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# ON ALGEBRAS OF BINARY FORMULAS FOR WEAKLY CIRCULARLY MINIMAL THEORIES OF FINITE CONVEXITY RANK

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**Abstract:** Algebras of binary isolating formulas are described for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of finite convexity rank with a trivial definable closure having a monotonic-to-right function to the definable completion of a structure and not having a non-trivial equivalence relation partitioning the universe of a structure into finitely many convex classes.

**Keywords:** algebra of binary formulas, weak circular minimality, $\aleph_0$-categorical theory, circularly ordered structure, convexity rank.

## 1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of an one-type at the binary level with respect to the superposition of binary definable sets. These algebras, as natural derivative structures with respect to initial ones, reflecting binary

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links, allow to clarify the structural behavior and to classify both the structures and their elementary theories. Essential structural properties of algebras of binary formulas include the (right) associativity, the (partial) commutativity, the (almost) determinacy, the (almost) absorbing, etc. These properties can be observed on a base of Cayley tables explicitly defining algebras.

The concepts and notations related to algebras of binary formulas can be found in [1, 2]. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter $a$ the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. Given a complete 1-type $p(x)$ over $\emptyset$ realized in a structure $M$, one considers the binary isolating formulas $\varphi(x, y)$. Then one considers an equivalence relation on such formulas (logical equivalence of $\varphi(a, y)$ and $\psi(a, y)$ for an element $a$ realizing $p(x)$). The classes of this equivalence are then embedded in an algebra of labels with a composition-like operation. Composing two (equivalence classes of) binary isolating formulas $\varphi(x, y)$ and $\psi(y, z)$ results in the set of all (classes of) isolating formulas $\chi(x, z)$ such that $\chi(a, z)$ implies $\exists y(\varphi(a, y) \wedge \psi(y, z))$. Thus the operator collecting labels for $\chi$ with respect to labels for $\varphi$ and $\psi$ generates the algebra on the Boolean of labels, which is the multi-algebra on the set of labels related to the type $p(x)$.

In recent years, algebras of binary formulas have been studied intensively for various classes of theories and have been continued in [3]–[11].

In the present paper, algebras of binary isolating formulas are described for the natural class of $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal theories of finite convexity rank with a trivial definable closure having a monotonic-to-right function to the definable completion of a structure and not having a non-trivial equivalence relation partitioning the universe of a structure into finitely many convex classes. The main result of the paper is the explicit characterization of the algebras of binary isolating formulas under the conditions above. In particular, a sharp bound is obtained for the degree of determinacy, i.e. the maximal size of a product of two labels.

Let $L$ be a countable first-order language. Throughout we consider $L$-structures and assume that $L$ contains a ternary relational symbol $K$, interpreted as a circular order in these structures (unless otherwise stated).

Let $M = \langle M, \leq \rangle$ be a linearly ordered set. If we connect two endpoints of $M$ (possibly, $-\infty$ and $+\infty$), then we obtain a circular order. More formally, the *circular order* is described by a ternary relation $K$ satisfying the following conditions:

(co1) $\forall x \forall y \forall z (K(x, y, z) \to K(y, z, x))$;

(co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;

(co3) $\forall x \forall y \forall z (K(x, y, z) \to \forall t [K(x, y, t) \vee K(t, y, z)])$;

(co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

**Fact 1.** [12] *(i) If $\langle M, \leq\rangle$ is a linear ordering and $K$ is the ternary relation derived from $\leq$ by the rule*

$$K(x,y,z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$$

*then $K$ is a circular order relation on $M$.*

*(ii) If $\langle N, K\rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a,y,z)$ is a linear order.*

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$-definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there exist circularly ordered structures that are not linearly ordered (in the sense that a linear ordering relation is not $\emptyset$-definable in an arbitrary circularly ordered structure).

**Example 1.** [13, 14] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K^3, S_0^2, S_1^2\rangle$ be a circularly ordered structure, where $K^3$ is a ternary relation expressing a circular order on $\mathbb{Q}_2$, and the following conditions hold:

(i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle $\pi$;

(ii) for distinct $a, b \in \mathbb{Q}_2$

$$S_0(a,b) \text{ iff } 0 < \arg(a/b) < \pi,$$

$$S_1(a,b) \text{ iff } \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between $a$ and $b$ clockwise.

Indeed, one can check that the linear order relation is not $\emptyset$-definable in this structure.

The notion of *weak circular minimality* was introduced in [15]. Let $A \subseteq M$, where $M$ is a circularly ordered structure. The set $A$ is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$ with $K(a,c,b)$, $c \in A$ holds, or for any $c \in M$ with $K(b,c,a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \ldots\rangle$ such that any definable (with parameters) subset of $M$ is a union of finitely many convex sets in $M$. The study of weakly circularly minimal structures was continued in the papers [16]–[21].

Let $M$ be an $\aleph_0$-categorical weakly circularly minimal structure, $G := \mathrm{Aut}(M)$. Following the standard group theory terminology, the group $G$ is called *$k$-transitive* if for any pairwise distinct $a_1, a_2, \ldots, a_k \in M$ and pairwise distinct $b_1, b_2, \ldots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \ldots, g(a_k) = b_k$. A *congruence* on $M$ is an arbitrary $G$-invariant equivalence relation on $M$. The group $G$ is called *primitive* if $G$ is 1-transitive and there are no non-trivial proper congruences on $M$.

**Notation 1.** (1) $K_0(x,y,z) := K(x,y,z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \dots, u_n)$ denotes a formula saying that all subtuples of the tuple $\langle u_1, \dots, u_n \rangle$ having the length 3 (in ascending order) satisfy $K$; similar notations are used for $K_0$.

(3) Let $A, B, C$ be disjoint convex subsets of a circularly ordered structure $M$. We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A$, $b \in B$, $c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

We say that $M = \langle M, K^3 \rangle$ is a *dense* circularly ordered structure if $M$ is a circularly ordered structure and for any $a, b \in M$ with $a \neq b$ there exists $c \in M$ such that $K_0(a, c, b)$.

Further, we need the notion of the definable completion of a circularly ordered structure, introduced in [15]. Its linear analog was introduced in [22]. A *cut* $C(x)$ in a circularly ordered structure $M$ is maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in $M$ is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\overline{M}$ of a structure $M$ consists of $M$ together with all definable cuts in $M$ that are irrational (essentially, $\overline{M}$ consists of endpoints of definable subsets of the structure $M$).

**Notation 2.** [15] Let $F(x, y)$ be an $L$-formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^{\ell}(y)$ be the formula saying $y$ is a left endpoint of $F(M, y)$:

$$\exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \to \neg F(t_1, y)) \wedge$$
$$\forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \to F(t_2, y))].$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x, y) \to F^l(y) \wedge \forall z (K(y, z, x) \to F(z, y))].$$

If $F_1(x, y), F_2(x, y)$ are arbitrary convex-to-right formulas we say $F_2$ is *bigger than* $F_1$ if there is $a \in M$ with $F_1(M, a) \subset F_2(M, a)$. If $M$ is 1-transitive and this holds for some $a$, it holds for all $a$. This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x, y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M, a)$ for arbitrary $a \in M$. In general, $F(M, a)$ has no the right endpoint in $M$. For example, if $dcl(\{a\}) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x, y)$ and any $a \in M$ the formula $F(M, a)$ has no the right endpoint in $M$. We write $f(y) := \text{rend } F(M, y)$, assuming that $f(y)$ is the right endpoint of the set $F(M, y)$ that lies in general in the definable completion $\overline{M}$ of $M$. Then $f$ is a function mapping $M$ in $\overline{M}$.

Let $F(x,y)$ be a convex-to-right formula. We say that $F(x,y)$ is *equivalence-generating* if for any $a,b \in M$ such that $M \models F(b,a)$ the following holds:

$$M \models \forall x(K(b,x,a) \wedge x \neq a \to [F(x,a) \leftrightarrow F(x,b)]).$$

Consider the following formula:

$$F(x,y) := y = x \vee S_0(y,x),$$

where the predicate $S_0$ from Example 1. It can be easily checked that $F(x,y)$ is a convex-to-right formula that is not equivalence-generating.

**Example 2.** Let $M = \langle M, K^3, E^2 \rangle$ be a dense circularly ordered structure, where $E$ is an equivalence relation partitioning $M$ into infinitely many convex classes without endpoints. Consider the following formula:

$$F(x,y) := \exists t[\forall u(K(y,u,t) \to E(y,u)) \wedge K(y,x,t)].$$

Also, it can be easily checked that $F(x,y)$ is a convex-to-right formula that is equivalence-generating.

**Lemma 1.** [20] *Let $M$ be an $\aleph_0$-categorical 1-transitive weakly circularly minimal structure, $F(x,y)$ be a convex-to-right formula that is equivalence-generating. Then $E(x,y) := F(x,y) \vee F(y,x)$ is an equivalence relation partitioning $M$ into infinite convex classes.*

**Notation 3.** Let $E(x,y)$ be an $\emptyset$–definable equivalence relation partitioning $M$ into infinite convex classes. Suppose that $y$ lies in $\overline{M}$ (not necessarily in $M$). Then

$$E^*(x,y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge \forall t(K(y_1,t,y_2) \to E(t,x)) \wedge K_0(y_1,y,y_2)].$$

**Example 3.** Let $M = \langle \mathbb{Q} \times \mathbb{Q}, K^3, E^2, R^2 \rangle$ be a circularly ordered structure, where $\mathbb{Q} \times \mathbb{Q}$ is ordered lexicographically. The symbol $E$ is defined as follows: $E(a,b)$ holds iff $a_1 = b_1$ for any $a = (a_1,a_2), b = (b_1,b_2) \in M$. The symbol $R$ is interpreted as follows: $R(a,b)$ holds iff $a_1 = b_1$ and $a_2 \leq b_2 < a_2 + \sqrt{2}$ for any $a = (a_1,a_2), b = (b_1,b_2) \in M$.

Obviously, the relation $E(x,y)$ is an equivalence relation partitioning $M$ into infinitely many infinite convex classes. Let $f(x) := \operatorname{rend} R(x,M)$. Obviously, $f(a) \notin M$ for any $a \in M$, but $E^*(a,f(a))$. If we consider arbitrary $a,b \in M$ such that $\neg E(a,b)$, we have both $\neg E^*(a,f(b))$ and $\neg E^*(b,f(a))$.

Let $E(x,y)$ be an equivalence relation partitioning $M$ into infinitely many infinite convex classes. We say that the induced order on $E$-classes is *dense* if $M/E$ is densely ordered, i.e. for any $a,b \in M$ with $\neg E(a,b)$ there exists $c \in M$ such that $K_0(a,c,b)$, $\neg E(a,c)$ and $\neg E(c,b)$.

In Example 3 the induced order on $E$-classes is dense.

Let $f$ be a unary function from $M$ to $\overline{M}$. We say that $f$ is *monotonic-to-right (left) on $M$* if it preserves (reverses) the relation $K_0$, i.e. for any $a, b, c \in M$ such that $K_0(a, b, c)$, we have $K_0(f(a), f(b), f(c))$ ($K_0(f(c), f(b), f(a))$).

The following definition can be used in a circular ordered structure as well.

**Definition 1.** [23], [24] Let $T$ be a weakly o-minimal theory, $M$ be a sufficiently saturated model of $T$, $A \subseteq M$. *The rank of convexity of the set $A$* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.

2) $RC(A) = 0$ if $A$ is finite and non-empty.

3) $RC(A) \geq 1$ if $A$ is infinite.

4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of $A$.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where $\delta$ is a limit ordinal.

If $RC(A) = \alpha$ for some $\alpha$, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all $\alpha$), we put $RC(A) = \infty$.

*The rank of convexity of a formula* $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

*The rank of convexity of an 1-type $p$* is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

Let $M$, $N$ be circularly ordered structures. The 2-*reduct* of $M$ is a circularly ordered structure with the same universe of $M$ and consisting of predicates for each $\emptyset$-definable relation on $M$ of arity $\leq 2$ as well as of the ternary predicate $K$ for the circular order, but does not have other predicates of arities more than two. We say that the structure $M$ is *isomorphic* to $N$ *up to binarity* or *binarily isomorphic to* $N$ if the 2-reduct of $M$ is isomorphic to the 2-reduct of $N$.

We say that a structure $M$ has *a trivial definable closure* if $dcl(A) = A$ for any $A \subseteq M$. If $Th(M)$ is binary, the condition of triviality of the definable closure is just the condition $dcl(\{a\}) = \{a\}$ for any $a \in M$. If additionally $Th(M)$ is 1-transitive, this condition is just $dcl(\{a\}) = \{a\}$ for some $a \in M$.

The following theorem characterizes up to binarity $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structures $M$ of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) := \mathrm{rend}\, R(M, y)$ is monotonic-to-right on $M$:

**Theorem 1.** [17] *Let $M$ be an $\aleph_0$–categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1, $dcl(\{a\})$ $= \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula*

*$R(x,y)$ such that $r(y) := \operatorname{rend} R(M,y)$ is monotonic-to-right on $M$. Then $M$ is isomorphic up to binarity to*

$$M'_{s,m,k} := \langle M, K^3, E_1^2, E_2^2, \ldots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

*where $M$ is a dense circularly ordered structure, $s \geq 1$; $E_{s+1}$ is an equivalence relation partitioning $M$ into $m$ infinite convex classes without endpoints; $E_i$ for every $1 \leq i \leq s$ is an equivalence relation partitioning every $E_{i+1}$-class into infinitely many infinite convex $E_i$-subclasses without endpoints so that the induced order on $E_i$-subclasses is dense without endpoints; $R(M,a)$ has no right endpoint in $M$ and $r^k(a) = a$ for all $a \in M$ and some $k \geq 2$, where $r^k(y) := r(r^{k-1}(y))$; for every $1 \leq i \leq s+1$ and any $a \in M$*

$$M'_{s,m,k} \models \neg E_i^*(a, r(a)) \wedge \forall y (E_i(y,a) \to \exists u [E_i^*(u, r(a)) \wedge E_i^*(u, r(y))]),$$

*$m = 1$ or $k$ divides $m$.*

In [7] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories with a primitive automorphism group. In [8] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [9]–[10] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [11] algebras of binary isolating formulas are described for $\aleph_0$-categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure.

Recently, in [25] we described algebras of binary isolating formulas for $\aleph_0$-categorical 1-transitive non-primitive weakly circularly minimal structures $M$ with the following restrictions:

- $RC(x = x) = 2$;
- $dcl(\{a\}) = \{a\}$ for some $a \in M$;
- there exists a monotonic-to-right function to $\overline{M}$;
- there is no non-trivial equivalence relation partitioning $M$ into finitely many convex classes.

Here we describe algebras of binary isolating formulas for these structures replacing the first condition $RC(x = x) = 2$ by $RC(x = x) = k$ for some $2 \leq k < \omega$.

## 2 Main Theorem

**Definition 2.** [2] Let $p \in S_1(\emptyset)$ be non-algebraic. The algebra $\mathcal{P}_{\nu(p)}$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_{\nu(p)}$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_{\nu(p)}$ is *$m$-deterministic* if the product $u_1 \cdot u_2$ consists of at most $m$ elements for any labels $u_1, u_2 \in \rho_{\nu(p)}$. We also say that an $m$-deterministic algebra $\mathcal{P}_{\nu(p)}$ is *strictly m-deterministic* if it is not $(m-1)$-deterministic. Obviously, strict 1-determinacy of an algebra is equivalent to its determinacy.

**Example 4.** Consider the structure $M'_{2,1,2} := \langle M, K^3, E_1^2, E_2^2, R^2\rangle$ from Theorem 1. Then we have the following: $E_2(x,y)$ is an equivalence relation partitioning $M$ into infinitely many infinite convex classes so that the induced order on $E_2$-classes is dense; $E_1(x,y)$ is an equivalence relation partitioning every $E_2$-class into infinitely many infinite convex $E_1$-subclasses without endpoints so that the induce order on $E_1$-subclasses is dense; $R(M,a)$ has no right endpoint in $M$, $r^2(a)=a$ and $\neg E^*(a,r(a))$ for all $a\in M$, and $r$ preserves $E_i$-classes for every $1\le i\le 2$, i.e. if $E_i(a,b)$ holds for $a,b\in M$ then both $r(a)$ and $r(b)$ lie in the same $E_i^*$-class. Whence we obtain that $Th(M'_{2,1,2})$ has eleven binary isolating formulas:

$$\theta_0(x,y) := x=y,$$
$$\theta_1(x,y) := K_0(x,y,r(x))\wedge E_1(x,y),$$
$$\theta_2(x,y) := K_0(x,y,r(x))\wedge \neg E_1(x,y)\wedge E_2(x,y),$$
$$\theta_3(x,y) := K_0(x,y,r(x))\wedge \neg E_2(x,y)\wedge \neg E_2^*(y,r(x)),$$
$$\theta_4(x,y) := K_0(x,y,r(x))\wedge E_2^*(y,r(x))\wedge \neg E_1^*(y,r(x)),$$
$$\theta_5(x,y) := K_0(x,y,r(x))\wedge E_1^*(y,r(x)),$$
$$\theta_6(x,y) := K_0(r(x),y,x)\wedge E_1^*(y,r(x)),$$
$$\theta_7(x,y) := K_0(r(x),y,x)\wedge E_2^*(y,r(x))\wedge \neg E_1^*(y,r(x)),$$
$$\theta_8(x,y) := K_0(r(x),y,x)\wedge \neg E_2(x,y)\wedge \neg E_2^*(y,r(x)),$$
$$\theta_9(x,y) := K_0(r(x),y,x)\wedge \neg E_2(x,y)\wedge \neg E_1(x,y),$$
$$\theta_{10}(x,y) := K_0(r(x),y,x)\wedge E_1(x,y).$$

Obviously, the following holds for any $a\in M$:

$$K_0(\theta_0(a,M),\theta_1(a,M),\theta_2(a,M),\theta_3(a,M),\ldots,\theta_9(a,M),\theta_{10}(a,M)).$$

Define labels for these formulas as follows:

$$\text{label } k \text{ for } \theta_k(x,y),\ \text{ where } 0\le k\le 10.$$

It easy to check that for the algebra $\mathfrak{P}_{M'_{2,1,2}}$ the following equalities hold:
$0\cdot k = k\cdot 0=\{k\}$ for every $0\le k\le 10$,
$1\cdot 1=\{1\}$, $1\cdot 2=\{2\}$, $1\cdot 3=\{3\}$, $1\cdot 4=\{4\}$, $1\cdot 5=\{5,6\}$, $1\cdot 6=\{6\}$, $1\cdot 7=\{7\}$, $1\cdot 8=\{8\}$, $1\cdot 9=\{9\}$, and $1\cdot 10=\{10,0,1\}$.
For example, let's consider the product $1\cdot 5$ in more detail: let

$$\phi(x,y) := \exists t[\theta_1(x,t)\wedge\theta_5(t,y)].$$

Take arbitrary $a,b\in M$ such that $b\in\theta_1(a,M)$ and consider $\theta_5(b,M)$. Obviously, $\theta_5(b,M)\cap\theta_5(a,M)\ne\emptyset$ and $\theta_5(b,M)\cap\theta_6(a,M)\ne\emptyset$, whence we obtain that each of $\theta_5(a,y)$ and $\theta_6(a,y)$ is consistent with $\phi(a,y)$. Also, we can easily see that each of the remaining $\theta_i(a,y)$ are not consistent with $\phi(a,y)$. Thus, we have $1\cdot 5=\{5,6\}$.

Further, $2\cdot 1=\{2\}$, $2\cdot 2=\{2\}$, $2\cdot 3=\{3\}$, $2\cdot 4=\{4,5,6,7\}$, $2\cdot 5=\{7\}$,

$2\cdot 6=\{7\}$, $2\cdot 7=\{7\}$, $2\cdot 8=\{8\}$, $2\cdot 9=\{9,10,0,1,2\}$, and $2\cdot 10=\{2\}$,
$3\cdot 1=\{3\}$, $3\cdot 2=\{3\}$, $3\cdot 3=\{3,4,5,6,7,8\}$, $3\cdot 4=\{8\}$, $3\cdot 5=\{8\}$,
$3\cdot 6=\{8\}$, $3\cdot 7=\{8\}$, $3\cdot 8=\{8,9,10,0,1,2,3\}$, $3\cdot 9=\{3\}$, and $3\cdot 10=\{3\}$,
$4\cdot 1=\{4\}$, $4\cdot 2=\{4,5,6,7\}$, $4\cdot 3=\{8\}$, $4\cdot 4=\{9\}$, $4\cdot 5=\{9\}$,
$4\cdot 6=\{9\}$, $4\cdot 7=\{9,10,0,1,2\}$, $4\cdot 8=\{3\}$, $4\cdot 9=\{4\}$, and $4\cdot 10=\{4\}$,
$5\cdot 1=\{5,6\}$, $5\cdot 2=\{7\}$, $5\cdot 3=\{8\}$, $5\cdot 4=\{9\}$, $5\cdot 5=\{10\}$,
$5\cdot 6=\{10,1,2\}$, $5\cdot 7=\{2\}$, $5\cdot 8=\{3\}$, $5\cdot 9=\{5\}$, and $5\cdot 10=\{5\}$,
$6\cdot 1=\{6\}$, $6\cdot 2=\{7\}$, $6\cdot 3=\{8\}$, $6\cdot 4=\{9\}$, $6\cdot 5=\{10,0,1\}$,
$6\cdot 6=\{1\}$, $6\cdot 7=\{2\}$, $6\cdot 8=\{3\}$, $6\cdot 9=\{4\}$, and $6\cdot 10=\{5,6\}$,
$7\cdot 1=\{7\}$, $7\cdot 2=\{7\}$, $7\cdot 3=\{8\}$, $7\cdot 4=\{9,10,0,1,2\}$, $7\cdot 5=\{2\}$,
$7\cdot 6=\{2\}$, $7\cdot 7=\{2\}$, $7\cdot 8=\{3\}$, $7\cdot 9=\{4,5,6,7\}$, and $7\cdot 10=\{7\}$,
$8\cdot 1=\{8\}$, $8\cdot 2=\{8\}$, $8\cdot 3=\{8,9,10,0,1,2,3\}$, $8\cdot 4=\{3\}$, $8\cdot 5=\{3\}$,
$8\cdot 6=\{3\}$, $8\cdot 7=\{3\}$, $8\cdot 8=\{3,4,5,6,7,8\}$, $8\cdot 9=\{8\}$, and $8\cdot 10=\{8\}$,
$9\cdot 1=\{9\}$, $9\cdot 2=\{9,10,0,1,2\}$, $9\cdot 3=\{3\}$, $9\cdot 4=\{4\}$, $9\cdot 5=\{4\}$,
$9\cdot 6=\{4\}$, $9\cdot 7=\{4,5,6,7\}$, $9\cdot 8=\{8\}$, $9\cdot 9=\{9\}$ and $9\cdot 10=\{9\}$,
$10\cdot 1=\{10,0,1\}$, $10\cdot 2=\{2\}$, $10\cdot 3=\{3\}$, $10\cdot 4=\{4\}$, $10\cdot 5=\{5\}$,
$10\cdot 6=\{5,6\}$, $10\cdot 7=\{7\}$, $10\cdot 8=\{8\}$, $10\cdot 9=\{9\}$ and $10\cdot 10=\{9\}$.

By these equalities the algebra $\mathfrak{P}_{M'_{2,1,2}}$ is commutative and strictly 7-deterministic.

**Theorem 2.** *The algebra $\mathfrak{P}_{M'_{s,1,k}}$ of binary isolating formulas with monotonic-to-right function $r$ has $2sk+k+1$ labels, is commutative and strictly $(2s+3)$-deterministic for every $s\geq 1$ and $k\geq 2$.*

Proof of Theorem 2. From Theorem 1 $M'_{s,1,k}=\langle M,K^3,E_1^2,\ldots,E_s^2,R^2\rangle$, where $E_s(x,y)$ is an equivalence relation partitioning $M$ into infinitely many infinite convex classes so that the induced order on $E_s$-classes is dense; $E_i(x,y)$ for every $1\leq i\leq s-1$ is an equivalence relation partitioning every $E_{i+1}$-class into infinitely many convex $E_i$-subclasses without endpoints so that the induced order on $E_i$-subclasses is dense; $R(M,a)$ has no right endpoint in $M$, $r^k(a)=a$ and $\neg E^*(a,r(a))$ for all $a\in M$; $r$ preserves $E_i$-classes for every $1\leq i\leq s$, i.e. if $E_i(a,b)$ holds for $a,b\in M$ then both $r(a)$ and $r(b)$ lie in the same $E_i^*$-class. Whence we obtain that the algebra $\mathfrak{P}_{M'_{s,1,k}}$ has $2sk+k+1$ binary isolating formulas:

$$\theta_0(x,y):=x=y,$$
$$\theta_{(2s+1)i+1}(x,y):=K_0(r^i(x),y,r^{i+1}(x))\wedge E_1^*(y,r^i(x)),\ \text{where } 0\leq i\leq k-1,$$
$$\theta_{(2s+1)i+j}(x,y):=K_0(r^i(x),y,r^{i+1}(x))\wedge E_j^*(y,r^i(x))\wedge\neg E_{j-1}^*(y,r^i(x)),$$
$$\text{where } 0\leq i\leq k-1, 2\leq j\leq s,$$
$$\theta_{(2s+1)i+s+1}(x,y):=K_0(r^i(x),y,r^{i+1}(x))\wedge\neg E_s^*(y,r^i(x))\wedge\neg E_s^*(y,r^{i+1}(x)),$$
$$\text{where } 0\leq i\leq k-1,$$
$$\theta_{(2s+1)(i+1)+1-j}(x,y):=K_0(r^i(x),y,r^{i+1}(x))\wedge E_j^*(y,r^{i+1}(x))$$
$$\wedge\neg E_{j-1}^*(y,r^{i+1}(x)),\ \text{where } 0\leq i\leq k-1, 2\leq j\leq s,$$
$$\theta_{(2s+1)(i+1)}(x,y):=K_0(r^i(x),y,r^{i+1}(x))\wedge E_1^*(y,r^{i+1}(x)),$$

$$\text{where } 0 \le i \le k-1.$$

Thus, we have $1+k+(s-1)k+k+(s-1)k+k = 2sk+k+1$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \ldots, \theta_{2sk+k-1}(a, M), \theta_{2sk+k}(a, M)).$$

We will now prove that the algebra $\mathfrak{P}_{M'_{s,1,k}}$ is commutative and strictly $(2s+3)$-deterministic for every $s \ge 1$ and $k \ge 2$.

First, obviously that $0 \cdot l = l \cdot 0 = \{l\}$ for any $0 \le l \le 3k$. Suppose further that $l_1 \ne 0$ and $l_2 \ne 0$.

Consider the following formula

$$\exists t[\theta_{l_1}(x, t) \wedge \theta_{l_2}(t, y)].$$

*Case 1:* $l_1 = (2s+1)i_1 + 1$ for some $0 \le i_1 \le k-1$.

We have: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$ and $E_1^*(t, r^{i_1}(x))$.

Let also $l_2 = (2s+1)i_2+1$ for some $0 \le i_2 \le k-1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E_1^*(y, r^{i_2}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)) \text{ and } E_1^*(y, r^{i_1+i_2}(x)).$$

Clearly, $0 \le i_1 + i_2 \le (k-1) + (k-1) = 2k-2$. If $i_1 + i_2 \le k-1$ then $l_1 \cdot l_2 = \{(2s+1)(i_1+i_2)+1\}$. If $i_1 + i_2 > k-1$ then we have $K_0(r^m(x), y, r^{m+1}(x))$ and $E_1^*(y, r^m(x))$, where $m = (i_1+i_2)[\text{mod k}]$. Then $l_1 \cdot l_2 = \{(2s+1)m+1\}$.

Let now $l_2 = (2s+1)i_2 + j$ for some $0 \le i_2 \le k-1$ and $2 \le j \le s$. Then we have: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E_j^*(y, r^{i_2}(t))$ and $\neg E_{j-1}^*(y, r^{i_2}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_j^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_{j-1}^*(y, r^{i_1+i_2}(x)).$$

If $i_1+i_2 \le k-1$ then $l_1 \cdot l_2 = \{(2s+1)(i_1+i_2)+j\}$. If $i_1+i_2 > k-1$ then we have $K_0(r^m(x), y, r^{m+1}(x))$, $E_j^*(y, r^m(x))$ and $\neg E_{j-1}^*(y, r^m(x))$, where $m = (i_1+i_2)[\text{mod k}]$. Then $l_1 \cdot l_2 = \{(2s+1)m+j\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_j^*(t, r^{i_2}(x))$, $\neg E_{j-1}^*(t, r^{i_2}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E_1^*(y, r^{i_1}(t))$.

Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_j^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_{j-1}^*(y, r^{i_1+i_2}(x)).$$

Then $l_2 \cdot l_1 = \{(2s+1)m+j\}$, where $m = (i_1+i_2)[\text{mod k}]$.

Let now $l_2 = (2s+1)i_2 + s + 1$ for some $0 \le i_2 \le k-1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E_s^*(y, r^{i_2}(t))$ and $\neg E_s^*(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), \neg E_s^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+1}(x)).$$

Let $m = (i_1+i_2)[\text{mod k}]$. then $l_1 \cdot l_2 = \{(2s+1)m+s+1\}$.

Consider the product $l_2 \cdot l_1$. We have: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $\neg E_s^*(t, r^{i_2}(x))$, $\neg E_s^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E_1^*(y, r^{i_1}(t))$.

Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_s^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_s^*(y, r^{i_1+i_2}(x)).$$

Then $l_2 \cdot l_1 = \{(2s+1)m + s + 1\}$, where $m = (i_1 + i_2)[\text{mod k}]$.

Let now $l_2 = (2s+1)(i_2+1)+1-j$ for some $0 \le i_2 \le k-1$ and $2 \le j \le s$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E_j^*(y, r^{i_2+1}(t))$ and $\neg E_{j-1}^*(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2}(t), y, r^{i_1+i_2+1}(t)), E_j^*(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E_{j-1}^*(y, r^{i_1+i_2+1}(t)).$$

Let $m = (i_1 + i_2)[\text{mod k}]$. Then $l_1 \cdot l_2 = \{(2s+1)(m+1) + 1 - j\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_j^*(t, r^{i_2}(x))$, $\neg E_{j-1}^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E_1^*(y, r^{i_1}(t))$.

Whence we obtain:

$$K_0(r^{i_1+i_2}(t), y, r^{i_1+i_2+1}(t)), E_j^*(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E_{j-1}^*(y, r^{i_1+i_2+1}(t)).$$

Then $l_2 \cdot l_1 = \{(2s+1)(m+1) + 1 - j\}$, where $m = (i_1 + i_2)[\text{mod k}]$.

Let now $l_2 = (2s+1)(i_2+1)$ for some $0 \le i_2 \le k-1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E_1^*(y, r^{i_2+1}(t))$. Whence we obtain: $E_1^*(y, r^{i_1+i_2+1}(x))$ and either $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ or $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$.

Let $m = (i_1+i_2+1)[\text{mod k}]$. Clearly, $0 \le m \le k-1$, since $1 \le i_1+i_2+1 \le (k-1)+(k-1)+1 = 2k-1$. If $m = 0$ then $l_1 \cdot l_2 = \{0, 1, 2sk+k\}$. If $m \ne 0$ then $l_1 \cdot l_2 = \{(2s+1)(m+1), (2s+1)(m+1)+1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_1^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$ and $E_1^*(y, r^{i_1}(t))$. Whence we obtain: $E_1^*(y, r^{i_1+i_2+1}(x))$ and either $K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x))$ or $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$.

Let $m = (i_1 + i_2 + 1)[\text{mod k}]$. Similarly, we have: if $m = 0$, $l_2 \cdot l_1 = \{0, 1, 2sk+k\}$. If $m \ne 0$, $l_2 \cdot l_1 = \{(2s+1)(m+1), (2s+1)(m+1)+1\}$.

*Case 2.* $l_1 = (2s+1)i_1 + j_1$ for some $0 \le i_1 \le k-1$ and $2 \le j_1 \le s$.

We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$, $E_{j_1}^*(t, r^{i_1}(x))$ and $\neg E_{j_1-1}^*(t, r^{i_1}(x))$.

Let also $l_2 = (2s+1)i_2 + j_2$ for some $0 \le i_2 \le k-1$ and $2 \le j_2 \le s$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E_{j_2}^*(y, r^{i_2}(t))$ and $\neg E_{j_2-1}^*(y, r^{i_2}(t))$.

Let $m = (i_1 + i_2)[\text{mod k}]$. If $j_1 \le j_2$, we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_{j_2}^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_{j_2-1}^*(y, r^{i_1+i_2}(x)),$$

whence $l_1 \cdot l_2 = \{(2s+1)m + j_2\}$. If $j_1 > j_2$, we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_{j_1}^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_{j_1-1}^*(y, r^{i_1+i_2}(x)),$$

whence $l_1 \cdot l_2 = \{(2s+1)m + j_1\}$.

Let now $l_2 = (2s+1)i_2 + s + 1$ for some $0 \le i_2 \le k-1$. Then we have $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E_s^*(y, r^{i_2}(t))$ and $\neg E_s^*(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), \neg E_s^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+1}(x)).$$

Then $l_1 \cdot l_2 = \{(2s+1)m+s+1\}$, where $m=(i_1+i_2)$[mod k].

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $\neg E_s^*(t, r^{i_2}(x))$, $\neg E_s^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $E_{j_1}^*(y, r^{i_1}(t))$ and $\neg E_{j_1-1}^*(y, r^{i_1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), \neg E_s^*(y, r^{i_1+i_2}(x)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+1}(x)).$$

Similarly, $l_2 \cdot l_1 = \{(2s+1)m+s+1\}$, where $m=(i_1+i_2)$[mod k].

Let now $l_2 = (2s+1)(i_2 = 1)+1-j_2$ for some $0 \le i_2 \le k-1$ and $2 \le j_2 \le s$. We have the following: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E_{j_2}^*(y, r^{i_2+1}(t))$ and $\neg E_{j_2-1}^*(y, r^{i_2+1}(t))$.

If $j_1 \le j_2$, we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_{j_2}^*(y, r^{i_1+i_2+1}(x)) \text{ and } \neg E_{j_2-1}^*(y, r^{i_1+i_2+1}(x)).$$

Then $l_1 \cdot l_2 = \{(2s+1)m+1-j_2\}$, where $m=(i_1+i_2+1)$[mod k]. If $j_1 > j_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_1}^*(y, r^{i_1+i_2+1}(x))$ and $\neg E_{j_1-1}^*(y, r^{i_1+i_2+1}(x))$, whence $l_1 \cdot l_2 = \{(2s+1)m+j_1\}$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_{j_2}^*(t, r^{i_2+1}(x))$, $\neg E_{j_2-1}^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $E_{j_1}^*(y, r^{i_1}(t))$ and $\neg E_{j_1-1}^*(y, r^{i_1}(t))$.

Let $m=(i_1+i_2+1)$[mod k]. If $j_1 \le j_2$, we obtain:

$$K_0(r^{i_1+i_2}(x), y, r^{i_1+i_2+1}(x)), E_{j_2}^*(y, r^{i_1+i_2+1}(x)) \text{ and } \neg E_{j_2-1}^*(y, r^{i_1+i_2+1}(x)),$$

whence $l_2 \cdot l_1 = \{(2s+1)m+1-j_2\}$. If $j_1 > j_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_1}^*(y, r^{i_1+i_2+1}(x))$ and $\neg E_{j_1-1}^*(y, r^{i_1+i_2+1}(x))$, whence $l_2 \cdot l_1 = \{(2s+1)m+j_1\}$.

Let now $l_2 = (2s+1)(i_2+1)$ for some $0 \le i_2 \le k-1$. Then we have: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E_1^*(y, r^{i_2+1}(t))$. Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_1}^*(y, r^{i_1+i_2+1}(x))$ and $\neg E_{j_1-1}^*(y, r^{i_1+i_2+1}(x))$. Then $l_1 \cdot l_2 = \{(2s+1)m+j_1\}$, where $m=(i_1+i_2+1)$[mod k].

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_1^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $E_{j_1}^*(y, r^{i_1}(t))$ and $\neg E_{j_1-1}^*(y, r^{i_1}(t))$.

Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_1}^*(y, r^{i_1+i_2+1}(x))$ and $\neg E_{j_1-1}^*(y, r^{i_1+i_2+1}(x))$, whence $l_2 \cdot l_1 = \{(2s+1)m+j_1\}$.

*Case 3.* $l_1 = (2s+1)i_1+s+1$ for some $0 \le i_1 \le k-1$.

We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$, $\neg E_s^*(t, r^{i_1}(x))$ and $\neg E_s^*(t, r^{i_1+1}(x))$.

Let also $l_2 = (2s+1)i_2+s+1$ for some $0 \le i_2 \le k-1$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $\neg E_s^*(y, r^{i_2}(t))$ and $\neg E_s^*(y, r^{i_2+1}(t))$.

Whence we obtain: $K_0(r^{i_1+i_2}(x), t, r^{i_1+i_2+2}(x))$.

Let $m=(i_1+i_2+1)$[mod k]. If $m=0$ then

$$l_1 \cdot l_2 = \{(2s+1)k-s, (2s+1)k-s+1, \dots, (2s+1)k, 0, 1, \dots, s, s+1\},$$

i.e. the product $l_1 \cdot l_2$ consists of $2s+3$ labels. If $m \neq 0$ then

$$l_1 \cdot l_2 = \{(2s+1)m+s+1, (2s+1)m+s+2, \ldots, (2s+1)m+3s+2\},$$

i.e. the product $l_1 \cdot l_2$ consists of $2s+1$ labels.

Let now $l_2 = (2s+1)(i_2 = 1) + 1 - j$ for some $0 \leq i_2 \leq k-1$ and $2 \leq j \leq s$. We have the following: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E_j^*(y, r^{i_2+1}(t))$ and $\neg E_{j-1}^*(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), \neg E_s^*(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+2}(t)).$$

Then $l_1 \cdot l_2 = \{(2s+1)m+s+1\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_j^*(t, r^{i_2+1}(x))$, $\neg E_{j-1}^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $\neg E_s^*(y, r^{i_1}(t))$ and $\neg E_s^*(y, r^{i_1+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), \neg E_s^*(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+2}(t)).$$

Then also $l_2 \cdot l_1 = \{(2s+1)m+s+1\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

Let now $l_2 = (2s+1)(i_2+1)$ for some $0 \leq i_2 \leq k-1$. Then we have: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E_1^*(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), \neg E_s^*(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+2}(t)).$$

Then $l_1 \cdot l_2 = \{(2s+1)m+s+1\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

Consider the product $l_2 \cdot l_1$. We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E_1^*(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $\neg E_s^*(y, r^{i_1}(t))$ and $\neg E_s^*(y, r^{i_1+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)), \neg E_s^*(y, r^{i_1+i_2+1}(t)) \text{ and } \neg E_s^*(y, r^{i_1+i_2+2}(t)).$$

Then also $l_2 \cdot l_1 = \{(2s+1)m+s+1\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

*Case 4.* $l_1 = (2s+1)(i_1+1)+1-j_1$ for some $0 \leq i_1 \leq k-1$ and $2 \leq j_1 \leq s$. We have the following:

$$K_0(r^{i_1}(x), t, r^{i_1+1}(x)), E_{j_1}^*(t, r^{i_1+1}(x)) \text{ and } \neg E_{j_1-1}^*(t, r^{i_1+1}(x)).$$

Let also $l_2 = (2s+1)(i_2+1)+1-j_2$ for some $0 \leq i_2 \leq k-1$ and $2 \leq j_2 \leq s$, i.e. $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$, $E_{j_2}^*(y, r^{i_2+1}(t))$ and $\neg E_{j_2-1}^*(y, r^{i_2+1}(t))$.

Let $m = (i_1+i_2+1)[\text{mod k}]$.

If $j_1 \leq j_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_2}^*(y, r^{i_1+i_2+2}(x))$ and $\neg E_{j_2-1}^*(y, r^{i_1+i_2+2}(x))$, whence $l_1 \cdot l_2 = (2s+1)(m+1)+1-j_2$.

If $j_1 > j_2$, we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_1}^*(y, r^{i_1+i_2+2}(x))$ and $\neg E_{j_1-1}^*(y, r^{i_1+i_2+2}(x))$, whence $l_1 \cdot l_2 = (2s+1)(m+1)+1-j_1$.

Let now $l_2 = (2s+1)(i_2+1)$ for some $0 \leq i_2 \leq k-1$. Then we have: $K_0(r^{i_2}(t), y, r^{i_1+1}(t))$ and $E_1^*(y, r^{i_2+1}(t))$.

Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E_{j_1}^*(y, r^{i_1+i_2+2}(x)$ and $\neg E_{j_1-1}^*(y, r^{i_1+i_2+2}(x))$. Then $l_1 \cdot l_2 = \{(2s+1)(m+1)+1-j_1\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

Consider the product $l_2 \cdot l_1$.

We have the following: $K_0(r^{i_2}(x), t, r^{i_2+1}(x))$, $E^*_1(t, r^{i_2+1}(x))$, $K_0(r^{i_1}(t), y, r^{i_1+1}(t))$, $E^*_{j_1}(y, r^{i_1+1}(t))$ and $\neg E^*_{j_1-1}(y, r^{i_1+1}(t))$.

Whence we obtain: $K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x))$, $E^*_{j_1}(y, r^{i_1+i_2+1}(x))$ and $\neg E^*_{j_1-1}(y, r^{i_1+i_2+2}(x))$.

Then also $l_2 \cdot l_1 = \{(2s+1)(m+1)+1-j_1\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

*Case 5.* $l_1 = (2s+1)(i_1+1)$ for some $0 \le i_1 \le k-1$.

We have the following: $K_0(r^{i_1}(x), t, r^{i_1+1}(x))$ and $E^*_1(t, r^{i_1+1}(x))$.

Let also $l_2 = (2s+1)(i_2+1)$ for some $0 \le i_2 \le k-1$. Then we have: $K_0(r^{i_2}(t), y, r^{i_2+1}(t))$ and $E^*_1(y, r^{i_2+1}(t))$. Whence we obtain:

$$K_0(r^{i_1+i_2+1}(x), y, r^{i_1+i_2+2}(x)) \text{ and } E^*_1(y, r^{i_1+i_2+2}(x)).$$

Then $l_1 \cdot l_2 = \{(2s+1)(m+1)\}$, where $m = (i_1+i_2+1)[\text{mod k}]$.

Thus, we established that the algebra $\mathfrak{P}_{M'_{s,1,k}}$ is commutative and strictly $(2s+3)$-deterministic for every $s \ge 1$ and $k \ge 2$. $\square$

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