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FIXED POINTS AND VARIATIONAL PRINCIPLE IN UNIFORM SPACES.

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Abstract. The main purpose of this paper is to extend the well known Caristi's fixed point result to the setting of uniform spaces. As application, we give an extended form of Takahashi's non-convex minimization theorem.

1. Introduction

In [4,5], Ekeland presented a variational principle for approximate solutions of minimization problems which contains many useful applications in various fields. Since then, there have appeared many extensions or equivalent formulations of Ekeland's variational principle in metric spaces and more general in quasi-metric spaces as seen in [6,7,8,9]. In 1989, Takahashi [11] proved an existence theorem for a certain class of non-convex minimization problem and mentioned that the variational principle of Ekeland and the result of Caristi [3], in which Caristi prove a fixed point theorem that requires no continuity of the mapping under consideration, are equivalent. Clearly, one would ask whether these celebrated results can be extended to uniform spaces. In a recent work of M.Aamri and D.El Moutawakil [1], the authors have introduced the concept of E-distance functions on uniform spaces and have generalized some well-known results in fixed point theory for both E-contractive or E-expansive maps. In this paper, we present some new results concerning fixed point theory and variational principles in uniform spaces.

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2. Preliminaries

We begin this section by recalling some basic concepts of the theory of uniform spaces needed in the sequal. For more information we refer the reader to the book by N. Bourbaki [2], chapter II. We call uniform space (X, ϑ) a nonempty set X endowed of an uniformity ϑ, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x,x)/x \in X\}$. If $V \in \vartheta$ and $(x,y) \in V$, $(y,x) \in V$, x and y are said to be V-close, and a sequence (x_n) in X is a Cauchy sequence for ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X/(x,y) \in V\}$ when V runs over ϑ. A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X, i.e., if $(x,y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1} = \{(y,x)/(x,y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x,y) \in W$ then x and y are both W and V-close, then for our purpose, we assume that each $V \in \vartheta$ is symmetical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they always refer to the topological space $(X, \tau(\vartheta))$.
Now we recall some definitions and properties given in [1].

Definition 2.1. *Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an E-distance if*

- **(p_1):** *For any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in V$,*
- **(p_2):** *$p(x,y) \leq p(x,z) + p(z,y), \quad \forall x, y, z \in X$.*

The following lemma contains some useful properties of E-distances.

lemma 2.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let (x_n), (y_n) be arbitrary sequences in X and (α_n), (β_n) be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds*

- **(a):** *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.*
- **(b):** *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then (y_n) converges to z.*
- **(c):** *If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then (x_n) is a Cauchy sequence in (X, ϑ).*

Let (X, ϑ) be a uniform space with an E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting:

- **(1):** X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim_{n \to \infty} p(x, x_n) = 0$.
- **(2):** X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim_{n \to \infty} x_n = x$ with respect to $\tau(\vartheta)$.

3. Main results

3.1. **Fixed point results.** Before we state our main theorem, we give the following new concept

Definition 3.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. A function $\phi : X \longrightarrow I\!R^+$ will be said to be p-lower semi-continuous at $x \in X$ if for all (x_n) in X, condition $\lim_{n\to\infty} p(x, x_n) = 0$ implies $\phi(x) \leq \liminf \phi(x_n)$.*

Examples 3.1. **1-:** *Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the functions p and ϕ defined as follows:*

$$p(x, y) = y \ \ and \ \ \phi(x) = \frac{1}{2}.$$

Then for all $x \in X$, the function ϕ is p-lower semi-continuous at x. ■

2-: *Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the functions p and ϕ defined as follows:*

$$p(x, y) = y \ \ and \ \ \phi(x) = x.$$

Then the function ϕ is not p-lower semi-continuous at 1. In fact, consider a sequence (x_n) that converges to 0. Then one has: $\lim_{n\to\infty} \phi(x_n) = 0$, $\lim_{n\to\infty} p(1, x_m) = 0$ and $\phi(1) = 1$. Therefore the function ϕ is not p-lower semi-continuous at 1. ■

Remark 3.1. *Let (X, ϑ) be a uniform space and let d be a distance on X. Clearly (X, ϑ_d) is a uniform space where ϑ_d is the set of all subsets of $X \times X$ containing a "band"$B_\epsilon = \{(x, y) \in X^2/d(x, y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E-distance on (X, ϑ) and each lower semi-continuous function $\phi : X \longrightarrow I\!R^+$ is d-lower semi-continuous at each $x \in X$.*

Let us recall that each decreasing family (A_n) of closed nonempty subsets of a complete metric space (X, d) such that $\lim_{n\to\infty} \delta(A_n) = 0$, where $\delta(A) = sup\{d(x, y) : x, y \in A\}$, has a nonempty intersection. It will be helpfull in the sequal to generalize this result to our setting. First, we give the following definition

Definition 3.2. *Let (X, ϑ) be an uniform space and p be an E-distance on X. We say that a nonempty subset A of X is p-closed iff*

$$\overline{A}^p = \{x \in X : p(x, A) = 0\} \subset A,$$

where $p(x, A) = inf\{p(x, y)/y \in A\}$.

Proposition 3.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X Such that X is S-complete. Let (A_n) be a family of decreasing p-closed nonempty subsets of X such that $\lim_{n\to\infty} \delta_p(A_n) = 0$. Then $\cap_{n\in I\!N} A_n = \{u\}$ for some $u \in X$.*

Now we are able to give our main theorem

Theorem 3.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X such that X is S-complete. Let T be a selfmapping of X and $\phi : X \longrightarrow I\!R^+$ a p-lower semi-continuous function such that:*

$$p(x, Tx) \leq \phi(x) - \phi(Tx), \ \ for\ all\ x \in X.$$

Then there exist $u \in X$ such that $Tu = u$.

Proof.
Consider on X the following relation: $x \prec_X y \Leftrightarrow x = y$ *or* $p(x, y) \leq \phi(x) - \phi(y)$. It is easy to see that $(X, \prec_X)$ is a partially ordered set (Note that if $p(x, y) = 0$ and

$p(y,x)=0$ then $p(x,x)=0$ and Lemma 2.1.a then gives $x=y$). We wish to show that $(X,\prec_X)$ has a maximal element. Let M be a totally ordered subset of X. It is clear that the function ϕ is decreasing on M. Let (x_n) be an increasing sequence of M such that $\lim_{n\to\infty}\phi(x_n)=inf\phi$. Let $F_n=\{x\in M:\phi(x)\le\phi(x_n)\}$. We have $F_n\neq\emptyset$ since $x_n\in F_n$, $F_{n+1}\subseteq F_n$ and $\overline{F_n}^p\subset F_n$. In fact, for each $x\in\overline{F_n}^p$, there exists a sequence (y_m) in F_n such that $\lim_{m\to\infty}p(x,y_m)=0$ and $\phi(y_m)\le\phi(x_n)$, which implies that $\phi(x)\le\lim_{m\to\infty}inf\phi(y_m)\le\phi(x_n)$ and therefore $x\in F_n$. Moreover, we have $\lim_{n\to\infty}\delta_p(F_n)=\lim_{n\to\infty}sup\{p(x,y)/x,y\in F_n\}=0$. Indeed, let $(x,y)\in F_n^2$ such that $x\prec_X y$, then $p(x,y)\le\phi(x_n)-inf\phi$, therefore $\delta_p(F_n)\le\phi(x_n)-inf\phi$, which implies $\lim_{n\to\infty}\delta_p(F_n)=0$. According to Proposition 3.1, there exists $u\in M$ such that $\cap F_n=\{u\}$. On the other hand, since $x_n\in F_n\subseteq\overline{F_n}$ $(n=1,2,...)$, then $\lim_{n\to\infty}p(x_n,u)=0$ and $\lim_{n\to\infty}p(u,x_n)=0$. Since the function ϕ is p-lower semi-continuous and $\lim_{n\to\infty}p(u,x_n)=0$, we have $\phi(u)\le\lim_{n\to\infty}inf\phi(x_n)$ and therefore $\phi(u)\le inf\phi$. Now we show that u is an upper bound of M. Let $x\in M$. There are two cases:

1. $x_n\prec_X x, (n=1,2,...)$. Then $\phi(x)\le\phi(x_n)$, i.e, $x\in F_n$, hence $x=u$.
2. $x\prec_X x_{n_0}$ for some $n_0\in I\!N$. Then $\forall n\ge n_0, x\prec_X x_n$, hence $\forall n\ge n_0, p(x,x_n)\le\phi(x)-\phi(x_n)$. Hence $p(x,u)\le p(x,x_n)+p(x_n,u)\le\phi(x)-\phi(u)+p(x_n,u)$. On letting n to infty, we get $p(x,u)\le\phi(x)-\phi(u)$, which implies $x\prec_X u$. Thus by Zorn's lemma, $(X,\prec_X)$ has a maximal element v. Lastly, we prove that v is the desired point. In fact, we have $p(v,Tv)\le\phi(v)-\phi(Tv)$, which implies that $v\prec_X Tv$, and therefore $Tv=v$. Hence we have the Theorem. ■

Example 3.1. *Let $X=[0,+\infty[$ and $d(x,y)=|x-y|$ the usual metric. Consider the functions p and ϕ defined as follows:*

$$p(x,y)=y \ \ and \ \ \phi(x)=\frac{1}{2}.$$

It is not hard to see that the function p is an E-distance on X such that X is S-complete. Also, the function ϕ is p-lower semi-continuous on X. Let T be the selfmapping of X defined by: $Tx=0$, for all $x\in X$. On the one hand, $d(1,T1)>\phi(1)-\phi(T1)$, and therefore we can not use Carist's fixed point theorem to prove the existence of a fixed point of T. On the other hand, for each $x\in X$, one has: $p(x,Tx)\le\phi(x)-\phi(Tx)$. According to our Theorem 3.1, there exists $u\in X$ such that $Tu=u$. Note that $T0=0$. ■

In [7], the authors have introduced the concept of a W-distance function for a metric space as follows: A function $w:X\times X\longrightarrow I\!R^+$ is said to be a W-distance if:

(1): $w(x,y)\le w(x,z)+w(z,y),\quad\forall x,y,z\in X$,
(2): For any ϵ, there exists $\delta>0$ such that if $w(z,x)\le\delta$ and $w(z,y)\le\delta$ for some $z\in X$, then $w(x,y)\le\epsilon$,
(3): for all $x\in X$, the function $w(x,.)$ is lower semi-continuous.

It is clear that a W-distance function in a metric case is w-lower semi-continuous. Therefore, we have the following new result:

Corollary 3.1. *Let (X,d) be a complete metric space and $w:X\times X\longrightarrow I\!R^+$ a W-distance function on X. Let T be a selfmapping of X and $\phi:X\longrightarrow I\!R^+$ a* *w-lower semi-continuous function such that:*

$$w(x, Tx) \leq \phi(x) - \phi(Tx), \quad \textit{for all } x \in X.$$

Then there exist $u \in X$ such that $Tu = u$.

In a metric setting, we get the well known Caristi's fixed point theorem as follows:

Corollary 3.2. [2] *Let (X, d) be a complete metric space. Let T be a selfmapping of X and $\phi : X \longrightarrow I\!R^+$ a lower semi-continuous function such that:*

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \textit{for all } x \in X.$$

Then there exist $u \in X$ such that $Tu = u$.

Following ideas in [7], J.R. Montes and J.A. Charris [10], gave a formulation of the concept of W-distance functions in unform spaces as follows: Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow I\!R^+$ is said to be a W-distance if:

(1)**:** For any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$,

(2)**:** $p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y, z \in X,$

(3)**:** For all $x \in X$, the function $p(x, .)$ is lower semi-continuous.

In [1], the authors noted that a W-distance function, introduced by J.R. Montes and J.A. Charris [10], is an E-distance. So, we have the following new result:

Corollary 3.3. *Let (X, ϑ) be a Hausdorff uniform space and p be an W-distance on X Such that X is S-complete. Let T be a selfmapping of X and $\phi : X \longrightarrow I\!R^+$ a p-lower semi-continuous function such that:*

$$p(x, Tx) \leq \phi(x) - \phi(Tx), \quad \textit{for all } x \in X.$$

Then there exist $u \in X$ such that $Tu = u$.

3.2. **Applications.** In this section, we look at an application of our main results to variational principles. Our main result in this section is the following:

Theorem 3.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X Such that X is S-complete and let $\phi : X \longrightarrow I\!R^+$ be a p-lower semi-continuous function. Suppose that, for each $u \in X$ with $\phi(u) > inf_{x \in X}\phi(x)$, there exists a $v \in X$ such that $u \neq v$ and $\phi(v) + p(u, v) \leq \phi(u)$. Then there exist an $u_0 \in X$ such that $\phi(u_0) = inf_{x \in X}\phi(x)$.*

Proof.
Suppose that there exists no $u_0 \in X$ such that $\phi(u_0) = inf_{x \in X}\phi(x)$. By hypothesis, for each $x \in X$, there exists an $y \in X$ satisfying $y \neq x$ and $\phi(y) + p(x, y) \leq \phi(x)$. Let $E(x)$ be the set of all $y \in X$ such that $y \neq x$ and $\phi(y) + p(x, y) \leq \phi(x)$. It is clear that for each $x \in X$, $E(x) \neq \emptyset$ and $x \notin E(x)$. Now we define $T : X \longrightarrow X$ by $Tx \in E(x)$. Clearly, for each $x \in X$, we have $p(x, Tx) \leq \phi(x) - \phi(Tx)$. So, by Theorem 3.1, we may infer that there exists $u \in X$ such that $Tu = u$ but T is fixed point free. Hence there exists an $u_0 \in X$ such that $\phi(u_0) = inf_{x \in X}\phi(x)$. ■

Corollary 3.4. [7] *Let (X, d) be a complete metric space, $w : X \times X \longrightarrow I\!R^+$ a W-distance function on X and $\phi : X \longrightarrow I\!R^+$ a w-lower semi-continuous function. Suppose that, for each $u \in X$ with $\phi(u) > inf_{x \in X}\phi(x)$, there exists a $v \in X$ such that $u \neq v$ and $\phi(v) + w(u, v) \leq \phi(u)$. Then there exist an $u_0 \in X$ such that $\phi(u_0) = inf_{x \in X}\phi(x)$.*

Corollary 3.5. [11] *Let* (X, d) *be a complete metric space and let* $\phi : X \longrightarrow I\!R^+$ *be a lower semi-continuous function. Suppose that, for each* $u \in X$ *with* $\phi(u) > inf_{x \in X}\phi(x)$*, there exists a* $v \in X$ *such that* $u \neq v$ *and* $\phi(v) + d(u, v) \leq \phi(u)$*. Then there exist an* $u_0 \in X$ *such that* $\phi(u_0) = inf_{x \in X}\phi(x)$*.*

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