function. Moreover, for the case of pure diffusion, the characteristic function and the fundamental solution of the equation for the probability density can be expressed in terms of the expectation $\mathbb{E}$ and the variance $\mathbb{V}$ of the process X_t, so that the moments of any order depend only on $\mathbb{E}$ and $\mathbb{V}$.

Keywords: mean field game equations, quadratic Hamiltonian, quadratic cost, moments of density.

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1 Introduction

Mean field game theory (MFG) has been intensively developed in the last decade in terms of various mathematical disciplines such as partial differential equations, control theory, probability theory and numerical methods. It also has numerous practical applications in all areas where the behavior of a large number of agents is critical, e.g. in economics, mobile network design, logistics, etc. [11], [10], [6], [3].

In this paper we consider the forward-backward MFG problem, which analytically reduces to the solution of the initial-terminal problem for coupled nonlocal Hamilton-Jacobi-Bellman (HJB) and Kolmogorov-Feller-Fokker-Planck (KFFP) equations for the density $m(t,x)$ and the value function $\Phi(t,x)$ [11], [12], [13]:

$$-\partial_t\Phi + \tfrac{1}{2}(|\nabla\Phi|)^2 - \tfrac{\delta^2}{2}\Delta\Phi - \tag{1}$$

$$\lambda\left(\int\limits_{\mathbb{R}}\Phi(t,x+z)p(z)dz - \Phi(t,x)\right) = g(t,x,m),$$

$$\partial_t m - \operatorname{div}(m\nabla\Phi) - \tfrac{\delta^2}{2}\Delta m - \lambda\left(\int\limits_{\mathbb{R}} m(t,x-z)p(z)dz - m(t,x)\right) = 0, \tag{2}$$

$$m(0,x) = m_0(x), \qquad \Phi(x,T) = K(x). \tag{3}$$

Here $x\in\mathbb{R}^n$, $t\in[0,T]$, $0<T<\infty$. Let us stress that we do not discuss here the existence and smoothness of the solution and only assume that the initial and terminal conditions, as well as the right-hand side g are compatible with the existence of moments of density of any order.

In our consideration $m(t,x)\geq 0$ is the density of a stochastic process X with dynamics given by

$$dX_s = \alpha_s\,ds + \delta\,dW_s + \lambda d\Gamma_s, \quad X_0 = x_0, \tag{4}$$

$x_0\in\mathbb{R}^n$ is a point in the space of states, $0\leq s\leq T$, W_s is a standard vectorial Brownian motion, Γ_s is is the compound Poisson process with the generator $\mathcal{L}f = \int\limits_{\mathbb{R}}(f(x+y)-f(x))p(y)dy$, where $p(z)$ is a probability density of jumps, $\int\limits_{\mathbb{R}^n} p(z)dz = 1$ $\delta\geq 0$, $\lambda\geq 0$ are constants, $\int\limits_{\mathbb{R}^n} m(t,x) = 1$. We assume $\mathcal{M}_1 = \int\limits_{\mathbb{R}^n} zp(z)dz < \infty$, $\mathcal{M}_2 = \int\limits_{\mathbb{R}^n} |z|^2p(z)dz < \infty$.

Heuristically, the problem is to minimize over all the progressively measurable admissible controls $\alpha_s\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ the cost function

$$\mathbb{E}\left[\int_0^T\left(\frac{|\alpha_s|^2}{2} + g(s,X_s,m)\right)ds + K(X_T)\right],$$

$g:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+\to\mathbb{R}$ and $K:\mathbb{R}^n\to\mathbb{R}$ are prescribed continuous functions, the process X_s obeys (4), the expectation is taken with respect to the filtration $\mathcal{F}$ generating by the jump-diffusion process.

The main interest is the study of density evolution in the process of control. We are going to show that despite the impossibility of obtaining an explicit form of the density, for a particular case the behavior of its moments is relatively simple and often amenable to analytical study.

2 Properties of solutions

2.1. Assumptions. The main assumption that allows us to prove our main result is as follows:

$$g = a(t)|x|^2 + b(t) \cdot x + c(t), \tag{5}$$

where a, $b = (b_1, \dots, b_n)$, c are smooth bounded on $[0, T]$ functions (b is a vector) that can depend on the expectation $\mathbb{E}(X_s)$ and variance $\mathbb{V}(X_s)$, and in this sense the control depends on the density, and

$$K(x) = A_T|x|^2 + B_T \cdot x + C_T, \tag{6}$$

with constant A_T, B_T, C_T.

Similar assumptions about the cost function were made, for example, in [9], [2], [17], [5] for the case of pure diffusion, where it was shown that the solution can be found by solving the matrix Riccati equation. Our results are also based on the fact that the solution can be partially expressed in terms of the solution of the Riccati equation.

We are going to show that, with this special choice of control, the first two density moments can be found as solutions to second-order ODEs, and the characteristic function and fundamental solution are expressed in terms of them (and, perhaps, in terms of other parameters of the problem). Therefore, for any initial density, its full behavior can be restored.

At this stage, we will not explicitly prescribe the possible dependence of the coefficients of function g on the distribution moments. Our main goal is to find the structure of the equations that determine the expectation and dispersion of the distribution in the case of any quadratic dependence on spatial coordinates (Theorem 2).

When the equations (18) and (19) are obtained, we can see that adding a dependence on the mathematical expectation and dispersion to the coefficients a and b complicates the equations or even couples them, forcing us to study a separate boundary value problem for the resulting system. This problem, generally speaking, is nonlinear and there are no general methods for its analysis. Studying such problems in the general case is not the purpose of this paper. By way of illustration, we deliberately choose a type of dependency for which there is an explicit solution.

2.2. The HJB equation. First we find the solution to the HJB equation.

Lemma 1. *Assume that g has the form (5). Then (1) has a solution of the form*

$$\Phi = A(t)|x|^2 + B(t) \cdot x + C(t). \tag{7}$$

If $A(t)$ is uniquely defined by the terminal condition $A(T) = A_T$, then the coefficients $B(t)$, $C(t)$ are uniquely defined by the terminal conditions (9).

Proof. We substitute (7) to (1) and get a polynomial of the second order with respect to x. Then we combine the coefficients at x^2, x and 1 and obtain a system of ODEs for A, B, C, subject to the terminal conditions,

$$\dot{A} = 2A^2 - a, \quad \dot{B} - 2AB = 2\lambda\mathcal{M}_1 A - b, \tag{8}$$
$$\dot{C} = -\frac{1}{2}B^2 + \delta^2 A + \lambda(\mathcal{M}_2 A + \mathcal{M}_1 B) - c,$$
$$A(T) = A_T, \quad B(T) = B_T, \quad C(T) = C_T. \tag{9}$$

The equation for A splits from the rest of system and defines the entire dynamics. The equation for B is linear and contain A in coefficients. In turn, the equation for C is also linear, since A and B have already been found. ☐

It what follows we assume that $a(t)$ and $b(t)$ are such that

$$e^{-2\int\limits_0^T A(\tau)d\tau} < \infty, \qquad \int_0^T e^{-2\int\limits_\eta^T A(\tau)d\tau} B(\eta)\, d\eta < \infty. \tag{10}$$

Remark 1. *Note that the equation defining A is the Riccati equation, therefore its solution, generally speaking, does not extend over the entire interval $(0,T)$ and goes to infinity in a finite time $0 < T_* < T$ (see the example of Sec.3.1 for the simplest case of $a = \mathrm{const}$). However, this is not an obstacle, since for further reasoning it is only necessary to satisfy condition (10), and it can also be fulfilled when A turns to infinity. If there is an analytical expression for $A(t)$ on $(T^*,T]$, then we assume that the same expression defines the solution on $[0,T]$. If such an expression is unknown, then we solve the Cauchy problem on the interval $(T_1,T]$, $T_1 = T^* + \varepsilon$, where $\varepsilon > 0$, sufficiently small, make a change of variable $A_1 = \frac{1}{A}$, (A_1 is also subordinated to some Riccati equation). The blow-up point for A is already the regular one for A_1. Then we solve the Cauchy problem for A_1 with the initial condition $A_1(T_1) = \frac{1}{A(T_1)}$ on the interval $(T_1^*,T_1]$, $T_* \in (T_1^*,T_1]$, where T_1^* is a blow-up point for A_1 (if any). If $T_1^* > 0$, we repeat this procedure as many times as necessary to attain $t = 0$.*

Another way to extend the solution through the blow-up point is to reduce the Riccati equation for A to the linear second-order equation $\ddot{u} - 2au = 0$, $A = -\dot{u}/u$, with the terminal conditions $u(T)$, $\dot{u}(T)$ such that $-\dot{u}(T)/u(T) = A_T$. The solution $u(t)$ is unique and exists for all $t \in [0,T]$; the zeros of $u(t)$ are the blow-up points for $A(t)$. Since there is no accumulation of zeros for u, there is no accumulation of blow-up points for A. This remark, in particular, explains why, using the procedure described above, we can always construct a solution on the whole $[0,T]$.

Thus, a numerical implementation of the solution can be arranged if $a(t)$ is known in advance. If $a(t)$ is assumed to be a function of expectation and *dispersion, then at this stage we will not be able to find A and all problems with solvability are transferred to problems (18) and (19).*

Remark 2. *Note also that since (2) includes only the gradient of Φ, the results do not depend on the function $C(t)$, and therefore on the coefficient $c(t)$.*

2.3. The KFFP equation. Now we find the fundamental solution of the KFFP equation. If $\Phi(t,x)$ is known, (2) takes a more specific form,

$$\partial_t m - (2A\,x + B)\cdot\nabla m - 2Am - \tfrac{\delta^2}{2}\Delta m - \lambda\left(\int_{-\infty}^{\infty} m(t,x-z)p(z)dz - m\right) = 0, \tag{11}$$

with initial condition

$$m_0(x) = \delta(x-y). \tag{12}$$

We denote the solution to (11), (12) as $\mathcal{G}(t,x,y)$.

Applying the normalized inverse Fourier transform $x \to \omega$, we obtain the Cauchy problem for the characteristic function $\psi = \psi(t,\omega,y)$:

$$\partial_t \hat m + 2A(t)\omega\partial_\omega \hat m + \left(\frac{\delta^2}{2}\omega^2 + iB(t)\omega - (\hat p(\omega)-1)\lambda\right)\hat m = 0, \qquad \hat m(0,\omega) = e^{iy\cdot w}, \tag{13}$$

where it is assumed that A and B are known from the previous step. The solution to (13) can be explicitly found, namely,

$$\psi(t,\omega,y) = \exp\left[-\int_0^t \left(\tfrac{\delta^2}{2}\mathcal{R}^2 + iB(\eta)\mathcal{R} - (\hat p(\mathcal{R})-1)\lambda\right)d\eta + iy\mathcal{R}(t,0,\omega)\right], \tag{14}$$

$$\mathcal{R} = \mathcal{R}(t,\eta,\omega) = \omega e^{2R(t,\eta)}, \quad R(t,\eta) = -\int\limits_\eta^t A(\tau)d\tau.$$

2.4. The expectation and variance. We find the expectation and variance in terms of A and B.

Lemma 2. *Let X_t be a random process with an initial probability density $m_0(x)$, $x_i^k m_0(x) \in L_1(\mathbb{R})$, $k = 1,2$, $i = 1,\dots,n$. The expectation and variance of X_t can be found as*

$$\mathbb{E}(X_t) = \int_0^t e^{-2\int\limits_\eta^t A(\tau)d\tau}(B(\eta) + \lambda\mathcal{M}_1)\,d\eta + \mathbb{E}(X_0)\,e^{-2\int\limits_0^t A(\tau)d\tau}, \tag{15}$$

$$\mathbb{V}(X_t) = (\delta^2 + \lambda\mathcal{M}_2)\int_0^t e^{-4\int\limits_\eta^t A(\tau)d\tau}d\eta + \mathbb{V}(X_0)\,e^{-4\int\limits_0^t A(\tau)d\tau}, \tag{16}$$

where $A(t), B(t)$ is a solution to (8), $t \in [0,T]$.

Proof. We denote as $m(t,x)$ the solution of (11) subject to the initial data $m_0(x)$. By means of standard computations we get from (14)

$$\mathbb{E}(X_t) = -i\frac{\partial \hat{m}(t,w)}{\partial \omega}\Big|_{w=0} = -i\int\limits_{\mathbb{R}^n} \frac{\partial \psi(t,w,y)}{\partial \omega}\Big|_{w=0} m_0(y)\,dy =$$

$$e^{-2\int\limits_0^t A(\tau)d\tau}\int\limits_{\mathbb{R}^n} y m_0(y)\,dy + \int_0^t e^{2R(t,\eta)}\left(B(\eta)+\lambda\mathcal{M}_1\right)\,d\eta,$$

$$\mathbb{V}(X_t) = \mathbb{E}(X_t^2) - (\mathbb{E}(X_t))^2 = -\int\limits_{\mathbb{R}^n} \frac{\partial^2 \psi(t,w,y)}{\partial \omega^2}\Big|_{w=0} m_0(y)\,dy - (\mathbb{E}(X_t))^2 =$$

$$e^{-4\int\limits_0^t A(\tau)d\tau}\Big(\int\limits_{\mathbb{R}^n} y^2 m_0(y)\,dy - (\mathbb{E}(X_t))^2\Big) + (\delta^2+\lambda\mathcal{M}_2)\int_0^t e^{-4\int\limits_\eta^t A(\tau)d\tau}\,d\eta.$$

Here we take into account that

$$m(t,x) = \int\limits_{\mathbb{R}^n} \mathcal{G}(t,x,y)\, m_0(y)\,dy$$

and

$$\int\limits_{\mathbb{R}^n} m_0(y)\,dy = 1.$$

□

2.5. The characteristic function in terms of the expectation and variance. Now we express ψ in terms of $\mathbb{E}$ and $\mathbb{V}$.

Theorem 1. *Assume that condition (10) holds. Then the characteristic function $\psi(t,w,y)$ can be expressed in terms of $\mathbb{E}$, $\mathbb{V}$ and A as follows:*

$$\psi(t,w,y) = e^{-\frac{1}{2}|w|^2(\mathbb{V}(X_t)-\mathbb{V}(X_0)\mathbb{K}^2(t))+iw\cdot(\mathbb{E}(X_t)-\mathbb{E}(X_0)\mathbb{K}(t))+\lambda\mathcal{Q}(t,w)+iy\cdot w\,\mathbb{K}(t)},$$

where

$$\mathbb{K}(t) = e^{-2\int\limits_0^t A(\tau)d\tau}, \quad Q(t,w) = \int\limits_0^t (\hat{p}(\mathcal{R}) - 1 - i\mathcal{M}_1\cdot\mathcal{R})d\eta,$$

$$\mathcal{R}(t,\eta,\omega) = \omega e^{-2\int\limits_\eta^t A(\tau)d\tau},$$

($\mathcal{R}$ and M_1 are vectors).

Moreover, $\mathbb{K}$ is a solution of the boundary problem

$$\ddot{\mathbb{K}} + 2a\mathbb{K} = 0, \quad \mathbb{K}(0) = 1, \quad \mathbb{K}(T) = e^{-2\int\limits_0^T A(\tau)d\tau}. \tag{17}$$

For $\lambda = 0$ and $y = 0$ the characteristic function depends on $\mathbb{E}$, $\mathbb{V}$ only, therefore every moment of X_t can be expressed in terms of its expectation and variance.

Proof. The result follows from (14) and Lemma 2 directly, since

$$\psi(t,w,y) = \exp\big[-\frac{1}{2}|w|^2\bar{\mathbb{V}}(X_t) + iw\cdot\bar{\mathbb{E}}(X_t) + \lambda\mathcal{Q}(t,w) + iy\cdot w\, e^{-2\int\limits_0^t A(\tau)d\tau}\big],$$

$\bar{\mathbb{E}}$ and $\bar{\mathbb{V}}$ are the expectation and variance of the process X_t, given as (4), such that $\mathbb{E}(X_0) = \mathbb{V}(X_0) = y = 0$. Note that A is defined by a and the terminal condition, see (8). Property (17) follows from the first equation of (8). □

Corollary 1. *In the case $\lambda = 0$ the fundamental solution is*

$$\mathcal{G}(t,x,y) = \frac{1}{(\sqrt{2\pi\bar{\mathbb{V}}(X_t)})^n}\exp\big[-\frac{|x - y\, e^{-2\int\limits_0^t A(\tau)d\tau} - \bar{\mathbb{E}}(X_t)|^2}{2\bar{\mathbb{V}}(X_t)}\big].$$

3 Equations for the expectation and variance

Now we derive equations for the expectation and variance.

Theorem 2. *Assume that condition (10) holds. The expectation of $\mathbb{E}(t) = \mathbb{E}(X_t)$ satisfies the following boundary value problems:*

$$\mathbb{E}''(t) + 2a(t)\mathbb{E}(t) = -b(t), \tag{18}$$
$$\mathbb{E}(0) = \mathbb{E}(X_0),$$
$$\mathbb{E}(T) = \int_0^T e^{-2\int\limits_\eta^T A(\tau)d\tau}\,(B(\eta) + \lambda\mathcal{M}_1)\,d\eta + \mathbb{E}(X_0)\,e^{-2\int\limits_0^T A(\tau)d\tau}.$$

In turn, the variance $\mathbb{V}(t) = \mathbb{V}(X_t)$ satisfies the following boundary value problems:

$$\mathbb{V}''(t) + 4a(t)\mathbb{V}(t) - \frac{(\mathbb{V}'(t))^2 - K^2}{2\mathbb{V}(t)} = 0, \quad K = \delta^2 + \lambda\mathcal{M}_2, \tag{19}$$
$$\mathbb{V}(0) = \mathbb{V}(X_0),$$
$$\mathbb{V}(T) = (\delta^2 + \lambda\mathcal{M}_2)\int_0^T e^{-4\int\limits_\eta^T A(\tau)d\tau}\,d\eta + \mathbb{V}(X_0)\,e^{-4\int\limits_0^T A(\tau)d\tau},$$
$$t \in [0,T].$$

Proof. The statement follows from (15) and (16) by direct computation. □

Note that equation (18) was obtained in [14].

So far, we have not indicated the dependence of a and b on $\mathbb{E}$ and $\mathbb{V}$. We can use different hypotheses about this relationship and obtain a nonlinear system (18), (19) in combination with (8) at the boundary conditions. Generally speaking, this problem cannot be solved analytically, and even a numerical solution is difficult. However, in several cases, the solution can be found explicitly. Corollary 2 [14] gives one of these possibilities.

3.1. Particular case.

Corollary 2. *If* $a(t) = \text{const}$ *and* $b(t) = b_0 + b_1\mathbb{E}(t) + b_2\mathbb{E}'(t)$*,* $b_0, b_1, b_2 = \text{const}$*, then (18) transforms into a linear second order ODE*

$$\mathbb{E}^{''} + b_2\mathbb{E}' + (2a + b_1)\mathbb{E} = -b_0,$$

which can be explicitly solved.

Example. The simplest case of an explicit solution is the case of constant a and b. Namely,

i) For $a > 0$

$$\mathbb{E}(t) = C_1 \sin \sqrt{2t} + C_2 \cos \sqrt{2a}t - \tfrac{b}{2a}, \tag{20}$$

$$\mathbb{V}(t) = \pm\tfrac{1}{a}\sqrt{a\left((C_1^2 + C_2^2)a + \tfrac{K^2}{8}\right)} + C_1 \sin(2\sqrt{2at}) + C_2 \cos(2\sqrt{2at}), \tag{21}$$

the sign was chosen for reasons of non-negativity of $\mathbb{V}$.

In this case

$$\begin{aligned} A(t) &= \sqrt{\frac{a}{2}}\tan\theta(t,T), \qquad \theta(t,T) = \arctan\sqrt{\frac{2}{a}}A_T + \sqrt{2a}(T-t),\\ B(t) &= -\frac{b}{\sqrt{2a}}\tan\theta(t,T) + \frac{bA_T + aB_T}{\sqrt{a(a+2A_T^2)}\cos\theta(t,T)}. \end{aligned}$$

Note that the solution of (8) may have singularities inside $(0, T)$, nevertheless condition (10) holds, so the singularities of A do not interfere with the calculation of the boundary conditions.

It is possible to show (see [15]), that $\mathbb{E}(t)$ is differentiable in the points t, where A and B fail to exist. In turn, $\mathbb{V}(t)$ is continuous but not differentiable when $\mathbb{V}(t) = 0$.

ii) For $a < 0$

$$\mathbb{E}(t) = C_1 \sinh \sqrt{2t} + C_2 \cosh \sqrt{2a}t - \tfrac{b}{2a}, \tag{22}$$

$$\mathbb{V}(t) = C_1 + \tfrac{8aC_1^2 + K^2}{32aC_2}\exp\left(2\sqrt{-2a}t\right) + C_2\exp\left(-2\sqrt{-2a}t\right). \tag{23}$$

Functions A and B, necessary for boundary conditions, can be found from (8), the problem of singularities does not arise.

iii) For $a = 0$ both $\mathbb{E}(t)$ and $\mathbb{V}(t)$ are polynomials of the second order with respect to t.

Remark 3. *The case considered in the example above seems almost trivial, since the equation (2) does not depend on m. Undoubtedly, much more interesting is the case of the dependence of g on m. However, the presence of m in the utility function of agents is not critical. Indeed, if $g(m)$ increases then this means that it is beneficial for agents to stay closer to the maximum of m. However, the presence of the term $-\frac{\alpha^2}{2}$ already means that agents tend to resemble each other, this creates movement in the same direction. Numerical computations show that $g(m)$ only forces agents to concentrate faster near the maximum of the distribution, but the result does not change qualitatively in comparison with the dependence of g only on x and t.*

4 Discussion

Choosing g in the form (5) seems artificial. However, it appears naturally in a number of problems related to behavioral economics, e.g. [4], [15], [16]. Note that in [4] a second-order equation was obtained for the expectation under different assumptions. The MPG equations with nonlocal terms were considered in [7].

It is well known that in the case of pure diffusion and a quadratic cost function, one can find an explicit solution for the density in the form of a Gaussian function [9]. In [15] an equation for the position of the maximum of density was obtained, which coincides for this case with the expectation. For the case of a jump diffusion, these two characteristics of the density function do not coincide.

Since $\mathbb{E}(t)$ and $\mathbb{V}(t)$ are observable for real processes and can be estimated by statistical methods, we can propose a procedure for restoring the parameters of the cost functional under the assumption that the penalty term g is quadratic. For example, assuming constant a and b, one can use the formulas (20), (21) or (22), (23) depending on the oscillatory or non-oscillatory character of $\mathbb{E}(t)$ and $\mathbb{V}(t)$ in time. We can refer to [8], [1] and their references to show that both types of behavior are possible.

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