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A NOTE ON WEAKLY HEREDITARILY CLOSURE-PRESERVING FAMILIES

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Abstract. In this brief note, we discuss weakly hereditarily closure-preserving families of subsets of a space and answer a question on this class of families posed by Z. Li.

Let $\mathcal{P}$ be a family of subsets of a space X. We denote the family $\{\overline{P} : P \in \mathcal{P}\}$ by $\overline{\mathcal{P}}$, where $\overline{P}$ is the closure of P in X.

In [8], L. Yang claimed that $\overline{\mathcal{P}}$ need not to be hereditarily closure-preserving for a hereditarily closure-preserving family $\mathcal{P}$ of subsets of a regular space. Unfortunately, Yang's claim was incorrect.

S. Lin [4] noticed the error and proved the following theorem.

Theorem 1. *Let X be a regular space. If $\mathcal{P}$ is a hereditarily closure-preserving family of subsets of X, then $\overline{\mathcal{P}}$ is hereditarily closure-preserving.*

However, he did not know if "regular" in Theorem 1 can be omitted (see [6, Question 4.3], for example). It is still an open question. Take these into account, recently Z. Li asked me the following question in a private communication.

Question 1. *Let $\mathcal{P}$ be a family of subsets of a space X.*

(1) Is $\overline{\mathcal{P}}$ weakly hereditarily closure-preserving if $\mathcal{P}$ is weakly hereditarily closure-preserving?

(2) Furthermore, if the answer of (1) is negative, is $\overline{\mathcal{P}}$ weakly hereditarily closure-preserving if $\mathcal{P}$ is hereditarily closure-preserving?

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In this brief note, we investigate the above question. We give an example to show that the answer for Question 1(1) is negative. We also prove that $\overline{\mathcal{P}}$ is hereditarily closure-preserving for each hereditarily closure-preserving family $\mathcal{P}$, which answers Question 1(2) affirmatively. Throughout this paper, all spaces are assumed to be Hausdorff.

Definition 1. *Let $\mathcal{P}$ be a family of subsets of a space X.*

(1) $\mathcal{P}$ is called closure-preserving [1] *if $\overline{\bigcup \mathcal{P}'} = \bigcup \overline{\mathcal{P}'}$ for each $\mathcal{P}' \subset \mathcal{P}$.*

(2) $\mathcal{P}$ is called hereditarily closure-preserving [2] *if a family $\{H(P) : P \in \mathcal{P}\}$ is closure-preserving for each $H(P) \subset P \in \mathcal{P}$.*

(3) $\mathcal{P}$ is called weakly hereditarily closure-preserving [7] *if a family $\{x_P : P \in \mathcal{P}\}$ is closure-preserving for each $x_P \in P \in \mathcal{P}$.*

Remark 1. *Obviously, each hereditarily closure-preserving family of subsets of a space X is closure-preserving and weakly hereditarily closure-preserving. But a closure-preserving family of subsets of a space X need not be weakly hereditarily closure-preserving, and so it need not be hereditarily closure-preserving. In fact, let X be the closed interval $[0, 1]$ and $\mathcal{P} = \{(0, 1/n) : n \in \mathbb{N}\}$. Then $\mathcal{P}$ is closure-preserving, but $\mathcal{P}$ is not weakly hereditarily closure-preserving. By the following Remark 6, we can also know that a weakly hereditarily closure-preserving family of subsets of a space X need not be closure-preserving, and so it need not be hereditarily closure-preserving. However, we do not know if there is a closure-preserving and weakly hereditarily closure-preserving family $\mathcal{P}$ of subsets of a space X, such that $\mathcal{P}$ is not hereditarily closure-preserving.*

Having gained some enlightenment from [5, Example 3.1], we give the following example, which answers negatively Question 1(1). The space X in the following example is a known space constructed in 1965 by S. P. Franklin in [3].

Example 1. *There is a weakly hereditarily closure-preserving family $\mathcal{P}$ of subsets of a normal space X such that $\overline{\mathcal{P}}$ is not weakly hereditarily closure-preserving.*

Proof 1. *Let $X = \{(0,0)\} \bigcup (\mathbb{N} \times \{0\}) \bigcup (\mathbb{N} \times \mathbb{N}) \subset \mathbb{R}^2$. For $n, m \in \mathbb{N}$, put $V(n,m) = \{(n,0)\} \bigcup \{(n,k) : k \geq m\}$. Define an open neighborhood base $\mathcal{B}_x$ for each $x \in X$ for the desired topology on X as follows.*

(1) If $x = (n,m) \in \mathbb{N} \times \mathbb{N}$, then $\mathcal{B}_x = \{\{(n,m)\}\}$.

(2) If $x = (n,0) \in \mathbb{N} \times \{0\}$, then $\mathcal{B}_x = \{V(n,m) : m \in \mathbb{N}\}$.

(3) If $x = (0,0)$, then $\mathcal{B}_x = \{\{(0,0)\} \bigcup (\bigcup \{V(n, m_n) : n \geq k\}) : k \in \mathbb{N}, m_n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, put $P_n = \{(n,m) : m \in \mathbb{N}\}$, then $\overline{P_n} = \{(n,0)\} \bigcup P_n$. Put $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$.

Claim 1. X is normal.

It is clear that X is Hausdorff, so it suffices to prove X is paracompact. Let $\mathcal{U}$ be an open cover of X. Choose $U_0 \in \mathcal{U}$ such that $(0,0) \in U_0$. Then $\mathbb{N} \times \{0\} - U_0$ is finite. Choose a finite subfamily $\{U_i : i = 1, 2, \cdots, l\}$ of $\mathcal{U}$ such that $\{U_i : i = 1, 2, \cdots, l\}$ covers $\mathbb{N} \times \{0\}$. Put $\mathcal{U}' = \{U_i : i = 0, 1, 2, \cdots, l\}$ and $\mathcal{V} = \mathcal{U}' \bigcup \{\{x\} : x \in X - \bigcup \mathcal{U}'\}$. It is not difficult to check that $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. So X is paracompact.

Claim 2. $\mathcal{P}$ is weakly hereditarily closure-preserving.

For each $n \in \mathbb{N}$, let $(n, m_n) \in P_n$. Put $F = \{(n.m_n) : n \in \mathbb{N}\}$. We only need to prove that F is closed in X. Let $x \notin F$. If $x = (0,0)$, for each $n \in \mathbb{N}$, choose $m'_n \in \mathbb{N}$ such that $m'_n > m_n$. Put $U_x = \{(0,0)\} \bigcup (\bigcup \{V(n, m'_n) : n \in \mathbb{N}\})$, then U is an open *neighborhood of* $(0,0)$ *and* $U\bigcap F=\emptyset$. *If* $x=(n',0)\in\mathbb{N}\times\{0\}$, *choose* $m'_{n'}\in\mathbb{N}$ *such that* $m'_{n'}>m_{n'}$. *Put* $U_x=V(n',m'_{n'})$, *then* U *is an open neighborhood of* $(n',0)$ *and* $U\bigcap F=\emptyset$. *If* $x=(n,m)\in\mathbb{N}\times\mathbb{N}$, *put* $U_x=\{(n,m)\}$, *then* U *is an open neighborhood of* (n,m) *and* $U\bigcap F=\emptyset$. *Thus, we prove that* F *is closed in* X.

Claim 3. $\overline{\mathcal{P}}$ *is not weakly hereditarily closure-preserving.*

For each $n\in\mathbb{N}$, *choose* $(n,0)\in\overline{P_n}$, *then* $(0,0)\notin\{(n,0):n\in\mathbb{N}\}$. *It is clear that* $(0,0)\in\overline{\{(n,0):n\in\mathbb{N}\}}$, *so* $\overline{\mathcal{P}}$ *is not weakly hereditarily closure-preserving.*

Remark 2. *In Example 5,* $\mathcal{P}$ *is also not closure-preserving. In fact,* $\bigcup\overline{\mathcal{P}}=X-\{(0,0)\}$ *and* $\overline{\bigcup\mathcal{P}}=X$, *so* $\mathcal{P}$ *is not closure-preserving. This shows that a weakly hereditarily closure-preserving family of subsets of a space* X *need not be closure-preserving.*

The following theorem give an affirmative answer for Question 1(2).

Theorem 2. *Let* X *be a space. If* $\mathcal{P}$ *is a hereditarily closure-preserving family of subsets of* X, *then* $\overline{\mathcal{P}}$ *is weakly hereditarily closure-preserving.*

Proof 2. *Let* $\mathcal{P}=\{P_\alpha:\alpha\in\Lambda\}$ *be a hereditarily closure-preserving family of subsets of* X. *If* $\overline{\mathcal{P}}=\{\overline{P_\alpha}:\alpha\in\Lambda\}$ *is not weakly hereditarily closure-preserving, then there is* $\Lambda'\subset\Lambda$ *such that* $\{x_\alpha:\alpha\in\Lambda'\}$ *is not closed in* X, *where* $x_\alpha\in\overline{P_\alpha}$ *for each* $\alpha\in\Lambda'$. *Choose* $x\in\overline{\{x_\alpha:\alpha\in\Lambda'\}}-\{x_\alpha:\alpha\in\Lambda'\}$. *Since* $x_\alpha\neq x$ *for each* $\alpha\in\Lambda'$, *there are open neighborhoods of* U_α *and* V_α *of* x_α *and* x *respectively, such that* $U_\alpha\bigcap V\alpha=\emptyset$. *Since* $x_\alpha\in U_\alpha\bigcap\overline{P_\alpha}\subset\overline{U_\alpha\bigcap P_\alpha}$ *for each* $\alpha\in\Lambda'$, $x\in\overline{\{x_\alpha:\alpha\in\Lambda'\}}\subset\overline{\bigcup\{\overline{U_\alpha\bigcap P_\alpha}:\alpha\in\Lambda'\}}$. *Note that* $U_\alpha\bigcap P_\alpha\subset P_\alpha$ *for each* $\alpha\in\Lambda'$ *and* $\mathcal{P}$ *is hereditarily closure-preserving. So* $\overline{\bigcup\{U_\alpha\bigcap P_\alpha:\alpha\in\Lambda'\}}=\bigcup\{\overline{U_\alpha\bigcap P_\alpha}:\alpha\in\Lambda'\}$, *that is* $\bigcup\{\overline{U_\alpha\bigcap P_\alpha}:\alpha\in\Lambda'\}$ *is closed in* X. *Therefore* $\overline{\bigcup\{\overline{U_\alpha\bigcap P_\alpha}:\alpha\in\Lambda'\}}=\bigcup\{\overline{U_\alpha\bigcap P_\alpha}:\alpha\in\Lambda'\}$. *Thus* $x\in\bigcup\{\overline{U_\alpha\bigcap P_\alpha}:\alpha\in\Lambda'\}$, *so there is* $\alpha\in\Lambda'$ *such that* $x\in\overline{U_\alpha\bigcap P_\alpha}$. *This shows that* $V_\alpha\bigcap U_\alpha\bigcap P_\alpha\neq\emptyset$. *It is impossible as* $U_\alpha\bigcap V\alpha=\emptyset$.

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