S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 22, No. 1, pp. 853–863 (2025) УДК 512.542
https://doi.org/10.33048/semi.2025.22.053 MSC 20D10

ON THE RECOGNITION OF FINITE SOLVABLE GROUPS WITH $\mathfrak{F}$-SUBNORMAL PROJECTORS

V.I. MURASHKA

Communicated by I.B. Gorshkov

Abstract: For saturated formations $\mathfrak{X} \subseteq \mathfrak{F}$ where $\mathfrak{F}$ is hereditary it is proved that the class $\mathfrak{X}_{sn_{\mathfrak{F}}}$ of groups all of whose $\mathfrak{X}$-projectors are $\mathfrak{F}$-subnormal is a saturated formation and its local definitions are found. The methods for calculating the $\mathfrak{X}_{sn_{\mathfrak{F}}}$-residual of a finite solvable group are discussed in the paper.

Keywords: Finite group; solvable group; local formation; $\mathfrak{F}$-residual; $\mathfrak{F}$-projector; $\mathfrak{F}$-subnormal subgroup.

1 Introduction and the Main Result

All the groups under consideration are finite. The classical method for studying the structure of a group is to consider the properties of its subgroup chains. For example, a group is nilpotent if every its cyclic primary (or Sylow) subgroup is subnormal. These results have been significantly developed within the framework of the theory of formations. The structure of groups with different systems of formational subnormal subgroups was studied, for example, in [1, 2, 3, 4, 5, 6, 7].

Murashka, V.I., On the Recognition of Finite solvable Groups with $\mathfrak{F}$-Subnormal Projectors.

The work was carried out with the financial support of a grant from the Ministry of Education for doctoral students, postgraduate students, and students (№ state reg. 20240567).

Received November, 5, 2024, Published July, 25, 2025.

Definition 1 ([8, Definition 6.1.2], [9, 1.2.8])**.** *Let $\mathfrak{F}$ be a class of groups. A subgroup H of G is called $\mathfrak{F}$-subnormal in G, if $H = G$ or there exists a maximal chain of subgroups $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ such that $H_i/\mathrm{Core}_{H_i}(H_{i-1}) \in \mathfrak{F}$ for $i = 1, \ldots, n$. Denoted by $H\mathfrak{F}$-$sn\,G$.*

In [10, Definition 1.2] the author proposed the following construction: *Let $\mathfrak{F}$ be a class of groups and $\mathfrak{X}$ be a saturated homomorph. Denote by $f_{\mathfrak{X}}(\mathfrak{F})$ the class of groups in which all $\mathfrak{X}$-subgroups are $\mathfrak{F}$-subnormal.* The constructions of the works [1, 2, 3, 4, 5] are particular cases of it with the right choice of $\mathfrak{X}$.

Note that in an arbitrary case it is difficult to find all $\mathfrak{X}$-subgroups of a group from the computational point of view. That is why in the theory of classes of groups the systems of special $\mathfrak{X}$-subgroups in each group are considered. In the case when $\mathfrak{X}$ is a saturated formation, all $\mathfrak{X}$-projectors of the group G can be taken as such system. For example, the set of all Sylow p-subgroups of a group coincides with the set of all its $\mathfrak{N}_p$-projectors.

Let $\mathfrak{X}$ be a class of groups. Recall [12, III, Definition 3.1] that a subgroup U of a group G is called $\mathfrak{X}$-*maximal* in G provided that (a) $U \in \mathfrak{X}$, and (b) if $U \leq V \leq G$ and $V \in \mathfrak{X}$, then $U = V$; a subgroup H of a group G is called $\mathfrak{X}$-*projector* [12, III, Definition 3.2] if HN/N is $\mathfrak{X}$-maximal in G/N for every $N \trianglelefteq G$. According to [12, III, Theorem 3.10] $\mathfrak{X}$-projectors exist in every group in the case when $\mathfrak{X}$ is a non-empty saturated formation.

In [11] the authors studied the question: how the structure of a group depends on the embedding of its $\mathfrak{X}$-projectors? In particular for the non-empty classes of groups $\mathfrak{X}$ and $\mathfrak{F}$ they denoted [11, Definition 2.5] by $\mathfrak{X}_{sn_{\mathfrak{F}}}$ *the class of groups all of whose $\mathfrak{X}$-projectors are $\mathfrak{F}$-subnormal.*

In the case when $\mathfrak{X}$ is a saturated formation and $\mathfrak{F}$ is a hereditary formation it was only proven that $\mathfrak{X}_{sn_{\mathfrak{F}}}$ is a formation [11, Theorem 2.14(2)]. At first we establish the connection between $\mathfrak{X}_{sn_{\mathfrak{F}}}$ and $f_{\mathfrak{X}}(\mathfrak{F})$. Recall that $\mathfrak{X}^S$ denotes the class of groups all of whose subgroups belong to $\mathfrak{X}$ for a class of groups $\mathfrak{X}$.

Theorem 1. *Let $\emptyset \neq \mathfrak{X}$ be a saturated formation and $\mathfrak{F}$ be a hereditary formation. If $\mathfrak{X} \subseteq \mathfrak{F}$ or $\mathfrak{F}$ is saturated, then $\mathfrak{X}^S_{sn_{\mathfrak{F}}} = f_{\mathfrak{X}}(\mathfrak{F})$.*

The aim of this paper is to describe the structure of $\mathfrak{X}_{sn_{\mathfrak{F}}}$ and to suggest the algorithms for computing the $\mathfrak{X}_{sn_{\mathfrak{F}}}$-residual of a (solvable) group, in particular for testing whether a group is an $\mathfrak{X}_{sn_{\mathfrak{F}}}$-group. At first we are interested in conditions for $\mathfrak{X}_{sn_{\mathfrak{F}}}$ to be saturated (local) and in its canonical local definition.

Theorem 2. *Let $\emptyset \neq \mathfrak{X} \subseteq \mathfrak{F}$ be saturated formations and F be the canonical local definition of $\mathfrak{F}$. If $\mathfrak{F}$ is hereditary, then $\mathfrak{X}_{sn_{\mathfrak{F}}}$ is a local (saturated) formation locally defined by X where $X(p)$ is a class of all $\mathfrak{X}_{sn_{\mathfrak{F}}}$-groups all of whose $\mathfrak{X}$-projectors belong to $F(p)$ for all prime p.*

Proposition 1. *If $\mathfrak{X} \supseteq \mathfrak{F}$ are saturated formations, then $\mathfrak{X}_{sn_{\mathfrak{F}}} = \mathfrak{X}$.*

Note that from Theorem 2 the key results of [4] follow.

Corollary 1 ([4, Theorems 3.4 and 3.6])**.** *Let π be a set of primes and $\mathfrak{F}$ be a hereditary saturated formation. Then the class $W_\pi\mathfrak{F}$ of all groups G, all of whose Sylow p-subgroups for every $p \in \pi \cap \pi(G)$ and the unit subgroup are $\mathfrak{F}$-subnormal, is a hereditary saturated formation. If F and H are the canonical local definitions of $\mathfrak{F}$ and $W_\pi\mathfrak{F}$ respectively, then $H(p)$ is the class of all $W_\pi\mathfrak{F}$-groups all of whose Sylow π-subgroups belong to $F(p)$ for all $p \in \pi(\mathfrak{F})$ and $H(p) = \emptyset$ otherwise.*

The main idea of a local formation is to reduce the verification that a group has a given property associated with it to verifying that the group has certain properties associated with the values of its local definition (simpler formations). Therefore the presented in Theorem 2 local definition of $\mathfrak{X}_{sn_\mathfrak{F}}$ is not good from the computational point of view because it requires to check that a group belongs to $\mathfrak{X}_{sn_\mathfrak{F}}$ for every its value. We obtained a simpler local definition of $\mathfrak{X}_{sn_\mathfrak{F}}$ in the solvable case.

Theorem 3. *Let $\emptyset \neq \mathfrak{X} \subseteq \mathfrak{F}$ be saturated formations of solvable groups, $\mathfrak{F}$ be hereditary and F be the canonical local definition of $\mathfrak{F}$. Then $\mathfrak{X}_{sn_\mathfrak{F}}$ is locally defined by X, where $X(p)$ is a class of all solvable groups all of whose $\mathfrak{X}$-projectors belong to $F(p)$ for all prime p.*

Recall that the $\mathfrak{F}$-*residual* of a group G is the smallest normal subgroup $G^\mathfrak{F}$ of G with $G/G^\mathfrak{F} \in \mathfrak{F}$. If $\mathfrak{F}$ is a local formation and f is its local definition, then from [12, IV, Theorem 3.2(b)] the formula for the $\mathfrak{F}$-residual follows:

$$G^\mathfrak{F} = \mathrm{O}^{\pi(G)\cap\pi(\mathfrak{F})}(G) \prod_{p\in\pi(G)\cap\pi(\mathfrak{F})} \mathrm{O}^{p'}(\mathrm{O}^p(G^{f(p)}))$$

This formula was used to compute the $\mathfrak{F}$-residual of a solvable group in [13]. If a test for a group to belong to $\mathfrak{F}$ is known, then the method for computing the $\mathfrak{F}$-residual was suggested in [14]. For the formations of groups $\mathfrak{F}$ defined by the action of a group on its chief series the polynomial time algorithms for the computation of the $\mathfrak{F}$-residual of a permutation group were suggested in [15]. Note that in the case $\mathfrak{F} \cap \mathfrak{X} \notin \{\mathfrak{F}, \mathfrak{X}\}$ we cannot use any of these methods to compute the $\mathfrak{X}_{sn_\mathfrak{F}}$-residual of a group.

Definition 2. *Let H be a subgroup of a group G, $\mathfrak{F}$ be a hereditary formation and a chain of subgroups $G = G_0 \geq G_1 \geq G_2 \geq \dots$ be defined by $G_i = HG_{i-1}^\mathfrak{F}$. We will denote the final term of this chain by $g(H, G, \mathfrak{F})$.*

Remark 1. *Note that H is $\mathfrak{F}$-subnormal in G if and only if $H = g(H, G, \mathfrak{F})$.*

Recall [12, III, Definition 3.5] that a subgroup H of a group G is called $\mathfrak{X}$-*covering* for a class of groups $\mathfrak{X}$ if H is an $\mathfrak{X}$-projector of V for every $H \leq V \leq G$. For example a Sylow p-subgroup is an $\mathfrak{N}_p$-covering subgroup. It is known [12, III, Theorem 3.21] that for a saturated formation $\mathfrak{X}$ and a solvable group G the sets of its $\mathfrak{X}$-projectors and $\mathfrak{X}$-covering subgroups coincide. Our second approach in the computation of the $\mathfrak{X}_{sn_\mathfrak{F}}$-residual is based on the following result.

Theorem 4. *Let $\emptyset \neq \mathfrak{X}$ be a saturated formation and $\mathfrak{F}$ be a hereditary formation. Assume that in a group G every $\mathfrak{X}$-projector is an $\mathfrak{X}$-covering subgroup. Then the $\mathfrak{X}_{sn_{\mathfrak{F}}}$-residual of G is $\langle g(H, G, \mathfrak{F})^{\mathfrak{X}} \mid H$ is an $\mathfrak{X}$-projector of G with $g(H, G, \mathfrak{F}) \neq H\rangle$.*

In Section 4 we present the algorithms for the computation of the $\mathfrak{X}_{sn_{\mathfrak{F}}}$-residual of a solvable group based on Theorems 3 and 4 and compare them to a known ones in case when they are applicable.

2 Preliminaries

We use standard definitions and notations (see [8, 12]). Recall some of them: $\mathbb{P}$ is the set of all primes; $\pi(G)$ is the set of all prime divisors of $|G|$; $\pi(\mathfrak{X}) = \underset{G\in\mathfrak{X}}{\cup}\pi(G)$; $\Phi(G)$ is the Frattini subgroup of G; $\mathrm{O}^{\pi}(G)$ is the smallest normal subgroup of G of π-index; $\mathfrak{N}_p$ is the class of all p-groups.

Recall that a *formation* is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images (i.e. from $G \in \mathfrak{F}$ and $N \trianglelefteq G$ it follows that $G/N \in \mathfrak{F}$) and subdirect products (i.e. from $G/N_1 \in \mathfrak{F}$ and $G/N_2 \in \mathfrak{F}$ it follows that $G/(N_1 \cap N_2) \in \mathfrak{F}$). For a formation $\mathfrak{F}$ let $E_{\Phi}\mathfrak{F} = (G \mid G/\Phi(G) \in \mathfrak{F})$. A formation $\mathfrak{F}$ is said to be: *saturated* if $\mathfrak{F} = E_{\Phi}\mathfrak{F}$; *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$.

A function of the form $f : \mathbb{P} \to \{formations\}$ is called a *formation function*. Recall [12, IV, Definition 3.1] that a formation $\mathfrak{F}$ is called *local* if

$$\mathfrak{F} = LF(f) = (G \,|\, G/C_G(H/K) \in f(p) \text{ for every } p \in \pi(H/K) \text{ and chief factor } H/K \text{ of } G)$$

for some formation function f. In this case f is called a *local definition* of $\mathfrak{F}$. By the Gaschütz–Lubeseder–Schmid theorem [12, IV, Theorem 4.6], a non-empty formation is local if and only if it is saturated. Recall [12, IV, Proposition 3.8] that if $\mathfrak{F}$ is a local formation, there exists a unique formation function F, defining $\mathfrak{F}$, such that $F(p) = \mathfrak{N}_pF(p) \subseteq \mathfrak{F}$ for every $p \in \mathbb{P}$. In this case F is called the *canonical local definition* of $\mathfrak{F}$.

The basic properties of $\mathfrak{F}$-subnormality are contained in the following two lemmas.

Lemma 1 ([8, Lemma 6.1.6])**.** *Let $\mathfrak{F}$ be a formation, $H, R \leq G$ and $N \trianglelefteq G$.*

(1) *If H $\mathfrak{F}$-sn G, then HN/N $\mathfrak{F}$-sn G/N.*
(2) *If $N \leq H$ and H/N $\mathfrak{F}$-sn G/N, then H $\mathfrak{F}$-sn G.*
(3) *If H $\mathfrak{F}$-sn R and R $\mathfrak{F}$-sn G, then H $\mathfrak{F}$-sn G.*

Lemma 2 ([8, Lemma 6.1.7])**.** *Let $\mathfrak{F}$ be a hereditary formation, G be a group and $H, R \leq G$.*

(1) *If H $\mathfrak{F}$-sn G, then $H \cap R$ $\mathfrak{F}$-sn R.*
(2) *If H $\mathfrak{F}$-sn G and R $\mathfrak{F}$-sn G, then $H \cap R$ $\mathfrak{F}$-sn G.*

Recall that the generalization $\tilde{\mathrm{F}}(G)$ of the Fitting subgroup is defined by $\Phi(G) \subseteq \tilde{\mathrm{F}}(G)$ and $\tilde{\mathrm{F}}(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$.

Lemma 3 ([3, Lemma 2.7])**.** *Let $\mathfrak{F}$ be a saturated formation, G be a group and H be an $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroup of G. If $G = H\tilde{\mathrm{F}}(G)$, then $G \in \mathfrak{F}$.*

3 Proofs of the Main Results

Recall that H/K is called a *primitive section* of a group G if $K \trianglelefteq H \leq G$ and H/K is a primitive group, i.e. a group with a core-free maximal subgroup [12, A, Definition 15.1].

3.1. Proof of Theorem 1. Since any $\mathfrak{X}$-projector is an $\mathfrak{X}$-subgroup, we see that $f_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$. Note that in the assumptions of the theorem every primitive section of an $E_\Phi\mathfrak{F} \cap \mathfrak{X}$-group belongs to $\mathfrak{F}$. Therefore $f_{\mathfrak{X}}(\mathfrak{F})$ is a hereditary formation by [10, Theorem 3.1]. Note that $\mathfrak{X}^S_{sn_{\mathfrak{F}}}$ is the greatest by inclusion hereditary subformation of $\mathfrak{X}_{sn_{\mathfrak{F}}}$ by its definition and [16, Lemma 25.4]. Hence $f_{\mathfrak{X}}(\mathfrak{F}) \subseteq \mathfrak{X}^S_{sn_{\mathfrak{F}}}$.

Assume that $f_{\mathfrak{X}}(\mathfrak{F}) \subset \mathfrak{X}^S_{sn_{\mathfrak{F}}}$. Let G be a minimal order group from $\mathfrak{X}^S_{sn_{\mathfrak{F}}} \setminus f_{\mathfrak{X}}(\mathfrak{F})$. It means that G has a non-$\mathfrak{F}$-subnormal $\mathfrak{X}$-subgroup H.

Let N be a minimal normal subgroup of G. Then $G/N \in \mathfrak{X}^S_{sn_{\mathfrak{F}}}$. Hence $G/N \in f_{\mathfrak{X}}(\mathfrak{F})$ by our assumption. If G has two different minimal normal subgroups N_1 and N_2, then from $G/N_i \in f_{\mathfrak{X}}(\mathfrak{F})$ it follows that $G \simeq G/(N_1 \cap N_2) \in f_{\mathfrak{X}}(\mathfrak{F})$, a contradiction. Thus N is the unique minimal normal subgroup of G.

If $G^{\mathfrak{F}} \simeq 1$, then $G \in \mathfrak{F}$. Since $\mathfrak{F}$ is a hereditary formation, every subgroup of G is $\mathfrak{F}$-subnormal in G. So $G \in f_{\mathfrak{X}}(\mathfrak{F})$, a contradiction. Hence $G^{\mathfrak{F}} \not\simeq 1$ and $N \leq G^{\mathfrak{F}}$. Let P be an $\mathfrak{X}$-projector of G. From $P\,\mathfrak{F}$-sn G it follows that $PG^{\mathfrak{F}} < G$. Hence $PN < G$. In particular, $G/N \not\in \mathfrak{X}$ by the definition of $\mathfrak{X}$-projector.

Assume that $HN < G$. Then $HN \in f_{\mathfrak{X}}(\mathfrak{F})$. It means that $H\,\mathfrak{F}$-sn HN. From $G/N \in f_{\mathfrak{X}}(\mathfrak{F})$ it follows that $HN/N\,\mathfrak{F}$-sn G/N. Hence $HN\,\mathfrak{F}$-sn G and therefore $H\,\mathfrak{F}$-sn G by Lemma 1, a contradiction. Thus $HN = G$. Hence $G/N \simeq H/(H\cap N) \in \mathfrak{X}$, the final contradiction. It means that $f_{\mathfrak{X}}(\mathfrak{F}) = \mathfrak{X}^S_{sn_{\mathfrak{F}}}$.

3.2. Proof of Theorem 2. According to [11, Theorem 2.14] $\mathfrak{X}_{sn_{\mathfrak{F}}}$ is a formation. Assume that $\mathfrak{X}_{sn_{\mathfrak{F}}}$ is nonsaturated. Let G be a minimal order group from $E_\Phi\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}}$. Recall [12, A, Theorem 9.2(e)] that $\Phi(G)N/N \leq \Phi(G/N)$ for any normal subgroup N of G. It means that $(G/N)/\Phi(G/N)$ is a quotient group of $G/\Phi(G) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$. Hence $(G/N)/\Phi(G/N) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$. If N is non-trivial, then from $|G/N| < |G|$ and our assumption it follows that $G/N \in \mathfrak{X}_{sn_{\mathfrak{F}}}$.

If N_1 and N_2 are different minimal normal subgroups of G, then from $G/N_1, G/N_2 \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ it follows that $G \simeq G/(N_1\cap N_2) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$, a contradiction. Thus G has the unique minimal normal subgroup N. It is clear that $\Phi(G) \neq 1$. Hence $N \leq \Phi(G)$. It means that N is a p-group for some prime p. Therefore $\Phi(G)$ is a p-group and $\mathrm{O}_{p'}(G) = 1$.

Let H be an $\mathfrak{X}$-projector of G. Note that $H\Phi(G)/\Phi(G)$ is an $\mathfrak{X}$-projector of $G/\Phi(G)$. From $G/\Phi(G) \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ it follows that $H\Phi(G)/\Phi(G)\,\mathfrak{F}$-sn $G/\Phi(G)$. From (2) of Lemma 1 it follows that $H\Phi(G)\,\mathfrak{F}$-sn G.

Let $K = H\tilde{\mathrm{F}}(G)$. Since $\tilde{\mathrm{F}}(G)/\Phi(G)$ is a direct product of simple groups by definition, it is a normal quasinilpotent subgroup of $K/\Phi(G)$. Therefore $\tilde{\mathrm{F}}(G)/\Phi(G) \leq \mathrm{F}^*(K/\Phi(G))$. Hence $\tilde{\mathrm{F}}(G)/\Phi(G) \leq \tilde{\mathrm{F}}(K/\Phi(G))$ by [3, Lemma 1.1(4)]. Now $K/\Phi(G) = (H\Phi(G)/\Phi(G))\tilde{\mathrm{F}}(K/\Phi(G))$. Since $\mathfrak{F}$ is hereditary and $H\Phi(G)/\Phi(G)\,\mathfrak{F}$-sn $G/\Phi(G)$, we see that $H\Phi(G)/\Phi(G)\,\mathfrak{F}$-sn $K/\Phi(G)$ by Lemma 2. From $\mathfrak{X} \subseteq \mathfrak{F}$ it follows that $H\Phi(G)/\Phi(G) \in \mathfrak{F}$. Therefore $K/\Phi(G) \in \mathfrak{F}$ by Lemma 3. It means that K acts F-hypercentrally on $\tilde{\mathrm{F}}(G)/\Phi(G)$. From $\mathrm{O}_{p'}(G) = 1$ and [16, Theorem 9.18] it follows that K acts F-hypercentrally on $\Phi(G)$. Thus $K \in \mathfrak{F}$. Since $\mathfrak{F}$ is hereditary, $H\Phi(G) \in \mathfrak{F}$ and therefore $H\,\mathfrak{F}$-sn $H\Phi(G)$. Thus $H\mathfrak{F}$-sn G by Lemma 1. It means that $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$, a contradiction. Thus $\mathfrak{X}_{sn_{\mathfrak{F}}}$ is a saturated formation.

According to [12, IV, Proposition 1.2] the class of groups all of whose $\mathfrak{X}$-projectors are $F(p)$-subgroups is a formation. Therefore $X(p)$ is the intersection of formations and hence a formation.

Let $\mathfrak{H} = LF(X)$. Assume that $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}} \neq \emptyset$ and G is a minimal order group from $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}}$. Since $\mathfrak{H}$ and $\mathfrak{X}_{sn_{\mathfrak{F}}}$ are saturated formations, we see that $\Phi(G) = 1$ and G has the unique minimal normal subgroup N. If N is non-abelian, then $C_G(N) = 1$ and for every $p \in \pi(N)$ the following holds $G \simeq G/C_G(N) \in X(p) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$, a contradiction. Thus N is abelian. Hence $C_G(N) = N$ and N is a p-group for some prime p. Let H be an $\mathfrak{X}$-projector of G. Note that HN/N is an $\mathfrak{X}$-projector of G/N. It means that $HN/N\,\mathfrak{F}$-sn G/N. Hence $HN\,\mathfrak{F}$-sn G. From $G/N = G/C_G(N) \in X(p)$ it follows that $HN/N \in F(p)$. Now $HN \in \mathfrak{N}_pF(p) = F(p) \subseteq \mathfrak{F}$. Since $\mathfrak{F}$ is hereditary, we see that $H\,\mathfrak{F}$-sn HN. Therefore $H\,\mathfrak{F}$-sn G. It means that $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$, a contradiction. Thus $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$.

Assume that $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H} \neq \emptyset$. Let G be a minimal order group from $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H}$. Since $\mathfrak{H}$ and $\mathfrak{X}_{sn_{\mathfrak{F}}}$ are saturated formations, we see that $\Phi(G) = 1$ and G has the unique minimal normal subgroup N. Let H be an $\mathfrak{X}$-projector of G. From $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ it follows that $H\,\mathfrak{F}$-sn G. Since $\mathfrak{F}$ is hereditary, we see that $H\,\mathfrak{F}$-sn HN. Now from $\mathfrak{X} \subseteq \mathfrak{F}$ and $N \leq \tilde{\mathrm{F}}(HN)$ it follows that $HN \in \mathfrak{F}$ by Lemma 3. Let $p \in \pi(N)$. Since N is the unique minimal normal subgroup of G, we see that $C_G(N) \subseteq N$. Hence $\mathrm{O}_{p'}(HN) = 1$. Now $HN \simeq HN/\mathrm{O}_{p'}(HN) \in F(p)$. Since $\mathfrak{F}$ is hereditary, $F(p)$ is hereditary for every p too by [12, IV, Proposition 3.16]. Thus $H \in F(p)$. It means that G belongs to a formation $X(p)$ for every $p \in \pi(N)$. Therefore $G/C_G(N) \in X(p)$ for every $p \in \pi(N)$. From $G/N \in LF(X)$ it follows that $G \in LF(X) = \mathfrak{H}$, a contradiction. Hence $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$. Thus $\mathfrak{H} = \mathfrak{X}_{sn_{\mathfrak{F}}}$.

Note that $X(p) \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$ by the definition of X. Hence X is an inner definition of $\mathfrak{X}_{sn_{\mathfrak{F}}}$. Let prove that X is a full definition of $\mathfrak{X}_{sn_{\mathfrak{F}}}$. Note that if $F(p) = \emptyset$, then $X(p) = \emptyset$. That is why we may assume that $F(p) \neq \emptyset$. Assume that $G \in \mathfrak{N}_pX(p)$. It means that G has a normal p-subgroup N with

$G/N \in X(p)$. Note that $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$ by [12, IV, Proposition 3.8(a)]. Let H be an $\mathfrak{X}$-projector of G. Then $HN/N \simeq H/(H \cap N)$ is an $\mathfrak{X}$-projector of G/N and $H \cap N$ is a p-group. From $G/N \in X(p)$ it follows that $H/(H \cap N) \in F(p)$. Now $H \in \mathfrak{N}_p F(p) = F(p)$. Thus $G \in X(p)$. So X is a full definition of $\mathfrak{X}_{sn_{\mathfrak{F}}}$. Thus X is the canonical local definition of $\mathfrak{X}_{sn_{\mathfrak{F}}}$.

3.2.1. Proof of Proposition 1. Assume that $\mathfrak{X} \supseteq \mathfrak{F}$ are saturated formations and $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$. Let H be an $\mathfrak{X}$-projector of G. Then $H\,\mathfrak{F}$-sn G. If $H \neq G$, then $HG^{\mathfrak{F}} < G$ by the definition of $\mathfrak{F}$-subnormality. Hence $HG^{\mathfrak{F}}/G^{\mathfrak{F}} < G/G^{\mathfrak{F}} \in \mathfrak{F} \subseteq \mathfrak{X}$, a contradiction with the definition of $\mathfrak{X}$-projector. Thus $H = G \in \mathfrak{X}$. So $\mathfrak{X}_{sn_{\mathfrak{F}}} \subseteq \mathfrak{X}$. On the other hand an $\mathfrak{X}$-group is the unique $\mathfrak{X}$-projector in itself and hence $\mathfrak{F}$-subnormal. It means that $\mathfrak{X} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$. Thus $\mathfrak{X} = \mathfrak{X}_{sn_{\mathfrak{F}}}$.

3.2.2. Proof of Corollary 1. Note that the unit group is the unique (1)-projector and all Sylow p-subgroups are all $\mathfrak{N}_p$-projectors in every group. It means that $\mathfrak{F}_0 = (1)_{sn_{\mathfrak{F}}}$ and $\mathfrak{F}_p = (\mathfrak{N}_p)_{sn_{\mathfrak{F}}}$ are saturated formations by Theorem 2. Then $W_\pi\mathfrak{F} = \cap_{i \in \pi \cup \{0\}} \mathfrak{F}_i$ is a saturated formation. A direct check or [4, Theorem 3.1(5)] can be used to prove that $W_\pi\mathfrak{F}$ is hereditary. Let $F_0(p) = \mathfrak{F}_0$ and $F_q(p)$ be the class of all $\mathfrak{F}_q$-groups all of whose Sylow q-subgroups belong to $F(p)$ for all $p \in \pi(\mathfrak{F})$ and $F_0(p) = F_q(p) = \emptyset$ otherwise. Then F_0 and F_q are the canonical local definitions of $\mathfrak{F}_0$ and $\mathfrak{F}_q$ respectively by Theorem 2. Let $H(p) = \cap_{i \in \pi \cup \{0\}} F_i(p) \subseteq W_\pi\mathfrak{F}$ be an inner local definition of $W_\pi\mathfrak{F}$. From $\mathfrak{N}_p F_i(p) = F_i(p)$ for every $i \in \pi \cup \{0\}$ it follows that $\mathfrak{N}_p H(p) = H(p)$. Thus H is the canonical local definition of $W_\pi\mathfrak{F}$.

3.3. Proof of Theorem 3. From $\emptyset \neq \mathfrak{X} \subseteq \mathfrak{F} \subseteq \mathfrak{S}$ it follows that $1\,\mathfrak{S}$-sn G for every $\mathfrak{X}_{sn_{\mathfrak{F}}}$-group G. Thus $\mathfrak{X}_{sn_{\mathfrak{F}}}$ consists of solvable groups by [3, Lemma 2.6]. Let $X(p)$ be the class of all solvable groups all of whose $\mathfrak{X}$-projectors are $F(p)$-groups. From [12, IV, Proposition 1.2] it follows that $X(p)$ is a formation. Let $\mathfrak{H} = LF(X)$. Then $\mathfrak{H}$ is a saturated formation of solvable groups.

Assume that $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}} \neq \emptyset$. Let G be a minimal order group from $\mathfrak{H} \setminus \mathfrak{X}_{sn_{\mathfrak{F}}}$. Since $\mathfrak{H}$ and $\mathfrak{X}_{sn_{\mathfrak{F}}}$ are saturated formations of solvable groups, we see that $\Phi(G) = 1$, G is solvable and has the unique minimal normal subgroup N. So N is abelian. Hence $C_G(N) = N$ and N is a p-group for some prime p. Let H be an $\mathfrak{X}$-projector of G. Note that HN/N is an $\mathfrak{X}$-projector of G/N. It means that $HN/N\,\mathfrak{F}$-sn G/N. Hence $HN\,\mathfrak{F}$-sn G by Lemma 1. From $G/N = G/C_G(N) \in X(p)$ it follows that $HN/N \in F(p)$. Now $HN \in \mathfrak{N}_p F(p) = F(p) \subseteq \mathfrak{F}$. Since $\mathfrak{F}$ is hereditary, we see that $H\,\mathfrak{F}$-sn HN by Lemma 2. Therefore $H\,\mathfrak{F}$-sn G. It means that $G \in \mathfrak{X}_{sn_{\mathfrak{F}}}$, a contradiction. Thus $\mathfrak{H} \subseteq \mathfrak{X}_{sn_{\mathfrak{F}}}$.

Assume that $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H} \neq \emptyset$. Let G be a minimal order group from $\mathfrak{X}_{sn_{\mathfrak{F}}} \setminus \mathfrak{H}$. Since $\mathfrak{H}$ and $\mathfrak{X}_{sn_{\mathfrak{F}}}$ are saturated formations of solvable groups, we see that $\Phi(G) = 1$, G is solvable and has the unique minimal normal subgroup N. Now N is abelian p-group for some $p \in \pi(G)$ and $C_G(N) = N$. Let H be an $\mathfrak{X}$-projector of G. From $G \in \mathfrak{X}_{sn_\mathfrak{F}}$ it follows that $H\,\mathfrak{F}$-sn G. Since $\mathfrak{F}$ is hereditary, we see that $H\,\mathfrak{F}$-sn HN by Lemma 2. Now from $\mathfrak{X} \subseteq \mathfrak{F}$ and $N \leq \mathrm{F}(HN) = \tilde{\mathrm{F}}(HN)$ it follows that $HN \in \mathfrak{F}$ by Lemma 3. Note that $\mathrm{O}_{p'}(HN) = 1$. Now $HN \simeq HN/\mathrm{O}_{p'}(HN) \in F(p)$. Since $\mathfrak{F}$ is hereditary, $F(p)$ is hereditary for every p too by [12, IV, Proposition 3.16]. Thus $H \in F(p)$. It means that G belongs to a formation $X(p)$. Therefore $G/C_G(N) \in X(p)$. From $G/N \in LF(X)$ it follows that $G \in LF(X) = \mathfrak{H}$, a contradiction. Hence $\mathfrak{H} \subseteq \mathfrak{X}_{sn_\mathfrak{F}}$. Thus $\mathfrak{H} = \mathfrak{X}_{sn_\mathfrak{F}}$.

3.4. Proof of Theorem 4. Assume that

$$D = \langle g(H,G,\mathfrak{F})^{\mathfrak{X}} \mid H \text{ is an } \mathfrak{X}\text{-projector of } G \text{ with } g(H,G,\mathfrak{F}) \neq H\rangle.$$

Note that $(S^{\mathfrak{F}})^x = (S^x)^{\mathfrak{F}}$ for any $x \in G$ and $S \leq G$. Therefore $g(H^x, G, \mathfrak{F}) = g(H,G,\mathfrak{F})^x$. Thus D is a normal subgroup of G.

Let H be an $\mathfrak{X}$-projector of G with $K = g(H,G,\mathfrak{F}) \neq H$. Note that $HK^{\mathfrak{F}} = K$ and $K\,\mathfrak{F}$-sn G. Assume that N is a normal subgroup of G such that $HN/N\,\mathfrak{F}$-sn G/N. Since $\mathfrak{F}$ is hereditary, $HN/N\,\mathfrak{F}$-sn KN/N. Note that $K^{\mathfrak{F}} \subseteq (KN)^{\mathfrak{F}}$ and $(KN/N)^{\mathfrak{F}} = (KN)^{\mathfrak{F}}N/N$. If $HN/N < KN/N$, then by the definition of $\mathfrak{F}$-subnormality

$$KN/N = HK^{\mathfrak{F}}N/N \subseteq (HN/N)(KN/N)^{\mathfrak{F}} < KN/N,$$

a contradiction. Thus $HN/N = KN/N \simeq K/(K \cap N) \in \mathfrak{X}$. Hence $K^{\mathfrak{X}} \leq K \cap N \leq N$. It means that $D \leq G^{\mathfrak{X}_{sn_\mathfrak{F}}}$.

Assume that $D < G^{\mathfrak{X}_{sn_\mathfrak{F}}}$. It means that there is an $\mathfrak{X}$-projector S/D which is not $\mathfrak{F}$-subnormal in G/D. Note that [12, III, Proposition 3.7] there is an $\mathfrak{X}$-projector H of G with $HD/D = S/D$. Let $K = g(H,G,\mathfrak{F})$. If $K = H$, then $H\,\mathfrak{F}$-sn G and hence $S/D = HD/D\,\mathfrak{F}$-sn G/D, a contradiction. So $H < K$. From the definition of D it follows that $K^{\mathfrak{X}} \leq D$. By the assumption of the theorem H is an $\mathfrak{X}$-covering subgroup of G. Therefore H is an $\mathfrak{X}$-projector of K. It means that $HK^{\mathfrak{X}} = K$. Hence $HD/D = KD/D\,\mathfrak{F}$-sn$G/D$, the final contradiction. Thus $D = G^{\mathfrak{X}_{sn_\mathfrak{F}}}$.

4 The computation of the $\mathfrak{X}_{sn_\mathfrak{F}}$-residual of a solvable group

Recall [17] that a subgroup H of a group G is called $\mathbb{P}$-*subnormal* if $H = G$ or there is a chain of subgroups $H = H_0 < H_1 < \cdots < H_n = G$ such that $|H_i : H_{i-1}|$ is a prime for $1 \leq i \leq n$. Groups with various systems of $\mathbb{P}$-subnormal subgroups were studied in [10, 17, 18]. In the universe of all solvable groups the concepts of $\mathbb{P}$-subnormal and $\mathfrak{U}$-subnormal subgroups coincide where $\mathfrak{U}$ denotes the class of all supersolvable groups. Recall that $\mathfrak{N}$-covering subgroup (of a solvable group) is called a Carter subgroup. According to Theorem 2 the class $\mathfrak{N}_{sn_\mathfrak{U}}$ of all solvable groups all of whose Carter subgroups are $\mathbb{P}$-subnormal is a saturated formation. Note that the symmetric group of degree 4 belongs to this class but the alternating group of degree 4 does not belong to it. Therefore $\mathfrak{N}_{sn_\mathfrak{U}}$ is not hereditary.

Let $\mathcal{A}$ denote the class of all solvable groups all of whose Sylow subgroups are abelian. The dihedral group of order 8 shows that $\mathcal{A}$ is a non-saturated formation. So $\mathfrak{U}_{sn_{\mathcal{A}}}$ is a formation but we cannot tell either it is saturated or not.

If not stated the opposite for the rest of the section ***all considered in this section groups are solvable***. It is known [12, III, Theorem 3.21] that in the universe of all solvable groups for a saturated formation $\mathfrak{F}$ in every group there is a single conjugacy class of $\mathfrak{F}$-projectors which coincides with the set of all $\mathfrak{F}$-covering subgroups.

Let $\mathfrak{X} \subseteq \mathfrak{F}$ be saturated formations of solvable groups, $\mathfrak{F}$ be hereditary and F be the canonical local definition of $\mathfrak{F}$. If we are able to compute $\mathfrak{X}$-projectors and the $F(p)$-residual, then we are able to compute the $X(p)$-residual of a group for all prime p where X is the local definition of $\mathfrak{X}_{sn_{\mathfrak{F}}}$ that is described in Theorem 3:

$$G^{X(p)} = (H^{F(p)})^G, H \text{ is an } \mathfrak{X}\text{-projector of } G. \tag{1}$$

From Theorem 4 the algorithm for the computation of the $\mathfrak{X}_{sn_{\mathfrak{F}}}$-residual of a group for a saturated formation $\mathfrak{X}$ and a hereditary formation $\mathfrak{F}$ follows.

Algorithm 1: ResidualXsnF(G, $\mathfrak{X}$, $\mathfrak{F}$)

Result: $G^{\mathfrak{X}_{sn_{\mathfrak{F}}}}$.
Data: G is a solvable group, functions to compute the $\mathfrak{X}$-residual, the $\mathfrak{F}$-residual and an $\mathfrak{X}$-projector.
$G_0 \leftarrow$ generators of an $\mathfrak{X}$-projector of G;
$G_1 \leftarrow$ generators of G;
$G_2 \leftarrow$ generators of $G^{\mathfrak{F}}$;
$L \leftarrow G_0 \cup G_2$;
while $|\langle L\rangle| \neq |\langle G_1\rangle|$ **do**
  $G_1 \leftarrow L$;
  $G_2 \leftarrow$ generators of $\langle L\rangle^{\mathfrak{F}}$;
  $L \leftarrow G_0 \cup G_2$;
end
if $|\langle L\rangle| \neq |\langle G_0\rangle|$ **then**
  return $(\langle L\rangle^{\mathfrak{X}})^G$;
else
  return 1;
end

In the computer system algebra GAP [19] there is two packages which deal with formations, their residuals and projectors: FORMAT [20] and CRISP [21]. We implemented the described above methods to compute the $\mathfrak{X}_{sn_{\mathfrak{F}}}$-residual of a group in GAP.

To help the reader understand the applicability of the obtained algorithms, we present the runtime for some groups. All groups of orders 1296, 1200 and 1120 are taken from the package SmallGroups while The Dark's group, the maximal subgroup MF_{22} of Fischer's group and the wreath product U of the subgroup of upper triangular matrices in $GL(4,7)$ with the cyclic group of order 3 are taken from FORMAT. The timings (in seconds) were obtained using GAP 4.11.0 started with 4 GB of RAM on Intel(R) Core(TM) i5-8250U CPU 1.60GHz 1.80GHz.

Group(s)	$\mathfrak{N}_{sn_{\mathfrak{U}}}$-residual	$\mathfrak{U}_{sn_{\mathcal{A}}}$-residual
All 3609 groups of order $2^4 \cdot 3^4$	17.61	18.89
All 1201 solvable groups of order $2^4 \cdot 3 \cdot 5^2$	5.42	5.11
All 1092 groups of order $2^5 \cdot 5 \cdot 7$	5.10	5.03
The Dark's group of order $2^3 \cdot 3^9 \cdot 5^{24} \cdot 7 \cdot 31^8$	0.19	0.31
MF_{22} of order $2^8 \cdot 3^9$	0.02	0.02
U of order $2^{12} \cdot 3^{13} \cdot 7^{18}$	0.03	0.06

Acknowledgments. I am grateful to the reviewer for his helpful report.

References

[1] A.F. Vasil'ev, T.I. Vasil'eva, *On finite groups with generally subnormal Sylow subgroups*, Probl. Fiz. Mat. Tekh., **2011**:4(9) (2011), 86–91. Zbl 1245.20021
[2] V.N. Semenchuk, S.N. Shevchuk, *Characterization of classes of finite groups with the use of generalized subnormal Sylow subgroups*, Math. Notes, **89**:1 (2011), 117–120. Zbl 1230.20018
[3] V.I. Murashka, *Classes of finite groups with generalized subnormal cyclic primary subgroups*, Sib. Math. J., **55**:6 (2014), 1105–1115. Zbl 1344.20028
[4] A.F. Vasil'ev, T.I. Vasil'eva, A.S. Vegera, *Finite groups with generalized subnormal embedding of Sylow subgroups*, Sib. Math. J., **57**:2 (2016), 200–212. Zbl 1393.20006
[5] V.S. Monakhov, I.L. Sokhor, *On groups with formational subnormal Sylow subgroups*, J. Group Theory, **21**:2 (2018), 273–287. Zbl 1468.20041
[6] T.I. Vasil'eva, A.G. Koranchuk, *Finite groups with subnormal residuals of Sylow normalizers*, Sib. Math. J., **63**:4 (2022), 670–676. Zbl 1554.20036
[7] W. Guo, A.N. Skiba, *Finite groups whose n-maximal subgroups are σ-subnormal*, Sci. China Math., **62**:7 (2019), 1355–1372. Zbl 1476.20020
[8] A. Ballester-Bolinches, L.M. Ezquerro, *Classes of finite groups*, Springer, Dordrecht, 2006. Zbl 1102.20016
[9] S.F. Kamornikov, M.V. Sel'kin, *Subgroup functors and classes of finite groups*, Bel. Nauka, Minsk, 2003.
[10] V.I. Murashka, *Finite groups with given sets of $\mathfrak{F}$-subnormal subgroups*, Asian-Eur. J. Math., **13**:4 (2020), Article ID 2050073. Zbl 1454.20028
[11] T.I. Vasil'eva, A.I. Prokopenko, *Finite groups with $\mathfrak{F}$-accessible projectors*, Izv. Gomel. Gos. Univ. Im. F. Skoriny, **2006**:5(38), 2006, 14–18. Zbl 1157.20009
[12] K. Doerk, T. Hawkes, *Finite solvable Groups*, Walter de Gruyter, Berlin etc., 1992. Zbl 0753.20001
[13] B. Eick, C.R.B. Wright, *Computing subgroups by exhibition in finite solvable groups*, J. Symb. Comput., **33**:2 (2002), 129–143. Zbl 0995.20004
[14] B. Höfling, *Computing projectors, injectors, residuals and radicals of finite soluble groups*, J. Symb. Comput., **32**:5 (2001), 499–511. Zbl 0990.20008
[15] V.I. Murashka, *Formations of finite groups in polynomial time: $\mathfrak{F}$-residuals and $\mathfrak{F}$-subnormality*, J. Symb. Comput., **122** (2024), Article ID 102271. Zbl 1529.20020
[16] L.A. Shemetkov, *Formations of finite groups*, Nauka, Moscow, 1978. Zbl 0496.20014

[17] A.F. Vasil'ev, T.I. Vasil'eva, V.N. Tyutyanov, *On the finite groups of supersolvable type*, Sib. Math. J., **51**:6 (2010), 1004–1012. Zbl 1226.20013

[18] V.S. Monakhov, V.N. Kniahina, *Finite groups with $\mathbb{P}$-subnormal subgroups*, Ric. Mat., **62**:2 (2013), 307–322. Zbl 1306.20015

[19] The GAP Group, *GAP – Groups, Algorithms, Programming – a system for computational discrete algebra*.

[20] B. Eick, C.R. Wright, and The GAP Team, *GAP package FORMAT – computing with formations of finite solvable groups, v.1.4.4, 2024*.

[21] B. Höfling, *GAP package CRISP – computing with radicals, injectors, schunck classes and projectors, v. 1.4.6, 2022*.

Viachaslau Igaravich Murashka
Francisk Skorina Gomel State University,
Sovetskaya 104,
246028, Gomel, Belarus
Email address: mvimath@yandex.ru