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THE VOLUME OF A SPHERICAL TRIRECTANGULAR TETRAHEDRON

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Abstract: We consider a three-parameter family of tetrahedra in the spherical space, in which three edges at one vertex are pairwise orthogonal. We obtain formulae for their dihedral angles and volume in terms of edge lengths.

Keywords: trirectangular tetrahedron, spherical space, dihedral angle, volume, Coxeter tetrahedron.

1 Introduction

Calculating the volume of polyhedra is a classical problem, well-known since the time of Euclid. This problem remains relevant even today, in particular because the volume of the fundamental polyhedron is the main geometric invariant of a three-dimensional manifold.

Detailed overview of the results on calculating the volumes of polyhedra in spaces of constant curvature can be found in [1].

Volume formulas for non-Euclidean tetrahedra in some important special cases have been known since the time of N. Lobachevsky, J. Bolyai and L. Schläfli (see, e.g., [1] and [6]). In particular, Schläfli found the volume of the orthoscheme in $\mathbb{S}^3$ [10]. *Orthoscheme* is an n-dimensional simplex

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in which there exists a path along the edges $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n)$, that connects all vertices and the two edges of the path are perpendicular to each other. A three-dimensional orthoscheme is also called *birectangular tetrahedron*. By definition, a three-dimensional orthoscheme has three right dihedral angles (in Fig. 1 the corresponding edges are highlighted with colored bold lines) and three others, which we will call *essential dihedral angles* (in Fig. 1 they are designated as A, B, D).

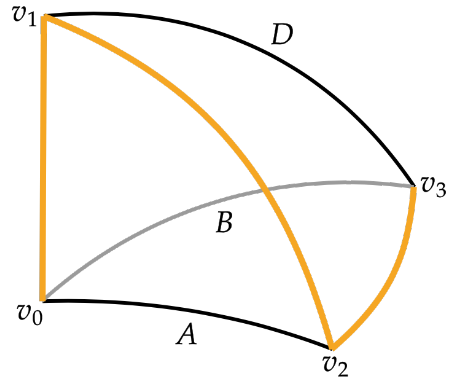

FIG. 1. Ortoscheme with essential dihedral angles A, B, D

Theorem 1 (Schläfli, 1858)**.** *Let T be a spherical orthoscheme with essential dihedral angles A, B, D. Then its volume $V = V(T)$ can be calculated by the following formula*

$$V = \frac{1}{4} S(A, B, D), \text{ where}$$

$$S\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right) = \widehat{S}(x, y, z) = \\ = \sum_{m=1}^{\infty} \left(\frac{D - \sin(x)\sin(z)}{D + \sin(x)\sin(z)}\right)^m \frac{\cos(2mx) - \cos(2my) + \cos(2mz - 1)}{m^2} - \\ - x^2 + y^2 - z^2$$

and $D = \sqrt{\cos^2 x \cos^2 z - \cos^2 y}$.

In the work of J. Murakami [7] formulae for the volume of an arbitrary spherical tetrahedron are proposed in terms of dihedral angles and edge lengths. However, each of them contains 16 multivalued complex dilogarithms of implicit functions, which makes them difficult to use. The purpose of this work is to obtain an explicit integral formula for the volume of a sufficiently wide three-parameter family of spherical tetrahedra, which will be convenient to use for further calculations and the study of the asymptotic behavior of the volume.

In this paper, the object of the study is a *trirectangular tetrahedron*, that is, a tetrahedron that has three pairwise orthogonal edges with a common vertex. In the work [2] a formula for the volume of a trirectangular tetrahedron was obtained in hyperbolic space.

Theorem 2 (Abrosimov, Stepanishchev, 2023). *Let $T = T(\ell_1, \ell_2, \ell_3)$ be a trirectangular hyperbolic tetrahedron given by lengths of pairwise orthogonal edges ℓ_1, ℓ_2, ℓ_3 at a common vertex. Then the volume $V = V(T)$ can be found by the formula*

$$V = \frac{1}{2}\int\limits_0^{\tanh \ell_1} \int\limits_0^{\frac{\tanh \ell_2 (\tanh \ell_1 - x)}{\tanh \ell_1}} \left[\frac{1}{1 - x^2 - y^2} + \frac{1}{x(x - 2x_0) + y(y - 2y_0) - e^{2\ell_3}}\right] dx\, dy,$$

where $x_0 = \dfrac{1 - e^{2\,\ell_3}}{2\tanh \ell_1}$ *and* $y_0 = \dfrac{1 - e^{2\,\ell_3}}{2\tanh \ell_2}$.

In the present paper we consider a trirectangular tetrahedron in spherical space. For this purpose, we introduce some notations and definitions. Consider 3-dimensional sphere

$$\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

The *distance* between two points $v_1, v_2 \in \mathbb{S}^3$ is the angle (in radians) between their radius vectors v_1 and v_2 in $\mathbb{R}^4$. Thus, the scalar product in $\mathbb{R}^4$ induces a metric in $\mathbb{S}^3$. We have

$$\cos a = < v_1, v_2 >,$$

where a is a distance between v_1 and v_2.

Geodesics are lines along which the shortest distances are achieved. Geodesics on a sphere are the so-called *great circles*, that is, circles whose center coincides with the center of the sphere.

A *spherical tetrahedron* is a convex hull (in the sense of the spherical metric) of four points in $\mathbb{S}^3$, called *vertices of the tetrahedron*.

In the book [3] one can find more detailed definitions, as well as the derivation of some classical formulas of spherical geometry.

2 Conditions for the existence of a trirectangular tetrahedron in $\mathbb{S}^3$

Consider a standard basis in $\mathbb{R}^4$

$$\begin{aligned} e_1 &= (1, 0, 0, 0), \\ e_2 &= (0, 1, 0, 0), \\ e_3 &= (0, 0, 1, 0), \\ e_4 &= (0, 0, 0, 1). \end{aligned}$$

Let P_{ij} denote the plane formed by the vectors e_i and e_j. Then the circle $S_{ij} := P_{ij} \cap \mathbb{S}^3$ is a geodesic.

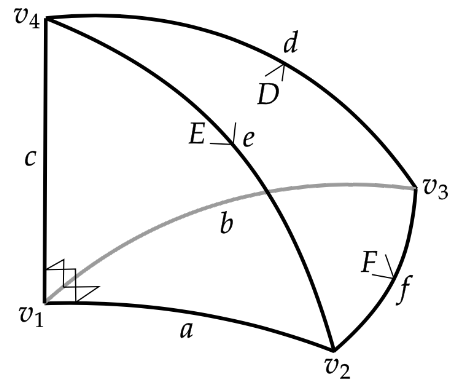

FIG. 2. Spherical trirectangular tetrahedron $T(a,b,c)$

Consider an arbitrary trirectangular tetrahedron $T = T(a,b,c)$ in $\mathbb{S}^3$ given by the lengths of its three pairwise orthogonal edges a,b,c with a common vertex (Fig. 2). Without loss of generality, we will assume that this vertex is placed at the point $v_1 = (1,0,0,0)$, and that the three orthogonal edges v_1v_2, v_1v_3, v_1v_4 are directed along the geodesics S_{12}, S_{13}, S_{14}, lying in the coordinate planes $P_{12}, P_{13}, P_{14} \subset \mathbb{R}^4$. Then the vertices of tetrahedron $T(a,b,c)$ have the following coordinates

$$\begin{aligned} v_1 &= (\ \ 1, \ \ 0, \ \ 0, \ \ 0), \\ v_2 &= (\cos a, \ \sin a, \ \ 0, \ \ 0), \\ v_3 &= (\cos b, \ \ 0, \ \sin b, \ \ 0), \\ v_4 &= (\cos c, \ \ 0, \ \ 0, \ \sin c). \end{aligned} \tag{1}$$

We have constructed a model of an arbitrary trirectangular tetrahedron $T(a,b,c)$ in $\mathbb{S}^3$ and thus proved the following proposition.

Proposition 1. *A trirectangular tetrahedron $T(a,b,c)$ exists in $\mathbb{S}^3$ for arbitrary $a,b,c \in (0,\pi)$.*

Proposition 2. *For a,b or $c = 0$, tetrahedron $T(a,b,c)$ in $\mathbb{S}^3$ loses its dimension (i.e. degenerates into a triangle, segment, or point). For a,b or $c = \pi$, tetrahedron $T(a,b,c)$ degenerates into a rectangular "spindle", i.e. a body with two vertices, at each of which three orthogonal edges meet.*

Proof. Indeed, let $a = 0$, then according to equalities (1), the vertex $v_2 = (1,0,0,0)$ coincides with the vertex v_1. In this case, the tetrahedron loses one dimension and degenerates into a triangle.

On the other hand, if $a = \pi$, then according to equalities (1), the vertex $v_2 = (-1,0,0,0) \in \mathbb{S}^3$ is diametrically opposite to vertex v_1. In this case, the edge v_1v_2 is a half of the great circle S_{12}. Then the two edges v_1v_3 and v_3v_2 form a half of the great circle S_{13}. The remaining two edges v_1v_4 and v_4v_2 form a half of the great circle S_{14}. Thus, tetrahedron $T(a,b,c)$ is a rectangular "spindle" with diametrically opposite vertices $v_1 = (1, 0, 0, 0)$, $v_2 = (-1, 0, 0, 0) \in \mathbb{S}^3$, at each of which three orthogonal edges meet. □

3 Dihedral angles in terms of edge lengths

Theorem 3. *Let $T(a, b, c)$ be a spherical trirectangular tetrahedron given by the lengths of its three pairwise orthogonal edges a, b, c with a common vertex. Then the dihedral angles opposite to these edges can be found by the formulas*

$$\cos D = \frac{\cos a \sin b \sin c}{\sqrt{R}},$$

$$\cos E = \frac{\sin a \cos b \sin c}{\sqrt{R}},$$

$$\cos F = \frac{\sin a \sin b \cos c}{\sqrt{R}},$$

where $R = 1 - \cos^2 a \cos^2 b - \cos^2 a \cos^2 c - \cos^2 b \cos^2 c + 2\cos^2 a \cos^2 b \cos^2 c$.

Proof. Consider a section of tetrahedron $T(a, b, c)$ by a sphere centered at vertex v_2 (Fig. 3).

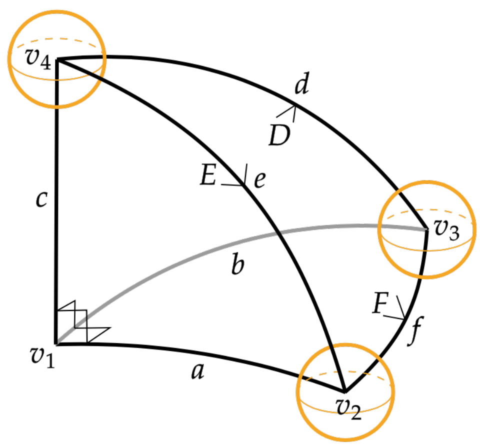

FIG. 3. Section of tetrahedron $T(a, b, c)$

As a section we get a spherical right triangle (Fig. 4).

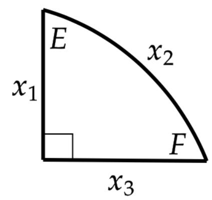

FIG. 4. Spherical triangle as a section of tetrahedron $T(a, b, c)$

By the first spherical cosine law for this triangle we have

$$\cos E = \frac{\cos x_3 - \cos x_1 \cos x_2}{\sin x_1 \sin x_2}.$$

Replacing in the last equality $\cos x_2 = \cos x_1 \cos x_3$ according to the spherical Pythagorean theorem, we obtain

$$\cos E = \frac{\cos x_3 \sin x_1}{\sin x_2}. \tag{2}$$

Let us write the spherical Pythagorean theorem for three right angled faces of tetrahedron $T(a, b, c)$

$$\cos d = \cos b \cos c, \quad \cos e = \cos a \cos c, \quad \cos f = \cos a \cos b. \tag{3}$$

Consider the lower face $v_1 v_2 v_3$ of tetrahedron $T(a, b, c)$. By the spherical law of sines in this right triangle we have

$$\sin x_3 = \frac{\sin b}{\sin f} = \frac{\sin b}{\sqrt{1 - \cos^2 f}}.$$

Substituting here $\cos f$ from (3), we get

$$\sin x_3 = \frac{\sin b}{\sqrt{1 - \cos^2 a \cos^2 b}}.$$

Hence,

$$\cos x_3 = \frac{\sin a \, \cos b}{\sqrt{1 - \cos^2 a \cos^2 b}}. \tag{4}$$

.

Renaiming b to c and x_3 to x_1 we obtain

$$\cos x_1 = \frac{\sin a \, \cos c}{\sqrt{1 - \cos^2 a \cos^2 c}}. \tag{5}$$

Hence,

$$\sin x_1 = \frac{\sin c}{\sqrt{1 - \cos^2 a \cos^2 c}}.$$

Now consider the face $v_2 v_3 v_4$ of tetrahedron $T(a, b, c)$. The first spherical cosine law gives

$$\cos x_2 = \frac{\cos d - \cos e \cos f}{\sin e \, \sin f}.$$

Substituting $\cos d, \cos e, \cos f$ from (3), by straightforward calculations we obtain

$$\cos x_2 = \frac{\sin^2 a \cos b \cos c}{\sqrt{(1-\cos^2 a \cos^2 b)(1-\cos^2 a \cos^2 c)}}.$$

Hence,

$$\sin x_2 = \sqrt{\frac{1-\cos^2 a\cos^2 b-\cos^2 a\cos^2 c-\cos^2 b\cos^2 c+2\cos^2 a\cos^2 b\cos^2 c}{(1-\cos^2 a\cos^2 b)(1-\cos^2 a\cos^2 c)}}. \quad (6)$$

It remains to substitute the found expressions (4), (5), (6) into the relation (2). Then we obtain

$$\cos E = \frac{\sin a\cos b\sin c}{\sqrt{1-\cos^2 a\cos^2 b-\cos^2 a\cos^2 c-\cos^2 b\cos^2 c+2\cos^2 a\cos^2 b\cos^2 c}}.$$

From the last formula one can also obtain the corresponding expressions for $\cos D, \cos F$ by renaming the edges. □

4 Edge lengths in terms of dihedral angles

Theorem 4. *Let T be a spherical trirectangular tetrahedron. Then the edge lengths a, b, c, d, e, f of T can be expressed in terms of dihedral angles D, E, F opposite to three pairwise orthogonal edges, by the formulas*

$$\cos a = \frac{\cos D}{\sqrt{\sin^2 E-\cos^2 F}}, \quad \cos d = \frac{\cos E\cos F}{\sqrt{(\sin^2 D-\cos^2 F)(\sin^2 E-\cos^2 D)}},$$
$$\cos b = \frac{\cos E}{\sqrt{\sin^2 D-\cos^2 F}}, \quad \cos e = \frac{\cos D\cos F}{\sqrt{(\sin^2 E-\cos^2 F)(\sin^2 E-\cos^2 D)}},$$
$$\cos c = \frac{\cos F}{\sqrt{\sin^2 E-\cos^2 D}}, \quad \cos f = \frac{\cos D\cos E}{\sqrt{(\sin^2 E-\cos^2 F)(\sin^2 D-\cos^2 F)}}.$$

Proof. Consider sections of tetrahedron T by three spheres centered at vertices v_2, v_3, v_4 (Fig. 3). We get three spherical right triangles (Fig. 5).

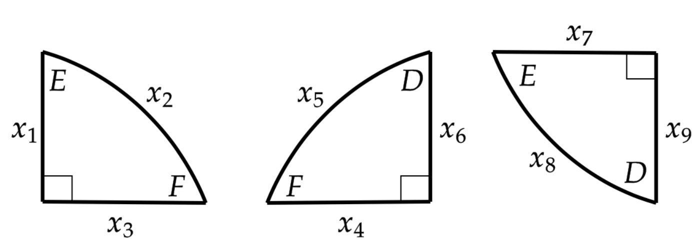

FIG. 5. Spherical triangles as sections of tetrahedron T

By the second spherical cosine law for these right triangles we have

$$\begin{aligned} \cos x_1 &= \frac{\cos F}{\sin E}, \quad \cos x_3 = \frac{\cos E}{\sin F}, \\ \cos x_4 &= \frac{\cos D}{\sin F}, \quad \cos x_6 = \frac{\cos F}{\sin D}, \\ \cos x_7 &= \frac{\cos D}{\sin E}, \quad \cos x_9 = \frac{\cos E}{\sin D}. \end{aligned} \tag{7}$$

By the second spherical cosine law for each of the three right angled faces of tetrahedron T we have

$$\begin{aligned} \cos a &= \frac{\cos x_7}{\sin x_1} = \frac{\cos x_7}{\sqrt{1-\cos^2 x_1}}, \\ \cos b &= \frac{\cos x_9}{\sin x_6} = \frac{\cos x_9}{\sqrt{1-\cos^2 x_6}}, \\ \cos c &= \frac{\cos x_1}{\sin x_7} = \frac{\cos x_1}{\sqrt{1-\cos^2 x_7}}. \end{aligned} \tag{8}$$

We substitute $\cos x_i$ in equalities (8) by expressions (7) and obtain

$$\begin{aligned} \cos a &= \frac{\cos D}{\sqrt{\sin^2 E - \cos^2 F}}, \\ \cos b &= \frac{\cos E}{\sqrt{\sin^2 D - \cos^2 F}}, \\ \cos c &= \frac{\cos F}{\sqrt{\sin^2 E - \cos^2 D}}. \end{aligned}$$

Then we apply spherical Pythagorean theorem for right angled faces of tetrahedron T and use the latter formulas for $\cos a, \cos b, \cos c$ to get the remaining formulas for $\cos d, \cos e, \cos f$. □

5 Volume of a trirectangular tetrahedron in $\mathbb{S}^3$

Theorem 5. *Let $T = T(a,b,c)$ be a spherical trirectangular tetrahedron given by the lengths of its pairwise orthogonal edges at a common vertex, equal to a, b, c. Then the volume $V = V(T)$ can be found by the formula*

$$\begin{aligned} V = \int_0^a \frac{1}{2R} \Bigg[& \frac{\arccos(\cos b \cos c)(1-\cos^2 b \cos^2 c)\sin\alpha \sin b \sin c}{\sqrt{R - \cos^2\alpha \sin^2 b \sin^2 c}} - \\ & - \cos\alpha (R - \sin^2\alpha(\cos^2 b + \cos^2 c - 2\cos^2 b \cos^2 c)) \cdot \\ & \cdot \left(\frac{\arccos(\cos\alpha \cos c)\cos b \sin c}{\sqrt{R - \sin^2\alpha \cos^2 b \sin^2 c}} + \frac{\arccos(\cos\alpha\cos b)\sin b \cos c}{\sqrt{R - \sin^2\alpha \sin^2 b \cos^2 c}} \right) \Bigg] d\alpha, \end{aligned}$$

where $R = 1 - \cos^2\alpha\cos^2 b - \cos^2\alpha\cos^2 c - \cos^2 b\cos^2 c + 2\cos^2\alpha\cos^2 b\cos^2 c$.

Proof. Consider a spherical trirectangular tetrahedron $T(a,b,c)$ with given edge lengths a, b, c. According to Proposition 1, the domain of existence of such a tetrahedron has the form $\Omega = \{(a, b, c) \in \mathbb{R}^3 : a, b, c \in [0, \pi]\}$ (Fig. 6).

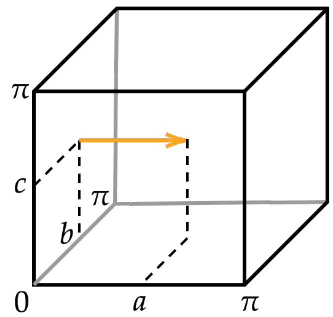

FIG. 6. Existence domain Ω of a spherical trirectangular tetrahedron $T(a, b, c)$

At the boundary $a = 0$ of the existence domain Ω, the tetrahedron loses its dimension, degenerating into a triangle, and its volume vanishes.

Let D, E, F denote the dihedral angles opposite to the edges a, b, c, respectively. According to Theorem 3, the dihedral angles are uniquely determined by the lengths of the edges a, b, c. Differentiating the volume as a function of a using the chain rule, we obtain

$$\frac{\partial V}{\partial a} = \frac{\partial V}{\partial D} \cdot \frac{\partial D}{\partial a} + \frac{\partial V}{\partial E} \cdot \frac{\partial E}{\partial a} + \frac{\partial V}{\partial F} \cdot \frac{\partial F}{\partial a}, \tag{9}$$

since $A = B = C = \dfrac{\pi}{2}$ and $\dfrac{\partial A}{\partial a} = \dfrac{\partial B}{\partial a} = \dfrac{\partial C}{\partial a} = 0$.

Recal the well-known Schläfli equation [9]

$$dV = \frac{1}{2} \sum_{\theta} l_{\theta} \, d\theta,$$

where the sum is taken over all edges of the tetrahedron, l_{θ} denotes the length of the edge and θ is the dihedral angle along it. Therefore,

$$\frac{\partial V}{\partial D} = \frac{d}{2}, \quad \frac{\partial V}{\partial E} = \frac{e}{2}, \quad \frac{\partial V}{\partial F} = \frac{f}{2}, \tag{10}$$

where $d = \arccos(\cos b \cos c)$, $e = \arccos(\cos a \cos c)$, $f = \arccos(\cos a \cos b)$, according to (3).

Note that

$$\frac{\partial \cos D}{\partial a} = -\sin D \, \frac{\partial D}{\partial a} \quad \text{or, equivlently,} \quad \frac{\partial D}{\partial a} = \frac{-1}{\sqrt{1 - \cos^2 D}} \cdot \frac{\partial \cos D}{\partial a}.$$

Using Theorem 3, by straightforward calculations we obtain

$$\frac{\partial D}{\partial a} = \frac{(1 - \cos^2 b \cos^2 c) \sin a \sin b \sin c}{R\sqrt{R - \cos^2 a \sin^2 b \sin^2 c}}, \tag{11}$$

where $R = 1 - \cos^2 a \cos^2 b - \cos^2 a \cos^2 c - \cos^2 b \cos^2 c + 2 \cos^2 a \cos^2 b \cos^2 c$.

Likewise,

$$\begin{aligned} \frac{\partial E}{\partial a} &= \frac{-(R-\sin^2 a(\cos^2 b+\cos^2 c-2\cos^2 b\cos^2 c))\cos a\cos b\sin c}{R\sqrt{R-\sin^2 a\cos^2 b\sin^2 c}}, \\ \frac{\partial F}{\partial a} &= \frac{-(R-\sin^2 a(\cos^2 b+\cos^2 c-2\cos^2 b\cos^2 c))\cos a\sin b\cos c}{R\sqrt{R-\sin^2 a\sin^2 b\cos^2 c}}. \end{aligned} \tag{12}$$

Integral of a differential form

$$dV = \frac{\partial V}{\partial a}da + \frac{\partial V}{\partial b}db + \frac{\partial V}{\partial c}dc \tag{13}$$

does not depend on the choice of the integration path in Ω connecting two fixed points. Since $V = 0$ on the boundary of Ω at $a = 0$, then by the Newton–Leibniz formula, the volume of tetrahedron $T(a, b, c)$ is equal to the integral of the form (13) over any piecewise smooth curve $\gamma \subset \Omega$ with origin $(0, b, c)$ and end at the point with coordinates (a, b, c). We will integrate over a straight line segment with origin $(0, b, c)$ and end (a, b, c), since in this case b, c are constant and the differential form (13) takes the form

$$dV = \frac{\partial V}{\partial a}da.$$

It remains to integrate both parts of the equation (9) with respect to the variable a.

Substituting the obtained derivatives (10), (11) and (12) into equation (9) and integrating it along the straight line segment with origin $(0, b, c)$ and end (a, b, c), we obtain the desired formula for the volume. To distinguish the length of the edge a and the integration variable, we denote the latter by α. □

6 Coxeter trirectangular tetrahedra in $\mathbb{S}^3$

Let T be a trirectangular tetrahedron in $\mathbb{S}^3$. Consider sections of tetrahedron T by three spheres centered at vertices v_2, v_3, v_4 (Fig. 3). As these sections, we get three spherical right triangles (Fig. 5).

Since the sum of angles in a spherical triangle is greater than π, then the dihedral angles of tetrahedron T satisfy the conditions

$$\begin{cases} E + F > \dfrac{\pi}{2}, \\ D + F > \dfrac{\pi}{2}, \\ D + E > \dfrac{\pi}{2}. \end{cases} \tag{14}$$

Coxeter tetrahedron is a tetrahedron whose dihedral angles are of the form $\dfrac{\pi}{n}$, where $n \geq 2$ is an integer. The list of spherical Coxeter tetrahedra was constructed by H.S.M. Coxeter [4]. He shown that there are 11 types of Coxeter tetrahedra in $\mathbb{S}^3$. In this section we will show that exactly 5 of these types belong to the family of trirectangular tetrahedra. We will calculate their volumes using Theorem 5 in order to check our formula.

Let $D = \dfrac{\pi}{\ell}, E = \dfrac{\pi}{m}, F = \dfrac{\pi}{n}$, where $l, m, n \geq 2$ are integers. Solving the system of inequalities (14) in integers ℓ, m, n we get the complete family of Coxeter trirectangular tetrahedra in $\mathbb{S}^3$ (see Table 1 below).

Here we should note that conditions (14) are necessary, but not sufficient for the existence of a trirectangular tetrahedron T in $\mathbb{S}^3$. That is why we also check the existence criterion for each of the solutions. The existence criterion for arbitrary tetrahedron in $\mathbb{S}^3$ in terms of dihedral angles was given by F. Luo [5].

Theorem 6 (F. Luo, 1997)**.** *The Gram matrix $G = [-\cos\theta_{ij}]$ of a spherical n-simplex is symmetric, positive definite with the diagonal entries equal to 1. Conversely, any positive definite symmetric matrix with diagonal entries equal to 1 is the Gram matrix of a spherical n-simplex unique up to isometry.*

A *Gram matrix G* of tetrahedron T is defined by its dihedral angles as follows

$$G = [-\cos\theta_{ij}]_{i,j=1,2,3,4} = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos C \\ -\cos A & 1 & -\cos F & -\cos E \\ -\cos B & -\cos F & 1 & -\cos D \\ -\cos C & -\cos E & -\cos D & 1 \end{pmatrix},$$

where θ_{ij} denotes a dihedral angle along the edge $v_i v_j$, we assume here that $-\cos\theta_{ii} = 1$.

This allows us to exclude example 7 in Table 1 with angles $D = E = \dfrac{\pi}{3}, F = \dfrac{\pi}{5}$ since the determinant of corresponding Gram matrix is negative.

According to Theorem 4, we calculate the cosines of edge lengths a, b, c for the given dihedral angles D, E, F. Then we use Theorem 5 to calculate the volumes.

Example 1 in Table 1 represents the infinite family of Coxeter trirectangular tetrahedra in $\mathbb{S}^3$. The next simple formula follows from Theorem 5 in this particular case.

Corollary 1. *The volume of a Coxeter trirectangular tetrahedron T in $\mathbb{S}^3$ with dihedral angles $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{n}\right), n \geq 2$, is equal $V(T) = \dfrac{\pi^2}{4n}$.*

Proof. Without loss of generality we set

$$A = B = C = D = E = \frac{\pi}{2}, \quad F = \frac{\pi}{n}.$$

Then by Theorem 4 we have

$$\cos a = \cos b = 0, \quad \cos c = \cos\frac{\pi}{n}.$$

Substituting the latter values into the volume formula from Theorem 5, we find

$$V(T) = \frac{\pi \sin \frac{\pi}{n}}{4} \int_0^{\frac{\pi}{2}} \frac{d\alpha}{1 + \cos \frac{\pi}{n} \cos \alpha} = \frac{\pi \sin \frac{\pi}{n}}{4} \cdot \frac{\pi}{n \sin \frac{\pi}{n}} = \frac{\pi^2}{4n}.$$

□

№	D	E	F	$\cos a$	$\cos b$	$\cos c$	*Existence*	*Volume*
1	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{n}$	0	0	$\cos \frac{\pi}{n}$	exist for $n \geq 2$	$\frac{\pi^2}{4n}$
2	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	exists	$\frac{\pi^2}{24}$
3	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	0	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{2}{3}}$	exists	$\frac{\pi^2}{48}$
4	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{5}$	0	$\sqrt{\frac{5+\sqrt{5}}{10}}$	$\sqrt{\frac{3+\sqrt{5}}{6}}$	exists	$\frac{\pi^2}{120}$
5	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	exists	$\frac{\pi^2}{96}$
6	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	1	1	1	degenerates	0
7	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{5}$	—	—	—	doesn't exist	—

TABLE 1. Coxeter trirectangular tetrahedra in $\mathbb{S}^3$

In example 6 we see that $\cos a = \cos b = \cos c = 1$, which means that tetrahedron $T(a, b, c)$ in this case degenerates into a single point.

The volumes in the last column of Table 1 coincide with well-known volumes of the respective Coxeter tetrahedra in $\mathbb{S}^3$ (see, e.g., [8], Table 1).

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