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PREGEOMETRIES ON SOME FINITELY GENERATED COMMUTATIVE SEMIGROUPS

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Abstract: We discuss the pregeometries of some finitely generated commutative semigroups. In this article, the case of finitely generated commutative semigroups having a unique extension is considered, and their pregeometries are studied. We prove that some such semigroups form a pregeometry with definable and algebraic closure operators. When the definable closure operator for such semigroups was studied, the degree of rigidity of these semigroups was evaluated. Moreover, it has been proven that a finitely generated, complete archimedean semigroup is a group, and its finite and infinite cases have been deterimined.

Keywords: pregeometry, rigidity, finitely generated commutative semigroups, definable closure operator, algebraic closure operator, archimedean semigroups.

1 Introduction and preliminaries

The main algebraic properties of finitely generated commutative semigroups are mentioned in the books [1]. In this article, we prove that some such semigroups form a pregeometry with definable and algebraic closure operators.

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Definition 1. [2, 3] Let S be a set, $\mathcal{P}(S)$ be the power set of S and let $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ be an operator. We say that (S, cl) is a *pregeometry* if the following conditions hold.

1) if $X \subseteq S$, then $X \subseteq \mathrm{cl}(X)$ and $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$;

2) if $X \subseteq Y \subseteq S$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;

3) if $X \subseteq S$, $a, b \in S$, and $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X \cup \{a\})$;

4) if $X \subset S$ and $a \in \mathrm{cl}(X)$, then there is a finite subset $X_0 \subseteq X$ such that $a \in \mathrm{cl}(X_0)$.

Definition 2. [2, 3] Let (S, cl) be a pregeometry, $A \subseteq S$.

a) We say that A is *closed* if $\mathrm{cl}(A) = A$.

b) We say that A *independent* if $a \notin \mathrm{cl}(A \setminus \{a\})$ for any $a \in A$ and say that A is a *basis* for X if $A \subseteq X$ is independent and $X \subseteq \mathrm{cl}(A)$.

Lemma 1. [3] *If (S, cl) is a pregeometry, $X \subseteq S$, $B_1, B_2 \subseteq X$, and B_1, B_2 are bases for X, then $|B_1| = |B_2|$.*

So the cardinality of a basis for X is an invariant, and if B is a basis for the set X, then we say that $|B|$ is the *dimension of* X and write $\dim(X) = |B|$.

Definition 3. [2, 3] Let (S, cl) be a pregeometry. We say that (S, cl) is *modular* if for finite-dimensional closed $A, B \subseteq S$,

$$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B).$$

Lemma 2. *a) Let the operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ for the set S satisfy the following two conditions:*

1) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$;

2) For any $a \in S$, $\mathrm{cl}(\{a\}) = S$.

Then, (S, cl) is a pregeometry.

b) The pregeometry in case a) is modular.

Proof. a) From the second condition, it can be seen that $\mathrm{cl}(X)$ satisfies all the conditions of a pregeometry for any non-empty $X \subseteq S$. Thus, if $X = \varnothing$, then from $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$, it follows that $\mathrm{cl}(\varnothing) = \varnothing$ or $\mathrm{cl}(\varnothing) = S$. In both cases, this operator satisfies all the conditions of a pregeometry.

b) If $\mathrm{cl}(\{a\}) = S$ for every $a \in S$, then in this pregeometry, the closed sets can only be of two types: $\varnothing$ or S. Thus, we can have 3 possible cases: 1) $A = B = \varnothing$; 2) either A or B is the empty set; 3) $A = B = S$. In all three cases, the formula $\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$ holds. Therefore, this pregeometry is modular. □

The following question arises from Lemma 2: If there exists an operator $cl : \mathcal{P}(S) \to \mathcal{P}(S)$ for the set S such that $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$ for all $X \subseteq S$, and there exists a finite natural number k such that for any subset $A \subseteq S$ with k elements, the condition $\mathrm{cl}(A) = S$, does this imply that $(S,)$ is a pregeometry? However, in general, this condition does not hold for $k > 1$. Let us consider the following example:

Example 1. Let $S = \{x_1, x_2, x_3, x_4, x_5\}$ be a set, and let the operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ be defined as follows:

$$\mathrm{cl}(A) = \begin{cases} A, & \text{if } |A| \leq 1 \text{ or } |A| = 3 \\ \{x_1, x_4, x_5\}, & \text{if } A = \{x_1, x_4\} \\ A, & \text{if } |A| = 2 \text{ and } A \neq \{x_1, x_4\} \\ S, & \text{if } |A| \geq 4 \end{cases}.$$

Then, it can be seen that the operator "cl" satisfies all axioms of pregeometry given in Definition 1, except for the third axiom. Indeed, if we take $X = \{x_1\}$ and $b = x_4$, we obtain $\mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X) = \{x_1, x_4, x_5\} \setminus \{x_1\} = \{x_4, x_5\}$. However, if we take $a = x_5$, then $\mathrm{cl}(X \cup \{a\}) = \{x_1, x_5\}$. Thus, $b \notin \mathrm{cl}(X \cup \{a\})$.

Definition 4. [2, 3] Let $\mathfrak{M} = \langle M, \Sigma \rangle$ be an Σ-structure and $A \subseteq M$. We say that $b \in M$ is *definable over* A if there is a Σ-formula $\varphi(x, \overline{y})$ and $\overline{a} \in A$ such that

$$\mathfrak{M} \models \varphi(b, \overline{a}) \wedge \forall t\big(\varphi(t, \overline{a}) \to t = b\big).$$

We denote the set of all elements that are definable over A by $\mathrm{dcl}(A)$. We say that $b \in M$ is *algebraic over* A if there is a Σ-formula $\varphi(x, \overline{y})$ and $\overline{a} \in A$ such that $\mathfrak{M} \models \varphi(b, \overline{a})$ and $\{t \in M \mid \mathfrak{M} \models \varphi(t, \overline{a})\}$ is a finite set, and denote the set of all elements that are algebraic over A by $\mathrm{acl}(A)$. The sets $\mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ for a set A are called *definable closure* and *algebraic closure of* A, respectively.

Lemma 3. [3] *Let* $\mathfrak{M} = \langle M, \Sigma \rangle$ *be an* Σ*-structure and* $\tau \in \{\mathrm{dcl}, \mathrm{acl}\}$.

a) $\tau(\tau(A)) = \tau(A)$.

b) If $A \subseteq B$*, then* $\tau(A) \subseteq \tau(B)$.

c) If $a \in \tau(A)$*, then there is a finite subset* $A_0 \subseteq A$ *such that* $a \in \tau(A_0)$.

It can be seen from Lemma 3 that the operators acl and dcl satisfy all conditions of pregeometry except the third one. Whether the operators acl and dcl satisfy the third condition depends on the model under consideration.

Example 2. a) Let us consider the structure $\mathfrak{M} = \langle M, s \rangle$. Here M is the set of non-negative integers, and s is a unary operation that maps each number in M to the next element, i.e., $s(0) = 1$, $s(1) = 2$, ..., $s(k) = k + 1$, ...

Here, for any element $k > 0$, if we take $\phi(x, y) \leftrightharpoons y \approx s^k(x)$, then $\mathfrak{M} \models \phi(0, k) \wedge \forall t(\phi(0, t) \to t \approx k)$. Thus, $k \in \mathrm{dcl}(\{0\})$ for any $k \in M$. Additionally, if we take $\varphi(x) \leftrightharpoons \neg\exists t(t \approx s(x))$, then $\mathfrak{M} \models \varphi(0) \wedge \forall t(\varphi(t) \to t \approx 0)$. Hence, $0 \in \mathrm{dcl}(\emptyset)$. From this, it can be concluded that $\mathrm{dcl}(\emptyset) = M$. Thus, it can be seen that $(\mathfrak{M}, \mathrm{dcl})$ is a pregeometry.

b) LetB's consider the structure $\mathfrak{M}_1 = \langle M, \rho \rangle$, where $\rho^{\mathfrak{M}} = \{(1, 0), (2, 0)\}$ and M is the set of non-negative integers. In this case, it can be seen that $\mathrm{dcl}(\{0\}) = \{0\}$, $\mathrm{dcl}(\{1\}) = \{1, 0\}$, $\mathrm{dcl}(\{2\}) = \{2, 0\}$ and $\mathrm{dcl}(\{k\}) = \{k\}$ for any $k > 2$. Let $X = \{3\}$, $a = 0$, $b = 1$. Then, it can be seen that $a \in \mathrm{dcl}(X \cup \{b\}) \setminus \mathrm{dcl}(X)$. However, $b \notin \mathrm{dcl}(X \cup \{a\})$. Thus, in this case, the third condition of pregeometry for the "dcl" operator is not satisfied.

Definition 5. [4] For a set A in a structure $\mathcal{M}$, $\mathcal{M}$ is called *semantically A-rigid* or *automorphically A-rigid* if any A-automorphism $f \in \mathrm{Aut}(\mathcal{M})$ is identical. The structure $\mathcal{M}$ is called *syntactically A-rigid* if $M = \mathrm{dcl}(A)$.

A structure $\mathcal{M}$ is called *$\forall$-semantically / $\forall$-syntactically n-rigid* (respectively, *$\exists$-semantically / $\exists$-syntactically n-rigid*), for $n \in \omega$, if $\mathcal{M}$ is semantically / syntactically A-rigid for any (some) $A \subseteq M$ with $|A| = n$.

The least n such that $\mathcal{M}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$Q$-semantical / Q-syntactical degree of rigidity*, it is denoted by $\deg_{\mathrm{rig}}^{Q\text{-sem}}(\mathcal{M})$ and $\deg_{\mathrm{rig}}^{Q\text{-synt}}(\mathcal{M})$, respectively. Here if a set A produces the value of Q-semantical / Q-syntactical degree then we say that A *witnesses* that degree. If such n does not exists we put $\deg_{\mathrm{rig}}^{Q\text{-sem}}(\mathcal{M}) = \infty$ and $\deg_{\mathrm{rig}}^{Q\text{-synt}}(\mathcal{M}) = \infty$, respectively.

2 u-semigroups and quasi u-semigroups and their pregeometries

Definition 6. [1] Let M be a set with a binary operation $*$ defined on it, forming the algebraic structure $\mathfrak{M} = \langle M, * \rangle$. If the operation is associative , meaning that $(a*b)*c = a*(b*c)$ for all a, b, c in M, then we call $\mathfrak{M} = \langle M, * \rangle$ a *semigroup*. If in the semigroup $\mathfrak{M} = \langle M, * \rangle$ the equality $a * b = b * a$ holds for all a, b in M, then this semigroup is called a *commutative semigroup.*

For the sake of convenience, we use the notation xy instead of the notation $x * y$ in the semigroup $\mathfrak{M}$, and instead of writing $\underbrace{x \dots x}_{n\ times}$, we write x^n. We assume that $xy^0 = x$, $x^0 y = y$ for any x, y in M.

Let $\mathfrak{M}$ be a commutative semigroup and $A \subseteq M$ be a finite subset. We denote the intersection of all subsemigroups of $\mathfrak{M}$ that contain A by $\langle A \rangle$. This means that $\langle A \rangle$ is the smallest subsemigroup of $\mathfrak{M}$ that contains A with respect to being a subsemigroup. If $A = \{u_1, \dots, u_k\}$ is a finite subset, then we can see that

$$\langle A \rangle = \left\{ u_1^{t_1} \dots u_k^{t_k} \mid t_i \in \omega, \sum_{i=1}^{k} t_i \neq 0 \right\}.$$

Definition 7. [1] If for a commutative semigroup $\mathfrak{M} = \langle M, * \rangle$, there exists a finite subset $P = \{p_1, p_2, \dots, p_n\} \subseteq M$ such that $M = \langle P \rangle$, then the semigroup $\mathfrak{M}$ is called a *finitely generated commutative semigroup generated by P.*

It can be seen from Definition 7 that if $P \subseteq K \subseteq M$ and K is a finite set, then $M = \langle K \rangle$, that is, $\mathfrak{M} = \langle M, * \rangle$ is also a finitely generated semigroup by K. Therefore, without loss of generality, when we refer to the semigroup $\mathfrak{M} = \langle M, * \rangle$ as finitely generated by P, we consider P as the set with the least number of elements that finitely generates $\mathfrak{M} = \langle M, * \rangle$.

Definition 8. Let $\mathfrak{M} = \langle M, * \rangle$ be a finitely generated commutative semigroup, $P = \{p_1, \ldots, p_n\} \subseteq M$ and M be generated by P. If $a = p_1^{s_1} \ldots p_n^{s_n} \in M$ and the equation $x_1^{s_1} \ldots x_n^{s_n} = a$ has a unique solution $(x_1, \ldots, x_n) = (p_1, \ldots, p_n)$ in $\mathfrak{M}$, then we say that $\mathfrak{M}$ is a *u-semigroup with respect to* P.

Let $P = \{q_1, q_2, \ldots, q_n\}$. Here, the set $\{q_1, q_2, \ldots, q_n\}$ is obtained by rearranging the elements of the set $\{p_1, p_2, \ldots, p_n\}$ in a different order. If for every $a = q_1^{r_1} q_2^{r_2} ... q_k^{r_k}$ in M the equation $x_1^{r_1} x_2^{r_2} ... x_k^{r_k} = a$ has a unique solution $(x_1, x_2, ..., x_k) = (q_1, q_2, ..., q_k)$, then we call the semigroup $\mathfrak{M}$ a *quasi u-semigroup with respect to* P, where $r_1, r_2, ..., r_k$ are natural numbers and $1 \leq k \leq n$.

The semigroup $\mathfrak{M}$ is called a *u-semigroup* [*quasi u-semigroup*] when there exists a finite set P such that $M = \langle P \rangle$ and $\mathfrak{M}$ is a *u-semigroup with respect to* P [*quasi u-semigroup with respect to* P].

Theorem 1. *If* $\mathfrak{M}$ *is a u-semigroup, then* $(\mathfrak{M}, \mathrm{dcl})$ *is a pregeometry.*

Proof. According to Lemma 3, it suffices to prove that the third condition of the pregeometry is satisfied for the operator $\mathrm{dcl} : \mathcal{P}(M) \to \mathcal{P}(M)$. Let M be generated by $P = \{p_1, \ldots, p_n\} \subseteq M$, $\mathfrak{M}$ is a u-semigroup with respect to P and $b = p_1^{r_1} \ldots p_n^{r_n}$. Let us take an arbitrary element $a = p_1^{s_1} \ldots p_n^{s_n} \in M$. Since $\mathfrak{M}$ is a u-semigroup, it follows that the formula

$$\Phi(x, b) \leftrightharpoons \exists x_1 \ldots \exists x_n \Big((x_1^{r_1} \ldots x_n^{r_n} \approx b) \wedge (x_1^{s_1} \ldots x_n^{s_n} \approx x) \Big)$$

has a unique solution $x = a$. It follows that $\mathrm{dcl}(\{b\}) = M$ for all $b \in M$. This proves that the third condition of pregeometry holds for a u-semigroup $\mathfrak{M}$. □

Corollary 1. *Let* $\mathfrak{M} = \langle M, * \rangle$ *be a u-semigroup.*

a) If A *is a non-empty closed set in the pregeometry* $(\mathfrak{M}, \mathrm{dcl})$*, then* $A = M$.

b) $(\mathfrak{M}, \mathrm{dcl})$ *is modular.*

*Proof.*The proof of this follows from Lemma2, Theorem1 and their proofs. □

Corollary 2. *If* $\mathfrak{M} = \langle M, * \rangle$ *is a u-semigroup then*

$$\max\{\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathfrak{M}), \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathfrak{M}), \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathfrak{M}), \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathfrak{M})\} \leq 1. \qquad (1)$$

Proof. By Corollary 1, $M = \mathrm{dcl}(A)$ for any nonempty $A \subseteq M$. In particular, $M = \mathrm{dcl}(\{a\})$ for any $a \in M$ implying $\deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathfrak{M}) \leq 1$. Since following [4],

$$\deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathfrak{M}) \leq \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathfrak{M}),\ \deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathfrak{M}) \leq \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathfrak{M}),$$
$$\deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathfrak{M}) \leq \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathfrak{M}),$$

we obtain the inequality (3). □

Example 3. Let us consider the following semigroup $\mathfrak{M} = \langle M, \cdot \rangle$, which is a subset of the set of natural numbers and is defined with the usual multiplication operation:

$$M = \langle P \rangle = \{2^\alpha \cdot 3^\beta | \alpha, \beta \in \{0, 1, 2, ...\}, \alpha^2 + \beta^2 \neq 0\},$$

where $P = \{2, 3\}$. It can be seen that the semigroup M is not u-semigroup with respect to P. For example, if we take the element $a = 2^7$, we can write it as $a = 2^7 \cdot 3^0$, but the equation $x_1^7 x_2^0 = a$ does not have a unique solution. In fact, we can see that any pair $(x_1, x_2) = (2, x)$ is a solution to this equation. However, it is not difficult to see that this semigroup is a quasi u-semigroup.

Proposition 1. *A semigroup $\mathfrak{M} = (M, *)$ is a countable u-semigroup if and only if it is a countable cyclic semigroup without an identity element.*

Proof. ($\Leftarrow$) First, let us prove that if $\mathfrak{M} = (M, *)$ is cyclic and does not have an identity element, it is a u-semigroup. In this case, we can assume $M = \{p, p^2, \dots, p^\alpha, \dots\}$. Now, consider the element $a = p^r$.

We can show that the equation $x^r = a$ has no solution other than $x = p$. Indeed, if this equation has another solution $x = p^s$, then we get the following equality: $p^{rs} = p^r$.

This means that the set M consists only of the elements

$$p, p^2, \dots, p^r, p^{r+1}, \dots, p^{sr-1},$$

which contradicts the assumption that M is infinite.

($\Rightarrow$) Therefore, it is sufficient to prove that for the semigroup $\mathfrak{M} = \langle M, * \rangle$, $M = \langle P \rangle$ and the set P consists of only one element. Let us assume the opposite. Without loss of generality, it is sufficient to consider the case $P = \{p_1, p_2\}$ and $\mathfrak{M}$ is u-semigroup with respect to P. For the element $a = p_1^\alpha$, we can express it as $a = p_1^\alpha \cdot p_2^0$. However, the equation $x_1^\alpha \cdot x_2^0 = a$ does not have a unique solution. In fact, any pair $(x_1, x_2) = (p_1, x)$ is a solution to this equation. From this, we can conclude that the set $P = \{p\}$ consists of a single element, and $M = \{p, p^2, \dots, p^\alpha, \dots\}$ holds only when $\mathfrak{M} = \langle M, * \rangle$ is a u-semigroup. Thus, $\mathfrak{M} = \langle M, * \rangle$ is a countable cyclic semigroup that does not have an identity element. $\square$

It is known that a group is also a special case of a semigroup. Therefore, a natural question arises: is it possible to introduce the concepts of a "u-group" and a "quasi u-group" for a group as described above? It can be seen that a u-group does not exist because a group contains an identity element. Moreover, it can be seen that a quasi u-group does not exist when $|P| \geq 2$. Indeed, for an element $a = p_1^r p_2^t$ in a group, the equation $x_1^r x_2^t = a$ does not have a unique solution. In fact, any pair $(x_1, x_2) = (p_1^r p_2^{\alpha t}, p_2^{t-\alpha t})$ is a solution to this equation. However, it can be seen that when $P = \{p\}$, this group is a quasi u-group.

Theorem 2. *If $\mathfrak{M} = \langle M, * \rangle$ is a quasi u-semigroup such that $M = \langle P \rangle$ and $P = \{p_1, p_2\}$, then $(\mathfrak{M}, \mathrm{dcl})$ is a pregeometry.*

Proof. First, we will prove that $\mathrm{dcl}(\{p_1\}) = M$. (Similarly, it can also be shown that $\mathrm{dcl}(\{p_2\}) = M$.) We introduce the following notation:

$$(x \mid y) \rightleftharpoons \exists t (x * t \approx y) \text{ and } (x \nmid y) \rightleftharpoons \neg \exists t (x * t \approx y).$$

Using these, we introduce the following formula:

$$\sigma(x,y) \rightleftharpoons \forall z\Big(\big((x \nmid y) \wedge (x \neq y) \wedge (x \nmid z) \wedge (x \neq z) \wedge (y \neq z)\big) \to (y \mid z)\Big).$$

From this, it can be seen that

$$\mathfrak{M} \models \sigma(p_1,p_2) \wedge \forall t\Big(\sigma(p_1,t) \to (t \approx p_2)\Big).$$

According to the definition 4, this implies that $p_2 \in \operatorname{dcl}(\{p_1\})$. Thus, it follows that $\operatorname{dcl}(\{p_1\}) = \operatorname{dcl}(\{p_1,p_2\}) = M$. Similarly, it can be shown that $p_1 \in \operatorname{dcl}(\{p_2\})$ and $\operatorname{dcl}(\{p_2\}) = M$.

Now, we will prove that $\operatorname{dcl}(\{a\}) = M$ for any $a \in M$. Since $\mathfrak{M}$ is a quasi u-semigroup, it does not have an identity element. Thus, there are three possible cases:

1) $a = p_1^{\alpha}p_2^{\beta}$; 2) $a = p_1^{\alpha}$; 3) $a = p_2^{\beta}$. (Here α and β are natural numbers greater than zero.) Let us take an arbitrary element b from M. We can assume that $b = p_1^r p_2^s$ and analyze each of the three cases above:

1) Let

$$\Upsilon(x,y) \rightleftharpoons \exists x_1 \exists x_2\Big((x_1^{\alpha}x_2^{\beta} \approx x) \wedge (x_1^r x_2^s \approx y)\Big).$$

Since $\mathfrak{M}$ is a quasi u-semigroup, we have

$$\mathfrak{M} \models \Upsilon(a,b) \wedge \forall t\Big(\Upsilon(a,t) \to (t \approx b)\Big).$$

It follows that $b \in \operatorname{dcl}(\{a\})$, which means that in this case $\operatorname{dcl}(\{a\}) = M$.

2) Let

$$\zeta(x,y) \rightleftharpoons \exists x_1 \exists x_2\Big((x_1^{\alpha} \approx x) \wedge \sigma(x,y) \wedge (x_1^r x_2^s \approx y)\Big).$$

Since $\mathfrak{M}$ is a quasi u-semigroup, we have

$$\mathfrak{M} \models \zeta(a,b) \wedge \forall t\Big(\zeta(a,t) \to (t \approx b)\Big).$$

So, we have shown that $\operatorname{dcl}(\{a\}) = M$ in this case as well.

3) This case is also proven analogously to case 2.

Thus, we have proven that $\operatorname{dcl}(\{a\}) = M$ for any $a \in M$. Therefore, according to Lemma 2, $(\mathfrak{M}, \operatorname{dcl})$ is a pregeometry. □

Corollary 3. *Let $\mathfrak{M} = \langle M, * \rangle$ be a quasi u-semigroup, and let $M = \langle P \rangle, P = \{p_1, p_2\}$. Then:*

a) $(\mathfrak{M}, \operatorname{dcl})$ *is a modular pregeometry;*

b) $\max\{\deg_{\mathrm{rig}}^{\exists\text{-sem}}(\mathfrak{M}), \deg_{\mathrm{rig}}^{\exists\text{-synt}}(\mathfrak{M}), \deg_{\mathrm{rig}}^{\forall\text{-sem}}(\mathfrak{M}), \deg_{\mathrm{rig}}^{\forall\text{-synt}}(\mathfrak{M})\} \leq 1.$

Proof. a) It follows directly from the above theorem and Lemma 2.

b) It can be seen that this can be proved as in Corollary 2. □

The proof of Theorem 2 can be briefly explained as follows: Since $\mathfrak{M}$ is a quasi u-semigroup, for any $a \in M$, we can determine its generator (i.e., p_1 or p_2). Using p_1 or p_2, the entire semigroup can be generated. The question arises, "Can Theorem 2 be generalized to the case where $P = \{p_1, p_2, ..., p_n\}$?"

Theorem 3. *If $\mathfrak{M} = \langle M, *\rangle$ is a quasi u-semigroup, then $(\mathfrak{M}, \mathrm{acl})$ is a pregeometry.*

Proof. We will prove this theorem by generalizing the proof of Theorem 2 mentioned above.

Since M is a quasi u-semigroup, it follows that there exists a finite set $P = \{p_1, \dots, p_n\}$ such that $M = \langle P\rangle$ and M is a quasi u-semigroup with respect to P. It is known that $\mathrm{dcl}(X) \subseteq \mathrm{acl}(X)$. Furthermore, it can be seen that $\mathrm{dcl}(\{a\}) = M$, hence $\mathrm{acl}(\{a\}) = M$ for every $a \in M$. From this, by Lemmas 2 and 3, it follows that for $n = 1$ and $n = 2$, (M, acl) is also a pregeometry.

Now, we will prove the case for $n > 2$. First, we will show that $P \subset \mathrm{acl}(\{p_i\})$ for any $p_i \in P$. Using this, we will then show that $\mathrm{acl}(\{a\}) = M$ for any $a \in M$.

Let $\mathrm{dfr}(\overline{x}) \rightleftharpoons \bigwedge_{\substack{1\le i,j\le k\\ i\neq j}}(x_i \neq x_j)$ for tuple $\overline{x} = (x_1, \dots, x_k)$ and let us introduce the following formulas:

$$\xi(\overline{t}, x, z) \rightleftharpoons \mathrm{dfr}(\overline{t}) \wedge \bigwedge_{i=1}^{n-1} \Big((x \nmid t_i) \wedge (x \neq t_i) \wedge (t_i \nmid z) \wedge (t_i \neq z)\Big),$$

$$\sigma_n(x, y) \rightleftharpoons \exists\overline{t}\,\forall z\Big((x \nmid y)\wedge(x \neq y)\wedge\xi(\overline{t}, x, z)\wedge(x \nmid z)\wedge(x \neq z)\wedge(y \neq z) \to (y \mid z)\Big),$$

where $\overline{t} = (t_1, t_2, \dots, t_{n-1})$. From the definition of these formulas, it can be seen that $\{b \mid \mathfrak{M} \models \sigma_n(p_1, b)\} = \{p_2, p_3, \dots, p_n\}$, meaning that $P \setminus \{p_1\} \subseteq \mathrm{acl}(\{p_1\})$. From this, it follows that $P \subseteq \mathrm{acl}(\{p_1\})$. Analogously, it can be seen that $P \subseteq \mathrm{acl}(\{p_i\})$ for any $p_i \in P$. Therefore, in general, if $A \subseteq P, A \neq \varnothing$, we can say that $P \subseteq \mathrm{acl}(A)$.

Now, since $\mathfrak{M}$ is a quasi u-semigroup, we will show that $\mathrm{dcl}(\{a\}) = M$ for any $a \in M$.

Let $a = p_{i_1}^{s_1} \dots p_{i_k}^{s_k}$, $p_{i_j} \in P$, $s_j \in \{1, 2, \dots, \}$, $j \in \{1, 2, \dots, k\}$, $k \le n$. For any $b \in M$, we will prove that $b \in \mathrm{dcl}(\{a\})$. Let $b = p_{u_1}^{r_1} \dots p_{u_m}^{r_m}$, $p_{u_l} \in P$, $r_l \in \{1, 2 \dots, \}$, $l \in \{1, 2, \dots, m\}$, $m \le n$. We will consider the following formula:

$$\theta(x, y) \rightleftharpoons \exists\overline{v}\,\exists\overline{w}\Big(\mathrm{dfr}(\overline{w})\wedge(v_1^{s_1} \dots v_k^{s_k} \approx x)\wedge(w_1^{r_1} \dots w_m^{r_m} \approx y)\wedge\bigwedge_{i=1}^{m} \sigma_n(v_1, w_i)\Big).$$

It can be observed that $\mathfrak{M}$ is a quasi u-semigroup, hence $\mathfrak{M} \models \theta(a, b)$, and moreover, the set $\{c \mid \mathfrak{M} \models \theta(a, c)\}$ is finite.

Thus, we have proven that $\mathrm{acl}(\{a\}) = M$. Therefore, according to Lemma 2, $(\mathfrak{M}, \mathrm{acl})$ is a pregeometry. $\square$

Corollary 4. *Let $\mathfrak{M} = \langle M, *\rangle$ be a quasi u-semigroup, and let $M = \langle P\rangle$, $P = \{p_1, \dots, p_n\}$, $n > 2$. Then $(\mathfrak{M}, acl)$ is a modular pregeometry.*

Proof. It follows directly from the above theorem and Lemma 2. □

Remark. It can be seen that the proof of Theorem 3 is a generalization of the proof of Theorem 2. However, in Theorem 2, we proved that $(\mathfrak{M}, \mathrm{dcl})$ is a pregeometry using this method, whereas in Theorem 3, this method shows us that $(\mathfrak{M}, \mathrm{acl})$ is a pregeometry. If we pay attention to the proof of Theorem 3, the formula $\sigma_n(p_i, y)$ allows us to ask the question: "If some $p_i \in P$ is known, what are the other elements of P? " In the proof of Theorem 3, when $n = 1$, $\sigma_1(x, y)$ can be considered as $\sigma(x, y)$ from the proof of Theorem 2. That is, using $\sigma_n(p_i, x)$, we can only find the elements of P other than p_i. As a result, when $P = \{p_1, p_2\}$, we can only find one element of P other than p_i. This is why we showed that $(\mathfrak{M}, \mathrm{dcl})$ is a pregeometry in the case where $P = \{p_1, p_2\}$. When $P = \{p_1, \ldots, p_n\}$ with $n > 2$, and we ask what the elements of P are other than p_i, we do not have a unique answer, but we do have a finite number of possible answers. Consequently, in this case, we proved that $(\mathfrak{M}, \mathrm{acl})$ is a pregeometry.

Corollary 5. *Let $\mathfrak{M} = \langle M, * \rangle$ be a quasi u-semigroup, and let $M = \langle P \rangle$, $P = \{p_1, \ldots, p_n\}$. Then $\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathfrak{M}) \leq \deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathfrak{M}) \leq n - 1$.*

Proof. 1) Let $n = 1, P = \{p\}$. Let's consider the following formula:

$$\phi(x) \rightleftharpoons \forall z((x \mid z) \vee (x \approx z)).$$

Then, it can be seen that $\mathfrak{M} \models \phi(p) \wedge \forall y(\phi(y) \to (p \approx y))$, which means $p \in \mathrm{dcl}(\varnothing)$ and from $p \in \mathrm{dcl}(\varnothing)$ and $\mathrm{dcl}(\{p\}) = M$, it follows that $\mathrm{dcl}(\varnothing) = M$.

2) Let $n = 2, P = \{p_1, p_2\}$. In this case, the validity of the assertion can be seen by Corollary 3.

3) Let $n > 2, P = \{p_1, \ldots, p_n\}$. Let's consider the following formula:

$$\chi(\overline{x}, y) \rightleftharpoons \forall z\Big(\mathrm{dfr}(\overline{x}) \wedge (y \neq z) \wedge \bigwedge_{i=1}^{n-1} \Big((x_i \nmid y) \wedge (x_i \neq y) \wedge (x_i \nmid z) \wedge (x_i \neq z)\Big) \to (y \mid z)\Big)$$

As we can see,

$$\mathfrak{M} \models \chi(p_1, p_2, \ldots, p_{n-1}, p_n) \wedge \forall t\big(\chi(p_1, p_2, \ldots, p_{n-1}, t) \to t \approx p_n\big)$$

for this formula, which means $p_n \in \mathrm{dcl}(\{p_1, p_2, \ldots, p_{n-1}\})$. Therefore, since $\mathrm{dcl}(\{p_1, \ldots, p_{n-1}, p_n\}) = M$ and $p_n \in \mathrm{dcl}(\{p_1, p_2, \ldots, p_{n-1}\})$, it follows that $\mathrm{dcl}(\{p_1, p_2, \ldots, p_{n-1}\}) = M$.

In all three cases above, we can see that

$$\deg^{\exists\text{-sem}}_{\mathrm{rig}}(\mathfrak{M}) \leq \deg^{\exists\text{-synt}}_{\mathrm{rig}}(\mathfrak{M}) \leq n - 1.$$

□

3 Finitely generated Archimedean semigroups and their pregeometries

Let $\mathfrak{M} = \langle M, * \rangle$ be commutative semigroup.

Definition 9. [1] Let g and h be elements of the commutative semigroup $\mathfrak{M}$. If there exists an element x in M such that $gx = h$, then we say that g *divides* h. Let $A \subseteq M$. The subset A is called an *Archimedean subset of* M if for any two elements a and b in A, one divides some power of the other . That is, for any $a, b \in A$, there exist an element x and a natural number α such that $bx = a^{\alpha}$. If in the above case $M = A$, then the semigroup $\mathfrak{M}$ is called an *Archimedean semigroup*. The subset A is called an *complete Archimedean subset of* M if, for any two elements a and b in A, one divides the other. That is, for any $a, b \in A$, there exists an element x such that $a = bx$. If in the above case $M = A$, then the semigroup $\mathfrak{M}$ is called an *complete Archimedean semigroup*.

Proposition 2. *a) If a finitely generated semigroup $\mathfrak{M} = \langle M, * \rangle$ is an Archimedean semigroup, then there exist α_i and a_{ij} such that*

$$\begin{cases} p_1^{\alpha_1} = p_1^{a_{11}} p_2^{a_{12}} \dots p_n^{a_{1n}} \\ p_2^{\alpha_2} = p_1^{a_{21}} p_2^{a_{22}} \dots p_n^{a_{2n}} \\ \dots\dots\dots \\ p_n^{\alpha_n} = p_1^{a_{n1}} p_2^{a_{n2}} \dots p_n^{a_{nn}} \end{cases}, \tag{2}$$

where $P = \{p_1, p_2, ..., p_n\}$, $M = \langle P \rangle$, $\alpha_i \in \omega \setminus \{0\}$, $a_{ij} \in \omega$, $\sum\limits_{\substack{1 \le i \le n \\ i \ne j}} a_{ij}^2 > 0$, $i, j \in \{1, ..., n\}$.

b) If a finitely generated semigroup $\mathfrak{M} = \langle M, * \rangle$ is an complete Archimedean semigroup, then there exist a_{ij} such that

$$\begin{cases} p_1 = p_1^{a_{11}} p_2^{a_{12}} \dots p_n^{a_{1n}} \\ p_2 = p_1^{a_{21}} p_2^{a_{22}} \dots p_n^{a_{2n}} \\ \dots\dots\dots \\ p_n = p_1^{a_{n1}} p_2^{a_{n2}} \dots p_n^{a_{nn}} \end{cases}, \tag{3}$$

where $P = \{p_1, p_2, ..., p_n\}$, $M = \langle P \rangle$, $a_{ij} \in \omega$, $\sum\limits_{\substack{1 \le i \le n \\ i \ne j}} a_{ij}^2 > 0$, $i, j \in \{1, ..., n\}$.

Proof. a) If $\mathfrak{M}$ is an Archimedean semigroup, then for p_i and p_j there exist $x \in M$ and $\alpha_i \in \omega \setminus \{0\}$ such that $p_j x = p_i^{\alpha_i}$, where $i, j \in \{1, \dots, n\}$, $i \ne j$. Since $\mathfrak{M}$ is a finitely generated semigroup, there exist $u_{ik} \in \omega$ ($k \in \{1, \dots, n\}$) such that $x = p_1^{u_{i1}} p_2^{u_{i2}} \dots p_n^{u_{in}}$. So, we come to the equality $p_i^{\alpha_i} = p_1^{u_{i1}} \dots p_j^{u_{ij}+1} \dots p_n^{u_{in}}$. By this method we form the equations (2) and we can see that $\sum\limits_{\substack{1 \le i \le n \\ i \ne j}} a_{ij}^2 > 0$, where $a_{ik} = u_{ik}, k \ne j$ and $a_{ij} = u_{ij} + 1$.

b) Assuming $\alpha_i = 1, i \in \{1, \dots, n\}$ for this case, it is proved as in case a). $\square$

Let $\mathfrak{M} = \langle M, * \rangle$ be a finitely generated complete Archimedean semigroup, $P = \{p_1, \ldots, p_n\} \subseteq M$ and M be generated by P. Thus, according to Proposition 2, we can consider that the elements of the set P are related to each other as follows:

$$\begin{cases} p_1 = p_1^{a_{11}} p_2^{a_{12}} \ldots p_n^{a_{1n}} \\ p_2 = p_1^{a_{21}} p_2^{a_{22}} \ldots p_n^{a_{2n}} \\ \ldots\ldots\ldots \\ p_n = p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}} \end{cases}, \qquad (4)$$

where $a_{ij} \in \omega$, $i, j \in \{1, \ldots, n\}$. Here, it is assumed that $a_{ii} > 0$, $\quad i \in \{1, \ldots, n\}$. Because if for some i, $a_{ii} = 0$, then the element p_i is expressed using other $p_j \in P$, $\quad j \neq i$. In this case, we can see that for the set $D = P \setminus \{p_i\}$, we have $M = \langle D \rangle$. The number of elements in set D is one less than the number of elements in set P. This contradicts our earlier assumption that the set P is the smallest set generating M (i.e., $M = \langle P \rangle$). Then, considering that $\mathfrak{M}$ is a finitely generated complete Archimedean semigroup and according to Proposition 2, $\sum\limits_{\substack{1 \leq i \leq n \\ i \neq j}} a_{ij}^2 > 0$ and $a_{ii} > 0$, we can see that $a_{ij} > 0$ $(i, j \in \{1, ..., n\})$ in the equations (4). If $\mathfrak{M} = \langle M, * \rangle$ is a complete Archimedean semigroup finitely generated by the set $P = \{p_1, \ldots, p_n\}$, then the elements of P are interconnected by a system of equations of the form (4), where all $a_{ij} > 0$. These relationships, as in (4), can be generated using the following algorithm:

First, we demonstrate how to construct the equation $p_1 = p_1^{a_{11}} p_2^{a_{12}} \ldots p_n^{a_{1n}}$ for p_1, where all $a_{1j} > 0$.

1) Since $\mathfrak{M}$ is a complete Archimedean semigroup, for p_1 and p_2, there exists an element x_1 such that $p_1 = p_2 x_1$. Thus, similarly, for the elements x_1 and p_1, there exists an element x_2 such that $x_1 = p_1 x_2$. From this, we obtain the equality $p_1 = p_1 p_2 x_2$.

2) Similarly, for x_2 and p_3, there exists an element x_3 such that the equality $x_2 = p_3 x_3$ holds and from this, we obtain the equality $p_1 = p_1 p_2 p_3 x_3$. Continuing this process, we see that $x_k = p_{k+1} x_{k+1}$. From this, we obtain the equality $p_1 = p_1 p_2 \ldots p_k p_{k+1} x_{k+1}$, where $k \in \{2, \ldots, n-1\}$. Thus, when $k = n - 1$, we arrive at the following equality: $p_1 = p_1 p_2 \ldots p_n x_n$.

3) Since x_{k+1} is also an element of the semigroup $\mathfrak{M}$, we can write $x_{k+1} = p_1^{c_1} p_2^{c_2} \ldots p_n^{c_n}$. From this, the last equality takes the form $p_1 = p_1^{a_{11}} p_2^{a_{12}} \ldots p_n^{a_{1n}}$, where $a_{1j} = c_j + 1$. Thus, it can be seen that $a_{1j} > 0$ for all $j \in \{1, \ldots, n\}$. By repeating the above algorithm for all p_i, we can see that the elements of the set P are related through a system of equations (4) where all $a_{ij} > 0$.

Thus, without loss of generality, when discussing a finitely generated complete Archimedean semigroup $\mathfrak{M} = \langle M, * \rangle$, we can assume that there exists a set P whose elements are related by a system of equations like (4), with all a_{ij} being positive integers, such that $M = \langle P \rangle$.

Theorem 4. *If $\mathfrak{M} = \langle M, *\rangle$ is a finitely generated complete Archimedean semigroup, $P = \{p_1, \ldots, p_n\}$ and M is generated by P, then $\mathfrak{M}$ is a finitely generated Abelian group.*

Proof. Let us consider an arbitrary element $g = p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n}$ from M. We can assume that $s_i > 0$ for p_i in the expansion of g. Since $a_{ij} > 0$, if this is not the case, using (4) as follows, we can make the numbers to the exponent of p_i in the expansion of g always greater than zero. For example, let $s_i = 0$ for some i. In this case, $g = p_1^{s_1} \ldots p_{i-1}^{s_{i-1}} p_{i+1}^{s_{i+1}} \ldots p_n^{s_n}$. Taking into account the equality $p_1 = p_1^{a_{11}} p_2^{a_{12}} \ldots p_n^{a_{1n}}$ (here all $a_{1j} > 0$) from the system of equations (4), we can write the element g as follows:

$$g = (p_1^{a_{11}} p_2^{a_{12}} \ldots p_i^{a_{1i}} \ldots p_n^{a_{1n}})^{s_1} p_2^{s_2} \ldots p_{i-1}^{s_{i-1}} p_{i+1}^{s_{i+1}} \ldots p_n^{s_n} =$$
$$= p_1^{a_{11}s_1} p_2^{a_{12}s_2} \ldots p_{i-1}^{a_{1i-1}s_{i-1}} p_i^{a_{1i}} p_{i+1}^{a_{1i+1}s_{i+1}} \ldots p_n^{a_{1n}s_n}.$$

Thus, we can assume without loss of generality that $g = p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n}$, $s_i > 0$. Then we write g as follows:

$$g = p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} = p_1 p_1^{s_1-1} p_2^{s_2} \ldots p_n^{s_n} = (p_1^{a_{11}} p_2^{a_{12}} \ldots p_n^{a_{1n}}) p_1^{s_1-1} p_2^{s_2} \ldots p_n^{s_n} =$$
$$= (p_1^{a_{11}-1} p_2^{a_{12}} \ldots p_n^{a_{1n}}) p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} = (p_1^{a_{11}-1} p_2^{a_{12}} \ldots p_n^{a_{1n}}) g,$$
$$g = p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} = p_1^{s_1} p_2 p_2^{s_2-1} \ldots p_n^{s_n} = p_1^{s_1} (p_1^{a_{21}} p_2^{a_{22}} \ldots p_n^{a_{2n}}) p_2^{s_2-1} \ldots p_n^{s_n} =$$
$$= (p_1^{a_{21}} p_2^{a_{22}-1} \ldots p_n^{a_{2n}}) p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} = (p_1^{a_{21}} p_2^{a_{22}-1} \ldots p_n^{a_{2n}}) g,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$g = p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} = p_1^{s_1} p_2^{s_2} \ldots p_n p_n^{s_n-1} = p_1^{s_1} p_2^{s_2} \ldots (p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}}) p_n^{s_n-1} =$$
$$= (p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}-1}) p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n} = (p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}-1}) g.$$

Therefore, the element g satisfies the relation $g = (p_1^{a_{11}} \ldots p_i^{a_{ii}-1} \ldots p_n^{a_{nn}}) g$, where $i \in \{1, \ldots, n\}$. By commutativity, the element $p_1^{a_{11}} \ldots p_i^{a_{ii}-1} \ldots p_n^{a_{nn}}$ is the right and left identity element of the semigroup $\mathfrak{M}$. It is known that if a semigroup has right and left identity elements, they are equal. It follows that

$$p_1^{a_{11}-1} p_2^{a_{12}} \ldots p_n^{a_{1n}} = p_1^{a_{21}} p_2^{a_{22}-1} \ldots p_n^{a_{2n}} = \ldots = p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}-1} = e. \quad (5)$$

Thus, we have proved that $\mathfrak{M}$ is a monoid.

Let us now show that for each element $g \in M$ there exists an element $g' \in M$ such that $gg' = g'g = e$. We find g' in the form $g' = p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n}$. Since (5), $g'g = e$ is equivalent to

$$g'g =$$
$$= (p_1^{a_{11}-1} p_2^{a_{12}} \ldots p_n^{a_{1n}})^{x_1} (p_1^{a_{21}} p_2^{a_{22}-1} \ldots p_n^{a_{2n}})^{x_2} \ldots (p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}-1})^{x_n} = e.$$

Here $x_1,\ x_2, \ldots, x_n$ are some natural numbers. Then: $g'g =$

$$= (p_1^{a_{11}-1} p_2^{a_{12}} \ldots p_n^{a_{1n}})^{x_1} (p_1^{a_{21}} p_2^{a_{22}-1} \ldots p_n^{a_{2n}})^{x_2} \ldots (p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}-1})^{x_n} \quad \Leftrightarrow$$
$$(p_1^{r_1} \ldots p_n^{r_n})(p_1^{s_1} \ldots p_n^{s_n}) =$$
$$= (p_1^{a_{11}-1} p_2^{a_{12}} \ldots p_n^{a_{1n}})^{x_1} (p_1^{a_{21}} p_2^{a_{22}-1} \ldots p_n^{a_{2n}})^{x_2} \ldots (p_1^{a_{n1}} p_2^{a_{n2}} \ldots p_n^{a_{nn}-1})^{x_n}$$
$$\Leftrightarrow \quad p_1^{r_1+s_1} p_2^{r_2+s_2} \ldots p_n^{r_n+s_n} =$$

$$p_1^{(a_{11}-1)x_1+a_{21}x_2+\ldots a_{n1}x_n} p_2^{a_{12}x_1+(a_{22}-1)x_2+\ldots a_{n2}x_n} \ldots p_n^{a_{1n}x_1+a_{2n}x_2+\ldots(a_{nn}-1)x_n}. \tag{6}$$

It can be seen that if

$$\begin{cases} r_1+s_1=(a_{11}-1)x_1+a_{21}x_2+\ldots a_{n1}x_n \\ r_2+s_2=a_{12}x_1+(a_{22}-1)x_2+\ldots a_{n2}x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ r_n+s_n=a_{1n}x_1+a_{2n}x_2+\ldots(a_{nn}-1)x_n \end{cases},$$

then $gg'=g'g=e$. Accordingly, since

$$\begin{cases} (a_{11}-1)x_1+a_{21}x_2+\ldots a_{n1}x_n-s_1=r_1 \\ a_{12}x_1+(a_{22}-1)x_2+\ldots a_{n2}x_n-s_2=r_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}x_1+a_{2n}x_2+\ldots(a_{nn}-1)x_n-s_n=r_n \end{cases}, \tag{7}$$

we can choose non-negative integers $x_1, x_2, \ldots, x_n$ such that $r_1, r_2, \ldots, r_n$ are non-negative integers. So, we have shown that for each element $g = p_1^{s_1}p_2^{s_2}\ldots p_n^{s_n} \in M$ there is an element $g' = p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n} \in M$ such that $gg'=g'g=e$.

Thus, we have proved that in this case $\mathfrak{M}$ is a finitely generated Abelian group. □

It can be seen that using the equations (4) it is possible to create different views for the element $g = p_1^{s_1}p_2^{s_2}\ldots p_n^{s_n} \in M$. For example, let $P=\{p_1,p_2\}$, $p_1 = p_1^a p_2^b, p_2 = p_1^c p_2^d$ and $g = p_1^r p_2^s$. In that case, g can be written as follows and described in another form:

$$g = p_1^r p_2^s = (p_1^a p_2^b)^r (p_1^c p_2^d)^s = p_1^{ar+cs} p_2^{br+ds}.$$

Thus, it can be seen that there are infinitely many different representations for each element g .

Theorem 5. *Let $\mathfrak{M} = \langle M, * \rangle$ be a finitely generated complete Archimedean semigroup and generated by $P = \{p_1, \ldots, p_n\}$, the elements of the set P be related to each other by the relations (4) and $g = p_1^{s_1}p_2^{s_2}\ldots p_n^{s_n}$, $h = p_1^{t_1}p_2^{t_2}\ldots p_n^{t_n}$. The system of equations*

$$\begin{cases} (a_{11}-1)x_1+a_{21}x_2+\ldots a_{n1}x_n=s_1-t_1 \\ a_{12}x_1+(a_{22}-1)x_2+\ldots a_{n2}x_n=s_2-t_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}x_1+a_{2n}x_2+\ldots(a_{nn}-1)x_n=s_n-t_n \end{cases} \tag{8}$$

has integer solutions if and only if $g = h$.

Proof. Let the system of equations (8) have an integer solution $(x_1 \ldots, x_n)$. By Theorem 4 $\mathfrak{M}$ is a finitely generated Abelian group. Then, if we find

$-t_1, -t_2, \ldots, -t_n$ through (8) as follows

$$\begin{cases} (a_{11}-1)x_1 + a_{21}x_2 + \cdots + a_{n1}x_n - s_1 = -t_1 \\ a_{12}x_1 + (a_{22}-1)x_2 + \cdots + a_{n2}x_n - s_2 = -t_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + (a_{nn}-1)x_n - s_n = -t_n \end{cases},$$

then according to (7) it follows that $g\cdot(p_1^{-t_1}p_2^{-t_2}\ldots p_n^{-t_n}) = e$. Hence, $gh^{-1} = e$, i.e. $g = h$.

Now, since $g = h$, let us prove that the system of equations (8) has an integer solution. We consider the set

$$\Gamma_g = \{(m_1, \ldots, m_n) \mid p_1^{m_1}\ldots p_n^{m_n} = g\}$$

for g. It can be seen that $(t_1, \ldots, t_n) \in \Gamma_g$. We assume that all representations of g are formed as above only using (4). As a result, the question arises of how another representation of g can be obtained. To do this, by selecting some integers $u_1, \ldots, u_n$, we can express g as follows:

$$g = p_1^{s_1}p_2^{s_2}\ldots p_n^{s_n} = (p_1^{u_1}\ldots p_n^{u_n})(p_1^{s_1-u_1}\ldots p_n^{s_n-u_n}) =$$
$$= (p_1^{a_{11}}\ldots p_n^{a_{1n}})^{u_1}\ldots(p_1^{a_{n1}}\ldots p_n^{a_{nn}})^{u_n}(p_1^{s_1-u_1}\ldots p_n^{s_n-u_n}) =$$
$$= p_1^{(a_{11}-1)u_1+a_{21}u_2+\cdots+a_{n1}u_n+s_1}\ldots p_n^{a_{1n}u_1+a_{2n}u_2+\cdots+(a_{nn}-1)u_n+s_n}.$$

Let us define $\alpha_i = (a_{1i}-1)u_1 + a_{2i}u_2 + \cdots + a_{ni}u_n + s_i$. Through this, we obtain the form $g = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_n^{\alpha_n}$ for g. Thus, to find all representations of g, it is necessary to determine all possible $(\alpha_1, \ldots, \alpha_n)$ that arise for all $(u_1, \ldots, u_n) \in \mathbb{Z}^n$. Thus, if we consider the following set

$$\Delta_g^1 = \{(\alpha_1, \ldots, \alpha_n) \mid (u_1, \ldots, u_n) \in \mathbb{Z}^n\},$$

we can observe that $\Gamma_g = \Delta_g^1$. It can be seen that after one step from the form $g = p_1^{s_1}p_2^{s_2}\ldots p_n^{s_n}$, the set Δ_1 that can be obtained is equal to the set Γ_g. As a result, the question arises whether a set different from Γ_g can be obtained in the next steps. Let us examine the next step:

Let $(\alpha_1, \ldots, \alpha_n) \in \Delta_g^1$. Then we do as above:

$$g = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_n^{\alpha_n} = (p_1^{v_1}\ldots p_n^{v_n})(p_1^{\alpha_1-v_1}\ldots p_n^{\alpha_n-v_n}) =$$
$$= (p_1^{a_{11}}\ldots p_n^{a_{1n}})^{v_1}\ldots(p_1^{a_{n1}}\ldots p_n^{a_{nn}})^{v_n}(p_1^{\alpha_1-v_1}\ldots p_n^{\alpha_n-v_n}) =$$
$$= p_1^{(a_{11}-1)v_1+a_{21}v_2+\cdots+a_{n1}v_n+\alpha_1}\ldots p_n^{a_{1n}v_1+a_{2n}v_2+\cdots+(a_{nn}-1)v_n+\alpha_n},$$

where $v_j \in \mathbb{Z},\ j \in \{1, \ldots, n\}$.

Let us define $\beta_i = (a_{1i}-1)v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n + \alpha_i$ and

$$\Delta_g^2 = \{(\beta_1, \ldots, \beta_n) \mid v_i \in \mathbb{Z},\ (\alpha_1, \ldots, \alpha_n) \in \Gamma_\alpha,\ i \in \{1, \ldots, n\}\}.$$

Then

$$\beta_i = (a_{1i}-1)v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n + \alpha_i =$$
$$= (a_{1i}-1)v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n + (a_{1i}-1)u_1 + a_{2i}u_2 + \cdots + a_{ni}u_n + s_i =$$
$$(a_{1i}-1)(u_1+v_1) + a_{2i}(u_2+v_2) + \cdots + a_{ni}(u_n+v_n) + s_i =$$

$$(a_{1i}-1)u'_1 + a_{2i}u'_2 + \cdots + a_{ni}u'_n + s_i,$$

where $u'_j = u_j + v_j,\ j \in \{1, \ldots, n\}$. Thus, it can be seen that

$$\Delta_g^2 = \left\{(\beta_1, \ldots, \beta_n) \mid (u'_1, \ldots, u'_n) \in \mathbb{Z}^n\right\} = \Delta_g^1 = \Gamma_g.$$

Therefore, by induction, it can be observed that

$$\Gamma_g = \Delta_g^1 = \Delta_g^2 = \cdots = \Delta_g^k = \ldots$$

holds for all steps.

Since $g = h$, it follows that $(t_1, \ldots, t_n) \in \Delta_g^k$. Thus, there exist integers $y_1, y_2, \ldots, y_n$ such that $t_i = (a_{1i}-1)y_1 + a_{2i}y_2 + \cdots + a_{ni}y_n + s_i$. This, in turn, implies that the system of equations (8) has an integer solution $(x_1, \ldots, x_n)$, where $x_i = -y_i$, due to $g = h$. □

It is known that if for each generating element $p_i,\ i \in \{1, \ldots, n\}$ there exists a positive integer $k_i \neq 1$ such that $p_i = p_i^{k_i}$, then the commutative semigroup $\mathfrak{M}$ is a finite semigroup. If in this case $\mathfrak{M}$ is a group, and for each p_i, there exists an integer $k_i \neq 1$ such that $p_i = p_i^{k_i}$, then $\mathfrak{M}$ is a finite group.

Theorem 6. *Let $\mathfrak{M} = \langle M, * \rangle$ be a finitely generated complete archimedean semigroup, $P = \{p_1, \ldots, p_n\}$ and M be generated by P, the elements of the set P be related to each other by the relations (4). Then*

a) If $\det(A - I) \neq 0$, then $\mathfrak{M}$ is a finite abelian group, where $A = (a_{ij})$ and I is the n-dimensional identity matrix.

b) If $\det(A - I) = 0$, then $\mathfrak{M}$ is an infinite abelian group where $A = (a_{ij})$ and I is the n-dimensional identity matrix.

Proof. a) By Theorem 4 it is clear that $\mathfrak{M}$ is a finitely generated abelian group. Let us now prove that the group $\mathfrak{M}$ is finite. Let us show that for each generating element p_i, there exists an integer $k_i \neq 1$ such that $p_i = p_i^{k_i}$. Let $g = p_i$ and $h = p_i^{k_i}$. Then the equations (8) for g and h look like this:

$$\begin{cases} (a_{11}-1)x_1 + a_{21}x_2 + \ldots a_{n1}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1i}x_1 + \cdots + (a_{ii}-1)x_i + \cdots + a_{ni}x_n = 1 - k_i \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}x_1 + a_{2n}x_2 + \ldots (a_{nn}-1)x_n = 0 \end{cases} . \tag{9}$$

If $\det(A - I) \neq 0$, then the system of equations (9) has a unique solution $(x_1, \ldots, x_i, \ldots, x_n)^T$. Using Cramer's formula, we find each $x_i,\ i \in \{1, \ldots, n\}$ as follows:

$$x_i = \frac{\det(D_i)}{\det(A-I)}.$$

Here the matrix D_i is formed by replacing the i-th column of the matrix $(A - I)$ with the column $(0, \ldots, 1 - k_i, \ldots, 0)^T$. Then we can find x_i as follows:

$$x_i = \frac{(1-k_i)\det(\overline{D}_i)}{\det(A-I)}, \tag{10}$$

where $\overline{D}_i$ is the matrix formed by replacing the i-th column of the matrix D_i with the column $(0, \ldots, 1, \ldots, 0)^T$. Since the elements of the matrices $(A - I)$ and D_i are integers, $\det(A - I)$ and $\det(\overline{D}_i)$ are also integers. Then, if we take $1 - k_i = \det(A - I)$, according to (10), we get an integer x_i. Hence, by theorem 5 the integer $k_i = 1 - \det(A - I)$ satisfies the equality $p_i = p_i^{k_i}$ for each $p_i \in P$, and this means that $\mathfrak{M}$ is a finite abelian group.

b) If $\det(A - I) = 0$, then it is clear that in order for (9) to have a solution, $k_i = 1$. So, by Theorem 5, for each p_i the number k_i satisfying the equality $p_i = p_i^{k_i}$ is only $k_i = 1$. Thus, in this case $\mathfrak{M}$ is an infinite abelian group. □

Corollary 6. *Let $\mathfrak{M} = \langle M, * \rangle$ be a finitely generated complete archimedean semigroup, $P = \{p_1, \ldots, p_n\}$ and M be generated by P, the elements of the set P be related to each other by the relations (4) and $\det(A - I) \neq 0$, where $A = (a_{ij})$. Then $(\mathfrak{M}, \mathrm{acl})$ is a pregeometry.*

Proof. By Theorem 6 $\mathfrak{M} = \langle M, * \rangle$ is a finite abelian group. It follows from this that $\mathrm{acl}(A) = M$ for each $A \in \mathcal{P}(M)$. So, the operator $\mathrm{acl}(A)$ satisfies all pregeometry conditions. Hence, $(\mathfrak{M}, \mathtt{acl})$ is a pregeometry. □

It is not difficult to see that if $\mathfrak{M} = \langle M, \Sigma \rangle$ is an Σ-structure and M is a finite set, then $(\mathfrak{M}, \mathrm{acl})$ is a modular pregeometry and $\dim(X) = 0$ for every $X \in \mathcal{P}(M)$. Thus, if $\det(A - I) = 0$, the question whether finitely generated complete archimedean semigroup as above form a pregeometry with dcl or acl operators remains open for now.

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