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A STUDY ASSOCIATED WITH CRITICAL LEVEL RESONANCE IN 2-D FLOW OVER ISOLATED RIDGES

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Abstract. A linear hydrostatic model of a stably stratified upwind profile for backward linear shear with a critical level (the wind speed $U = 0$) over a two-dimensional orographic barrier having two infinitely long ridges is considered. Analytical expressions for the mountain wave drag, energy flux and surface pressure perturbation are obtained. The results are illustrated by graphs of the mountain drag and the surface pressure for the flow with the Richardson number $R_i = 1$ and $R_i >> 1$.

1. Introduction

When stably stratified flow moves across an orographic barrier, gravity waves are generated; they propagate upwards, transferring horizontal momentum vertically. Because of the orographic waves the pressure is systematically higher on the upwind slopes rather than on downward slopes and thus exerting a net force on the ground. This force is known as mountain drag. This drag is an important part of the atmospheric momentum balance. Various theoretical studies are made for stably stratified airflow across orographic barrier related to mountain drag like [13], [14], [9], [2], [3], [10], [19], [1] etc. In [9], a two-dimensional model with arbitrary vertical distribution of density and velocity is considered. The author has developed a criterion giving a sufficient condition for the motion to be uniquely determined by the configuration of the topography over which the fluid moves. He has shown that internal Froude number of about 1/3 divides the motion into two states, one

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of which is called super critical, the other sub critical. Mountain wave problem addressing properties of mountain waves over Indian region were studied by many authors like [15], [16], [17], [18], [8], [5], [6] etc.
Recent study in [7] is devoted to fluxes of momentum and energy generated by mountain waves over Assam-Burma hills of India for constant wind speed. Next, in [12] the influence of Coriolis force on the fluxes of momentum and energy across the orographic barrier of Khasi- Jayantia hills of India is considered.
Still there is no works related to upwind profile for backward linear shear with a critical level for orographic barrier of India. Therefore the aim of the present paper is to develop a mathematical model to obtain the analytical expressions for mountain drag, energy flux and surface pressure perturbation for upwind profile for backward linear shear with a critical level across Assam- Burma hills of India.

2. The Mathematical Model

We consider a steady, frictionless, adiabatic flow of a vertically unbounded stratified, inviscid and Boussinesq fluid across a two-dimensional orography. The wind profile is assumed to have following form

$$U = \left[\begin{array}{rl} U_0 & \text{if} \quad z \le z_1 \\ U_0 \frac{z_c - z}{z_c - z_1} & \text{if} \quad z > z_1 \end{array} \right.$$

where U_0 is the surface wind, z_1 is the level where the wind changes from constant to backward linear and z_c is the critical level (where $U = 0$). The horizontal dimension of the hills as well as the disturbance is taken to be small enough so that the effect of Coriolis force may be neglected. We further assume that flow is independent of y coordinate. Under the above assumption, the linearized governing equations may be written as:

$$\rho_0 U \frac{\partial u'}{\partial x} + \frac{\partial p'}{\partial x} + \rho_0 w' \frac{\partial U}{\partial z} = 0 \tag{2.1}$$

$$\rho_0 U \frac{\partial w'}{\partial x} + \frac{\partial p'}{\partial z} + \rho' g = 0 \tag{2.2}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{2.3}$$

$$U \frac{\partial \rho'}{\partial x} + w' \frac{d\rho}{dz} = 0 \tag{2.4}$$

where u' , w' , p' and ρ' are respectively the perturbations of zonal wind, vertical wind, pressure and density. The mean density ρ_0' , gravitational acceleration g and density gradient $\frac{d\rho}{dz}$ are taken as constant. $N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz}$ is the Brunt-Vaisala frequency.

The vertical velocity satisfies the boundary condition near the ground

$$w'(x, z = 0) = U_0 \frac{\partial h}{\partial x} \tag{2.5}$$

where $h(x)$ is the profile of orographic barrier.
Now if $\hat{f}(k, z)$ is the Fourier transform of the function $f(x, z)$, then they are related by

$$\hat{f}(k, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x, z)\, e^{-ikx}\, dx \tag{2.6A}$$

$$f(x,z) = \int_{-\infty}^{\infty} \hat{f}(k,z)\, e^{ikx}\, dx \qquad (2.6B)$$

Now applying the Fourier transform to (2.1)– (2.4), we obtain

$$ik\rho_0 U\hat{u} + ik\hat{p} + \rho_0 \hat{w}\frac{\partial U}{\partial z} = 0 \qquad (2.7)$$

$$ik\rho_0 U\hat{w} + \frac{\partial \hat{p}}{\partial z} + \hat{\rho} g = 0 \qquad (2.8)$$

$$ik\hat{u} + \frac{\partial \hat{w}}{\partial z} = 0 \qquad (2.9)$$

$$ikU\hat{\rho} + \hat{w}\frac{d\rho}{dz} = 0 \qquad (2.10)$$

where, $\hat{u}$, $\hat{w}$, $\hat{\rho}$ are the Fourier transform of u', w', ρ' respectively. The equations (2.7)–(2.10) reduces to

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \left(\frac{N^2}{U^2} - k^2 - \frac{1}{U}\frac{d^2 U}{dz^2}\right) = 0 \qquad (2.11)$$

Under hydrostatic assumption, (2.11) becomes

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \left(\frac{N^2}{U^2} - \frac{1}{U}\frac{d^2 U}{dz^2}\right) = 0 \qquad (2.12)$$

Substituting $U = U_0 \frac{z_c - z}{z_c - z_1}$ into (2.12), we have

$$(z_c - z)^2 \frac{\partial^2 \hat{w}}{\partial z^2} + R_i \hat{w} = 0 \qquad (2.13)$$

where R_i is the Richardson number, defined by $R_i = N^2 (z_c - z_1)^2 / U_0^2$.

We solve solve (2.13) for $R_i > 1/4$ since it is the necessary condition for the flow to be hydrostatic stable and also for the energy to propagate at great height (see [11]). Under this assumption, the solution of (2.13) may be written as

$$\hat{w}(k,z) = \hat{w}(k,0)\, (z_c - z)^{1/2} exp(i\lambda \ln(z_c - z)) \qquad (2.14)$$

where $\lambda = \sqrt{(}R_i - 1/4)$
Substituting the Fourier transform of (2.5) into (2.14), we obtain

$$\hat{w}(k,z) = ikU_0 \hat{h}(k)(1 - z/z_c)^{1/2} exp(i\lambda \ln(1 - z/z_c)) \qquad (2.15)$$

here $\hat{w}(k,z)$ is the expression of perturbation vertical velocity in the Fourier form.

3. Mountain Drag and Energy Flux

The expression of mountain drag is

$$F = \int_{-\infty}^{\infty} p'\, dx = \int_{-\infty}^{\infty} p' \frac{d\eta'}{dx}\, dx = -\int_{-\infty}^{\infty} \frac{dp'}{dx} \eta'\, dx$$

$$= -2\pi \int_{-\infty}^{\infty} ik\hat{p}\hat{\eta}* \, dk \tag{3.1}$$

$\eta'(x, z)$ is the height of the streamline above undisturbed level and $\hat{\eta}*$ is the complex conjugate of $\hat{\eta}$.
Since $w' = U\frac{\partial \eta'}{\partial x}$, therefore

$$\hat{w} = ikU\hat{\eta} \tag{3.2}$$

Now using (2.7), (2.9) and (3.2), we rewrite (3.1)

$$F = -2\pi\rho_0 R[\int_{-\infty}^{\infty} i\frac{1}{k}(\frac{\partial \hat{w}}{\partial z} - \hat{w}\frac{1}{U}\frac{dU}{dz})\hat{w}^* \, dk] \tag{3.3}$$

where $\hat{w}*$ is the complex conjugate of $\hat{w}$.
We substitute (2.15) into (3.3)

$$F = -2\pi\rho_0 U_0^2 \frac{1}{z_c} R[\lambda + i\frac{1}{2}][\int_{-\infty}^{\infty} k\hat{h}(k)\hat{h}^*(k) \, dk] \tag{3.4}$$

Now let the 2-D profile of Assam-Burma hills (see [4]) is

$$h(x) = \frac{a^2 b_1}{a^2+x^2} + \frac{a^2 b_2}{a^2+(x-D)^2}$$

where $a = 20km$, $b_1 = 0.9km$, $b_2 = 0.7km$ and $D = 55.0km$
The Fourier transform of $h(x)$

$$\hat{h}(k) = ae^{-ak}(b_1 + b_2 e^{-iDk}) \tag{3.5}$$

Substituting (3.5) into (3.4), we have for positive and real solutions

$$F = -2\pi\rho_0 a^2 U_0^2 \frac{1}{z_c} R[\lambda + i\frac{1}{2}][\int_{-\infty}^{\infty} k(b_1^2 + b_2^2 + 2b_1 b_2 cosDk)e^{-2ak} \, dk]$$

$$= -\frac{1}{2}\pi\rho_0 N U_0(1 - \frac{z_1}{z_c})[1 - \frac{1}{4R_i}]^{1/2}[(b_1^2 + b_2^2) + 8b_1 b_2 a^2 \frac{(4a^2 - D^2)}{(4a^2+D^2)^2}] \tag{3.6}$$

Now the expression of energy flux is

$$E = \int_{-\infty}^{\infty} p'w' \, dx$$

$$= 2\pi \int_{-\infty}^{\infty} \hat{p}\hat{w}* \, dk \tag{3.7}$$

Applying (2.7), (2.9) to (3.7), we have

$$E = -2\pi\rho_0 \mathcal{R} \int_{-\infty}^{\infty} i\frac{1}{k}(U\frac{\partial \hat{w}}{\partial z} - \hat{w}\frac{dU}{dz})\hat{w}* \, dk \tag{3.8}$$

Applying (2.15) to (3.8)

$$E = -2\pi\rho_0 U_0^3 \frac{1}{z_c} R[\lambda + i\frac{1}{2}][\int_{-\infty}^{\infty} k\hat{h}(k)\hat{h}^*(k) \, dk] \tag{3.9}$$

Finally applying (3.5) to (3.9), for real and positive solutions we have

$$E = -\frac{1}{2}\pi\rho_0 N U_0^2 (1 - \frac{z_1}{z_c})[1 - \frac{1}{4R_i}]^{1/2}[(b_1^2 + b_2^2) + 8b_1 b_2 a^2 \frac{(4a^2 - D^2)}{(4a^2+D^2)^2}] \qquad (3.10)$$

4. The Surface Pressure Perturbation

In order to understand the behavior of the mountain drag, the surface pressure perturbation may be evaluated from equations (2.7) and (2.9)

$$\hat{p} = -i\rho_0 \frac{1}{k}(U\frac{\partial \hat{w}}{\partial z} - \hat{w}\frac{dU}{dz}) \qquad (4.1)$$

We substitute the value of U and $\hat{w}$ into (4.1) for surface pressure perturbation and have

$$\hat{p}(z=0) = \frac{a\rho_0 U_0^2}{(z_c - z_1)}(\frac{1}{2} - i\lambda)e^{-ak}(b_1 + b_2 e^{-iDk}) \qquad (4.2)$$

By the inverse Fourier transform of (4.2), we have

$$p'(z=0) = \rho_0 U_0 N[\frac{1}{2R_i^{1/2}}(\frac{a^2 b_1}{a^2+x^2} + \frac{a^2 b_2}{a^2+(x-D)^2}) + (1 - \frac{1}{4R_i})^{1/2}(\frac{ab_1 x}{a^2+x^2} + \frac{ab_2(x-D)}{a^2+(x-D)^2})] \qquad (4.3)$$

where, $P'(z=0)$ is the expression of surface pressure perturbation which contains two parts. The first part is symmetric with respect to ridges, and the second part is antisymmetric with respect to ridges. Variation of the surface pressure perturbation with respect to distance is shown in Figure 1.

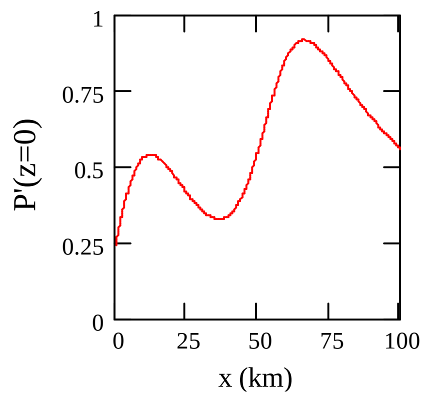

Fig.1 Variations of the surface pressure perturbation $P'(z=0)$ with respect to x at $R_i = 1$

5. Results and Conclusions

Using the profile of Assam- Burma hills of India in [4], we calculated the analytical expressions for mountain drag (eqn 3.6) and energy flux (eqn 3.10).

For $z_1 = 0$, the equations (3.6) and (3.10) become the drag and the flux for wind decaying linearly with height.

If the wind velocity changes from constant to backward linear flow at a greater height i.e. $z_1 >> 1$ implies that $R_i >> 1$, then equations (3.6) and (3.10) reduce to following form

$F_0 = -\frac{1}{2}\pi\rho_0 N U_0[(b_1^2 + b_2^2) + 8b_1 b_2 a^2 \frac{(4a^2-D^2)}{(4a^2+D^2)^2}]$
and
$E_0 = -\frac{1}{2}\pi\rho_0 N U_0^2[(b_1^2 + b_2^2) + 8b_1 b_2 a^2 \frac{(4a^2-D^2)}{(4a^2+D^2)^2}]$

These results agree with the results in [7]. Now, calculating the following ratios, we have

$\frac{F}{F_0} = \frac{E}{E_0} = (1 + \frac{Nz_1}{U_0 R_1^{1/2}})^{-1/2}(1 - \frac{1}{4R_i})^{1/2}$

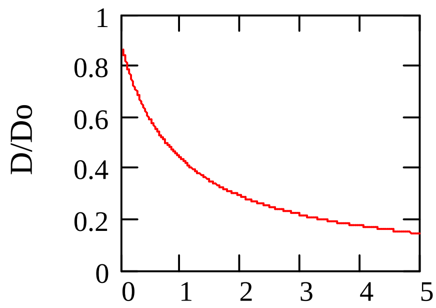

Fig.2 Variations of the normalized drag with Nz_1/U_0 at $R_i = 1$

Figure 2 shows that as Nz_1/U_0 (along x axis) increases, the normalized drag and flux both decreases.

It also can be seen that for increasing $1/R_i$, the normalized drag and flux both decrease and

$\frac{1}{R_i} \to \frac{1}{4}, \quad \frac{F}{F_0} = \frac{E}{E_0} \to 0$

From the equation (4.3), if R_i increases, the symmetric part of the surface pressure perturbation decreases and its antisymmetric part increases and at very high value of Richardson number i.e. for $R_i >> 1$, (4.3) reduces to

$P'(z = 0) = (p'(z = 0))_{z_1>>1} = a\rho_0 N U_0(\frac{b_1 x}{a^2+x^2} + \frac{b_2(x-D)}{a^2+(x-D)^2})$

The pressure perturbation for $R_i >> 1$ and its variation with respect to x are shown at Figure 3.

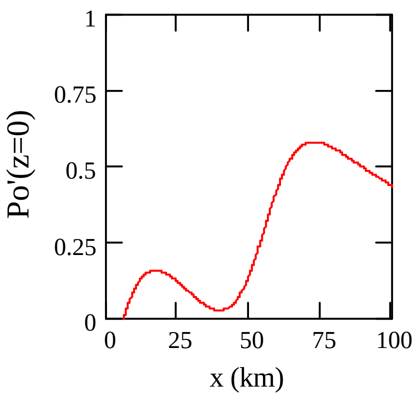

Fig.3 Variation of surface pressure perturbation with x for $R_i >> 1$

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