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ASYMPTOTIC PROFILE OF SOLUTIONS FOR THE CRITICAL SOBOLEV TYPE EQUATION ON A HALF-LINE

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Abstract. We study nonlinear Sobolev type equations on half-line

$$\begin{cases} \partial_t u + \mathbb{L}u = \lambda \left|u\right|^{\rho} u_x^{\sigma},\ x \in \mathbf{R}^+,\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^+, \end{cases}$$

with $\rho + \sigma = \frac{5}{2}, \rho > 0, \sigma > 0, \lambda \in \mathbf{C}$.The linear operator $\mathbb{L}$ is defined as

$$\mathbb{L} = \mathcal{L}^{-1} K(p) \mathcal{L}.$$

Here $\mathcal{L}^{-1}$ and $\mathcal{L}$ are Laplace transform and inverse Laplace transform with respect to space variable x and

$$K(p) = p^2 \sum_{j=0}^{m} a_j p^{2j} \left(\sum_{l=0}^{m+1} b_l p^{2l} \right)^{-1},$$

$m > 0$ is integer number.The aim of this paper is to prove the global existence of solutions to the initial-boundary value problem and to find the main term of the asymptotic representation of solutions in the critical convective case.

1. Introduction

This paper is devoted to the study of the initial-boundary value problem for the Sobolev type equation on half-line in the critical case

$$\begin{cases} \partial_t u + \mathbb{L}u = \lambda \left|u\right|^{\rho} u_x^{\sigma},\ x \in \mathbf{R}^+,\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^+, \end{cases} \tag{1.1}$$

with $\rho, \sigma \geq 0, \lambda \in \mathbb{C}$. The linear operator $\mathbb{L}$ is defined as

$$\mathbb{L} = \mathcal{L}^{-1} K(p) \mathcal{L}.$$

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Here and below $\mathcal{L}^{-1}$ and $\mathcal{L}$ are Laplace transform and inverse Laplace transform with respect to space variable x and

$$K(p) = p^2 \sum_{j=0}^{m} a_j p^{2j} \left(\sum_{l=0}^{m+1} b_l p^{2l} \right)^{-1},$$

where $m > 0$ is the integer . The constants $a_j \in \mathbb{C}$ and $b_j \in \mathbb{C}$ are such that $ReK(p) \geq 0$ for all $Rep = 0$ and

$$|K(p)| \leq C, |p| \leq 1.$$

The equation (1.1) include many model wave equations for media with a strong spatial dispersion,which appear in the nonlinear theory of quasy-stationary processes in electric media with account for the spatial dispersion (see [34]). For results concerning the Cauchy problem for nonlinear pseudoparabolic type equations see [14]-[16], [19], [20],[21], [36], [37]. The large time asymptotic of solutions to the Cauchy problem was obtained in papers [28], [26], [35].

In this paper we study the initial-boundary value problem (1.1) in the critical case, when the nonlinear term of equation (1.1) has the same time decay rate as that of the linear terms. Recently much attention was drown to the study of the global existence and large time asymptotic behavior of solutions to the Cauchy problems for nonlinear local and nonlocal equations (see papers [2]-[13], [17] , [18],[23],[24], [25], [27],[38]-[41] and literature cited therein). A general theory of nonlinear nonlocal equations on a half-line was developed in book [22]. The papers [31]-[29] present a further development of this theory for the case of nonanalytic symbols of pseudodifferential operator, where it was proposed the general method to construct the Green operator, which consists in the introduction of new necessary conditions at the singularity points of the symbol $K(p)$. Also it was found that critical and subcritical nonconvective types of the nonlinearity define different behavior of the solutions on a half-line, compared to the corresponding Cauchy problem.

In this paper we give some general approach to obtain global existence of solution of initial-boundary value problem in the critical case with convective type of the nonlinearity such that

$$\int_0^{+\infty} x\, |u|^{\sigma}\, u_x^{\rho} dx = 0.$$

We elaborate general sufficient conditions for obtaining the large time asymptotic expansion of solutions.

Below $\hat{\phi}$ is the Laplace transform of ϕ defined by

$$\hat{\phi}(\xi) = \int_{\mathbf{R}^+} e^{-x\xi} \phi(x)\, dx$$

and

$$\mathcal{L}^{-1}\hat{\phi}(\xi) = (2\pi i)^{-1} \int_{-\mathbf{i}\infty}^{i\infty} e^{x\xi} \hat{\phi}(\xi)\, d\xi$$

is the inverse Laplace transform of ϕ. Also we introduce the weighted Sobolev space

$$\mathbf{W}_p^{k,a} = \left\{ \phi \in \mathbf{L}^{p,a}, \sum_{j=0}^{k} \left\| \partial_x^j \phi \right\|_{\mathbf{L}^{p,a}} < \infty \right\}.$$

By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denote by $\mathbf{L}^p$, $1 \le p \le \infty$, the weighted Lebesgue space $\mathbf{L}^{1,a}$ is defined by

$$\mathbf{L}^{p,a} = \left\{ \phi \in \mathbf{L}^p(\mathbf{R}^+); \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle x\rangle^a \phi\|_{\mathbf{L}^p} < \infty \right\},$$

where $\langle x\rangle = \sqrt{1+x^2}$, $a \ge 0$.

Now we state the main result of this paper.

Theorem 1. *Let $\rho + \sigma = \frac{5}{2}$. We assume that the initial data $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}$, $a \in (0,1)$ are sufficiently small $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,1+a}} \le \varepsilon$. Then the initial-boundary value problem (1.1) has a unique global solution*

$$u(x,t) \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}\right) \cap \mathbf{C}\left((0,\infty); \mathbf{W}^{1,0}_\infty \cap \mathbf{W}^{1,1+a}_1\right)$$

Furthermore there exist a number A and a function $V \in \mathbf{L}^{1,1+a} \cap \mathbf{L}^\infty$ such that the asymptotic formula

$$u(x,t) = At^{-1}V\left(xt^{-\frac{1}{2}}\right) + O\left(t^{-1-\frac{a}{2}}\right), \tag{1.2}$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^+$.

2. Preliminary lemmas

We denote Green operator $\mathcal{G}(t)$ by

$$\mathcal{G}(t)\, g = \theta(x)\mathcal{L}^{-1}\left\{ e^{-K(p)t}\left(\widehat{g}(p) - \widehat{g}(-p)\frac{1}{m+1}\sum_{j=0}^{m+1}\frac{p+a_j}{p-a_j}\right)\right\}, \tag{2.1}$$

where a_j are simple poles of the symbol $K(p)$. From paper [31] we obtain the following

Proposition 1. *Let $u_0 \in \mathbf{L}^{1,1+a}(\mathbf{R}^+) \cap \mathbf{L}^\infty(\mathbf{R}^+)$, $a \ge 0$. Then for some $T > 0$ there exists an unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{L}^{1,1+a}(\mathbf{R}^+) \cap \mathbf{L}^\infty(\mathbf{R}^+)\right)$ to the problem (1.1). Moreover the solution of (1.1) has the following integral representation*

$$u(x,t) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau)\mathbb{N}(\tau)\, d\tau.$$

We denote by

$$G_1(s,q) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p^2}(e^{p(s-q)} - e^{p(s+q)})dp \tag{2.3}$$

and

$$G_0(x,t) = \frac{1}{\sqrt{t}}\lim_{y\to 0}\partial_y G_1(xt^{-\frac{1}{2}}, yt^{-\frac{1}{2}}). \tag{2.4}$$

Using the result of paper [30] we have

Lemma 1. *We suppose that the function $\phi \in \mathbf{L}^\infty(\mathbf{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbf{R}^+)$, where $a \in (0,1)$. Then the estimates*

$$\left\|\partial_x^k \mathcal{G}(t)\, \phi\right\|_{\mathbf{L}^r} \le C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{r_1}-\frac{1}{r}\right)-\frac{k}{2}}\|\phi\|_{\mathbf{L}^{r_1}} + e^{-t}\|\phi\|_{\mathbf{L}^r},$$

$$\left\|\partial_x^k\left(\mathcal{G}(t)\,\phi - \vartheta G_0(t)\right)\right\|_{\mathbf{L}^\infty} \le Ct^{-1-\frac{a}{2}-\frac{k}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-t}\|\phi\|_{\mathbf{L}^\infty},$$

and

$$\left\|(\cdot)^b\partial_x^k\left(\mathcal{G}\left(t\right)\phi-\vartheta G_0\right)\right\|_{\mathbf{L}^1}\leq Ct^{-\frac{1+k}{2}+\frac{b-a}{2}}\left\|\phi\right\|_{\mathbf{L}^{1,1+a}}+e^{-t}\left\|\phi\right\|_{\mathbf{L}^1}$$

are valid for all $t>0$*, where* $1\leq r\leq\infty$ *,*$0<b\leq a$

We now introduce operator $\mathcal{G}_0\left(t\right)$ by

$$\mathcal{G}_0\left(t\right)f=t^{-\frac{1}{2}}\int_{R^+}G_1(xt^{-\frac{1}{2}},yt^{-\frac{1}{2}})f(y)dy \tag{2.5}$$

Lemma 2. *Let* $\phi\in\mathbf{L}^\infty\cap\mathbf{L}^{1,1+a}$ *,*$k=0,1$

$$\left\|\partial_x^k\mathcal{G}_0\left(t\right)\phi\right\|_{\mathbf{L}^q}\leq Ct^{-\frac{k}{2}}\left\langle t\right\rangle^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\left\|\phi\right\|_{\mathbf{L}^r},$$

is true for all $t>0$*,* $1\leq q\leq\infty$*,*$1\leq r\leq\infty$*. Furthermore we assume that* $\phi\in\mathbf{L}^{1,1+a}$*, then the estimate*

$$\left\|\left(\cdot\right)^b\partial_x^k\left(\mathcal{G}_0\left(t\right)\phi-\vartheta G_0\left(t\right)\right)\right\|_{\mathbf{L}^q}\leq Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}-\frac{k}{2}}\left\|\phi\right\|_{\mathbf{L}^{1,1+a}}$$

is valid for all $t>0$*, where* $1\leq q\leq\infty$*,* $b\in[0,1+a]$ *and*

$$\vartheta=\int_0^{+\infty}x\phi\left(x\right)dx.$$

Доказательство. Since for $k=0,1$

$$\left|\partial_x^kG_1\left(t,x,y\right)\right|\leq Ct^{-\frac{1+k}{2}}e^{-\frac{C}{t}|x-y|^2}$$

for all $x,y\in\mathbf{R}^+$, by the Young inequality we have for $p=\frac{qr+r-q}{qr}$

$$\begin{aligned}\left\|\partial_x^k\mathcal{G}_0\left(t\right)\phi\right\|_{\mathbf{L}^q}&\leq Ct^{-\frac{1+k}{2}}\left\|\int_0^{+\infty}e^{-\frac{C}{t}|x-y|^2}\phi\left(y\right)dy\right\|_{\mathbf{L}^q}\\&\leq Ct^{-\frac{1+k}{2}}\left\|e^{-\frac{C}{t}|x|^2}\right\|_{\mathbf{L}^p}\left\|\phi\right\|_{\mathbf{L}^r}\leq Ct^{-\frac{k}{2}}\left\langle t\right\rangle^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\left\|\phi\right\|_{\mathbf{L}^r}\end{aligned}$$

for all $t>0$, where $1\leq q\leq\infty$. Hence the first estimate of the lemma follows. For the second estimate we write

$$x^b\left(\mathcal{G}_0\left(t\right)\phi-\vartheta G_0\left(t,x\right)\right)=\int_0^{+\infty}x^b\left(G_1\left(t,x,y\right)-G_0\left(t,x\right)y\right)\phi\left(y\right)dy$$

for any $b\in[0,1+a]$. Applying Taylor expansion, we obtain

$$\left|G_1\left(t,x,y\right)-G_0\left(t,x\right)y\right|\leq Ct^{-1-\frac{a}{2}}y^{1+a}\left(e^{-\frac{C}{t}|x-y|^2}+e^{-\frac{C}{t}|x|^2}\right)$$

for all $x,y\in\mathbf{R}^+$. Hence in the domain $y\leq\frac{x}{2}$

$$\begin{aligned}x^b\left|G_1\left(t,x,y\right)-G_0\left(t,x\right)y\right|&\leq Ct^{-1-\frac{a}{2}}y^{a+1}x^be^{-\frac{C}{t}|x|^2}\\&\leq Ct^{-1+\frac{b-a}{2}}y^{a+1}e^{-\frac{C}{t}|x|^2}.\end{aligned}$$

By the Lagrange finite differences Theorem we have

$$\left|G_1\left(t,x,y\right)\right|\leq Ct^{-\frac{1+\nu}{2}}y^\nu e^{-\frac{C}{t}|x-y|^2}$$

for all $x, y \in \mathbf{R}^+$, where $\nu \in [0,1]$. Taking $\nu = 1 + a - b$, in the case $b \in [1, a+1]$ we get for $y \geq \frac{x}{2}$

$$\begin{aligned} & x^b \left| G_1(t,x,y) - G_0(t,x)y \right| \\ \leq\ & x^b \left(\left| G_1(t,x,y) \right| + \left| G_0(t,x)y \right| \right) \\ \leq\ & Ct^{-1+\frac{b-a}{2}} x^b y^{a+1-b} e^{-\frac{C}{t}|x-y|^2} + Ct^{-\frac{3}{2}} x^{b+1} y e^{-\frac{C}{t}|x|^2} \\ \leq\ & Ct^{-1+\frac{b-a}{2}} y^{a+1} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right), \end{aligned}$$

and in the case $b \in [0,1]$ we write

$$\begin{aligned} & x^b \left| G_1(t,x,y) - G_0(t,x)y \right| \\ \leq\ & x^b \left(\left| G_1(t,x,y) \right| + \left| G_0(t,x)y \right| \right)^b \left| G_1(t,x,y) - G_0(t,x)y \right|^{1-b} \\ \leq\ & Ct^{-b} |y|^{(1+a)b} t^{-\left(1+\frac{a}{2}\right)(1-b)} |y|^{(a+1)(1-b)} \\ & \times \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right) \\ \leq\ & Ct^{-1+\frac{b-a}{2}} y^{1+a} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right), \end{aligned}$$

for all $x, y \in \mathbf{R}^+$, $y \geq \frac{x}{2}$. Thus we obtain the estimate

$$x^b \left| G_1(t,x,y) - G_0(t,x)y \right| \leq Ct^{-1+\frac{b-a}{2}} |y|^{a+1} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right)$$

for all $x, y \in \mathbf{R}^+$, and for any $b \in [0, 1+a]$. Applying the above estimate with Young inequality we find

$$\begin{aligned} & \left\| (\cdot)^b \left(\mathcal{G}_0(t)\phi - \vartheta G_0(t) \right) \right\|_{\mathbf{L}^p} \\ =\ & \left\| \int_0^{+\infty} x^b \left(G_1(t,x,y) - G_0(t,x)y \right) \phi(y)\, dy \right\|_{\mathbf{L}^q_x} \\ \leq\ & Ct^{-1+\frac{b-a}{2}} \left\| \int_0^{+\infty} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right) y^{1+a} \left| \phi(y) \right| dy \right\|_{\mathbf{L}^q_x} \\ \leq\ & Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,1+a}}. \end{aligned}$$

In the same way we prove second estimate for case $k = 1$. Thus the second estimate of the lemma follows. Lemma is proved. □

We introduce the norms

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left(\sum_{j=0}^{m+1} \left(\{t\}^{\frac{k}{2}} \langle t \rangle^{\frac{1+k}{2}+\frac{1}{2}\left(1-\frac{1}{r}\right)} \left\| \partial_x^k \phi(t) \right\|_{\mathbf{L}^r} + \{t\}^{\frac{k}{2}} \langle t \rangle^{\frac{k-1}{2}} \left\| \partial_x^k \phi(t) \right\|_{\mathbf{L}^{1,a+1}} \right) \right)$$

and

$$\|\phi\|_{\mathbf{Y}} = \sup_{t>0} \left(\{t\}^{\frac{\sigma}{2}} \langle t \rangle^{\frac{1}{2}+1+\frac{1}{2}\left(1-\frac{1}{r}\right)} \|\phi(t)\|_{\mathbf{L}^r} + \{t\}^{\frac{\sigma}{2}} \langle t \rangle^{\frac{1}{2}} \|\phi(t)\|_{\mathbf{L}^{1,a+1}} \right),$$

where $a \in (0,1)$.

Lemma 3. *Let the function $f(x,t)$ satisfy $\int_0^{+\infty} x f(x,t)\, dx = 0$. Then the following inequality*

$$\left\| \int_0^t \mathcal{G}_0(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{X}} \leq C \|f\|_{\mathbf{Y}},$$

$$\left\| \langle t \rangle^{\gamma_2} \int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)) f(\tau) d\tau \right\|_{\mathbf{X}} \le C \left\| f(t)) \right\|_{\mathbf{Y}}, \gamma_2 \in \left[0, 1 - \frac{\sigma}{m}\right)$$

is valid for $l = 0, 1$, provided that the right-hand side is finite.

Доказательство. By the Lemma 2 and Lemma 1 we get

$$\left\| \int_0^t \partial_x^k \mathcal{G}_0(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^\infty} + \left\| \int_0^t \partial_x^k \mathcal{G}_0(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^{1,1+a}}$$
$$\le C \left\| f(t) \right\|_{\mathbf{Y}} \int_0^1 (1+\tau)^{-1} \{\tau\}^{-\frac{\sigma}{2}} \{t-\tau\}^{-\frac{1}{2}} d\tau \le C \left\| f(t) \right\|_{\mathbf{Y}}$$

for all $0 \le t \le 1$. We now consider $t > 1$. By virtue of Lemma 2 with $\omega = 0$ we obtain

$$\left\| \int_0^t \partial_x^k \mathcal{G}_0(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^r}$$
$$\le C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{2+k}{2} + \frac{1}{n}\left(\frac{1}{r}-1\right)} \langle \tau \rangle^{-1+\frac{1}{2}} \{\tau\}^{-\frac{\sigma}{2}} d\tau \left\| f(\tau) \right\|_{\mathbf{L}^{1,a+1}}$$
$$+ C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{k}{2} - \frac{1}{2}(1-\frac{1}{r})} \langle t \rangle^{-\frac{1}{2}-1} d\tau \sup_{\tau>0} \left(\left\| f(\tau) \right\|_{\mathbf{L}^\infty} + \left\| f(\tau) \right\|_{\mathbf{L}^1} \right)$$
$$+ C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{k+1}{2} - \frac{1}{2}(1-\frac{1}{r})} \langle t \rangle^{-\frac{1}{2}-1+\frac{1}{4}} d\tau \sup_{\tau>0} \left\| f(\tau) \right\|_{\mathbf{L}^{1,1}}$$
$$\le C \{t\}^{-\frac{k}{2}} \langle t \rangle^{-\frac{1+k}{2} + \frac{1}{2}(\frac{1}{r}-1)} \left\| f \right\|_{\mathbf{Y}}$$

for $1 \le r \le \infty$ and using the second estimate of Lemma 1 we get

$$\left\| \int_0^t \partial_x^k \mathcal{G}_0(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{L}^{1,Q+1}}$$
$$\le C \int_0^{\frac{t}{2}} \{\tau\}^{-\frac{\sigma}{2}} \langle \tau \rangle^{-1+\frac{1}{2}} (t-\tau)^{\frac{1}{2}\left(\frac{1}{r}-1-k\right)} d\tau \left\| f(\tau) \right\|_{\mathbf{L}^{1,a+1}}$$
$$+ C \int_{\frac{t}{2}}^t \langle \tau \rangle^{-1+\frac{1}{2}} (t-\tau)^{\frac{1}{2}\left(\frac{1}{r}-1-k\right)} d\tau \left\| f(\tau) \right\|_{\mathbf{L}^{1,a+1}}$$
$$\le C \{t\}^{-\frac{k}{2}} \langle t \rangle^{\frac{1-k}{2}} \left\| f \right\|_{\mathbf{Y}}.$$

for all $t > 1$. Hence the result of the lemma follows. In the same way we can prove second estimate of this Lemma. Lemma 3 is proved. □

3. Proof of Theorem 1

By Proposition 1 , it follows that the global solution (if it exists) is unique. So our main purpose in the proof of Theorem 1 is to show the global in time existence of solutions. We use method of paper [33]. We denote by

$$\left\| g \right\|_{\mathbf{Z}} = \left(\left\| \phi(t) \right\|_{\mathbf{L}^\infty} + \left\| \phi(t) \right\|_{\mathbf{L}^{1,1+a}} \right),$$

and by

$$\left\| \phi \right\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t \rangle \left\| \phi(t) \right\|_{\mathbf{L}^\infty} + \langle t \rangle^{-\frac{a}{2}} \left\| \phi(t) \right\|_{\mathbf{L}^{1,1+a}} \right),$$

where $a \in (0,1)$. We note that the $\mathbf{L}^1$ - norm is estimated by the norm $\mathbf{X}$

$$\begin{aligned} \langle t\rangle^{\frac{1}{2}} \|\phi(t)\|_{\mathbf{L}^1} &= \int_0^{\langle t\rangle} |\phi(t,x)|\, dx + \int_{\langle t\rangle}^{+\infty} |1+x|^{-1-\alpha} |x|^{1+\alpha} |\phi(t,x)|\, dx \\ &\leq C\langle t\rangle \|\phi(t)\|_{\mathbf{L}^\infty} + C\langle t\rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,1+a}} \leq C\|\phi\|_{\mathbf{X}}. \end{aligned}$$

From Lemmas 2 and 1 we easily get the following estimates

$$\|\mathcal{G}_0(t)\phi\|_{\mathbf{X}} + \left\|\langle t\rangle^{\frac{a}{2}} (\mathcal{G}(t) - \mathcal{G}_0(t))\phi\right\|_{\mathbf{X}} \leq C\|\phi\|_{\mathbf{Z}}. \tag{3.1}$$

Also by direct calculation we get

$$\|(\mathbb{N}(u_1) - \mathbb{N}(u_2))\|_{\mathbf{Y}} \leq C\|u_1 - u_2\|_{\mathbf{X}} (\|u_1\|_{\mathbf{X}}^{\sigma_1} + \|u_2\|_{\mathbf{X}}^{\sigma_1}), \tag{3.2}$$

where $\sigma_1 = \rho + \sigma - 1$. Since that if $\int_{R^+} x\mathbb{N}dx = 0$ from Lemma 3 we have

$$\begin{aligned} \left\|\int_0^t \mathcal{G}_0(t-\tau)\mathbb{N}(\tau)d\tau\right\|_{\mathbf{X}} &\leq C\|\mathbb{N}(t))\|_{\mathbf{Y}} \\ &\leq C\|u\|_{\mathbf{X}}^{\sigma_1+1}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \left\|\langle t\rangle^{\frac{a}{2}} \int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau))\mathbb{N}(\tau)d\tau\right\|_{\mathbf{X}} &\leq C\|\mathbb{N}(t))\|_{\mathbf{Y}} \\ &\leq C\|u\|_{\mathbf{X}}^{\sigma_1+1}. \end{aligned} \tag{3.4}$$

Now we prove that

$$\int_{\mathbf{R}^+} x\mathcal{G}_0\phi dx = \int_{\mathbf{R}^+} x\phi dx. \tag{3.5}$$

Note that $\mathcal{G}_0\phi$ is solution the following initial-boundary value problem

$$\begin{cases} (\mathcal{G}_0\phi)_t - (\mathcal{G}_0\phi)_{xx} = 0, \quad t > 0, x > 0, \\ (\mathcal{G}_0\phi)(x,0) = \phi,\ x > 0;\ (\mathcal{G}_0\phi)(0,t) = 0,\ t > 0. \end{cases}$$

Therefore we get

$$\frac{d}{dt}\int_0^{+\infty} x\mathcal{G}_0\phi dx = \int_0^{+\infty} x(\mathcal{G}_0\phi)_{xx} dx.$$

Since

$$\int_0^{+\infty} x(\mathcal{G}_0\phi)_{xx} dx = 0$$

we easily see that

$$\int_0^{+\infty} x\mathcal{G}_0\phi dx = \int_0^{+\infty} yF(y)\, dy.$$

We prove the existence of the solution $u(x,t)$ for the initial-boundary value problem (1.1) by the successive approximations $u_m(t,x)$, $m = 1, 2, ...$, defined as follows

$$\partial_t u_m + \mathbb{K}(u_m) = \mathcal{N}(u_{m-1}),\ v_m(0,x) = v_0(x), \tag{3.7}$$

for all $m \geq 2$, where

$$u_1 = \mathcal{G}(t)u_0.$$

The integral equation associated with (1.1) is written as

$$u_m(t) = \mathcal{G}(t)u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u_{m-1})\, d\tau. \tag{3.8}$$

We now prove by induction the following estimates for all $m \geq 1$

$$\|u_m\|_{\mathbf{X}} \leq C\varepsilon, \ \ \|u_m(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{X}} \leq C\varepsilon, \tag{3.9}$$

From 3.1 we have

$$\|\mathcal{G}(t)\, u_0\|_{\mathbf{X}} \leq C\varepsilon,$$
$$\|u_1 - \mathcal{G}(t)\, u_0\|_{\mathbf{X}} \leq C\varepsilon.$$

Therefore estimates (3.9) are valid for $m = 1$. We assume that estimates (3.9) is true with m replaced by $m-1$. Due to (3.2) we have

$$\|x\mathcal{N}(u)\|_{\mathbf{Y}} \leq C\, \|u\|_{\mathbf{X}}^{\sigma_1+1}$$

Since $x\mathcal{N}(u_{m-1}(\tau))$ have the zero mean value via (3.3) and (3.4) we get

$$\left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u_{m-1}(\tau))\, d\tau \right\|_{\mathbf{X}}$$
$$\leq \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u_{m-1}(\tau))\, d\tau \right\|_{\mathbf{X}} \leq \|\mathcal{N}(u_{m-1})\|_{\mathbf{Y}} \leq C\varepsilon.$$

It follows that

$$\|u_m\|_{\mathbf{X}} \leq C\varepsilon, \|u_m(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{X}} \leq C\varepsilon. \tag{3.10}$$

Thus by induction we see that estimates (3.9) are valid for all $m \geq 1$. In the same way by induction we can prove that

$$\|u_m - u_{m-1}\|_{\mathbf{X}} \leq \frac{1}{4}\, \|u_{m-1} - u_{m-2}\|_{\mathbf{X}}\, ,$$

for all $m > 2$. Therefore taking the limit $m \to \infty$, we obtain a unique solution $\lim_{m\to\infty} u_m(x,t) = u(x,t) \in \mathbf{X}$, satisfying equations

$$u(t) = \mathcal{G}(t)\, u_0 - \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(u)d\tau,$$

and estimates for $t > 1$

$$\|u(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{X}} \leq C\varepsilon, \|u(t)\|_{\mathbf{X}} \leq C\varepsilon.$$

We now compute the asymptotics of the solution. First we show the existence of solutions to the integral equation

(3.12)

$$V(\xi) = V_0(\xi) - \int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{yz^{\frac{1}{2}}}{(1-z)^{\frac{1}{2}}} \right) \mathcal{N}(V(y))dy,$$

where $V_0(\xi) = \mathcal{G}_0 u_0$. We define successive approximations $V_{k+1} = \mathcal{R}(V_k)$ for $k = 0, 1, 2, ...$, where

$$\mathcal{R}(V_k)(\xi) = V_0(\xi) - \int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{yz^{\frac{1}{2}}}{(1-z)^{\frac{1}{2}}} \right) \mathcal{N}(V_k(y))dy,$$

By induction we prove the estimates

$$\|V_k - V_0\|_{\mathbf{Z}} \leq C\varepsilon, \ \ \|V_k\|_{\mathbf{Z}} \leq C$$

and

$$\|V_{k+1} - V_k\|_{\mathbf{Z}} \leq \frac{1}{2}\, \|V_k - V_{k-1}\|_{\mathbf{Z}} \tag{3.14}$$

for all $k \geq 1$. Firstly we have to show that

$$\int_{\mathbf{R}^+} y\mathcal{N}(V_k(y))dy = 0 \text{ and } \int_{\mathbf{R}^+} yV_k(y)\,dy = \theta. \tag{3.15}$$

Since due to (3.5)

$$\int_{\mathbf{R}^+} yV_0(y)\,dy = \int_{\mathbf{R}^+} xu_0 dx = \theta$$

by the definition of $\mathcal{N}(V(y))$ and (3.5), we see that (3.15) is true for $k = 0$. We assume that (3.15) holds for some k. Then via (3.5) we have

$$\begin{aligned}\int_{\mathbf{R}^+} \xi V_{k+1}(\xi)\,d\xi &= \theta - \int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}^+} \xi d\xi \\ &\times \int_{\mathbf{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{yz^{\frac{1}{2}}}{(1-z)^{\frac{1}{2}}}\right)\mathcal{N}(V_k(y))dy \\ &= 0\end{aligned}$$

hence it follows that $\int_{\mathbf{R}^+} y\mathcal{N}(V_{k+1}(y))dy = 0$. Thus we get (3.15) for any k. Estimates (3.14)-(3.15) are valid for $k = 0$. Denote $\mathcal{N}(V(y)) = F(y)$. Changing $\tau = tz$ and $y_1 = \tau^{-\frac{1}{2}}y$ and using

$$\left\|t^{-1}V((\cdot)\,t^{\frac{1}{2}})\right\|_{\mathbf{X}} = \|V(\cdot)\|_{\mathbf{Z}}$$

we get

$$\begin{aligned}&\|V_{k+1} - V_0\|_{\mathbf{Z}} \\ &= C\left\|\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}^+} G_1\left(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{yz^{\frac{1}{2}}}{(1-z)^{\frac{1}{2}}}\right) F_k(y)\,dy\right\|_{\mathbf{Z}} \\ &= C\left\|t^{-\frac{3}{2}}\int_0^t \tau^{-\frac{3}{2}}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\int_{\mathbf{R}^+} dyG_1\left(\frac{(\cdot)\,t^{-\frac{1}{2}}}{(t-\tau)^{\frac{1}{2}}}, \frac{y}{(t-\tau)^{\frac{1}{2}}}\right) F_k\left(y\tau^{-\frac{1}{2}}\right)\right\|_{\mathbf{W}} \\ &\leq C\left\|\int_0^t \mathcal{G}_0(t-\tau)\,\tau^{-\frac{2(\sigma_1+1)-1}{2}} F_k\left(\cdot\tau^{-\frac{1}{2}}\right)d\tau\right\|_{\mathbf{X}}\end{aligned}$$

Since

$$\left\|\int_0^t \mathcal{G}_0(t-\tau)\,\tau^{-\frac{2(\sigma_1+1)}{2}} F_k\left(\cdot\tau^{-\frac{1}{2}}\right)d\tau\right\|_{\mathbf{X}} \leq C\left\|t^{-\frac{2(\sigma_1+1)}{2}}F_k(\cdot t^{-\frac{1}{2}})\right\|_{\mathbf{Y}}$$

and

$$\begin{aligned}\left\|t^{-\frac{2(\sigma_1+1)}{2}}F_k(\cdot t^{-\frac{1}{2}})\right\|_{\mathbf{Y}} &\leq C\left\|t^{-1}V_k(\cdot t^{-\frac{1}{2}})\right\|_{\mathbf{X}}^{\sigma_1+1} \\ &\leq C\,\|V_k(\cdot)\|_{\mathbf{Z}}^{\sigma_1+1}\end{aligned}$$

we get

$$\|V_{k+1} - V_0\|_{\mathbf{Z}} \leq C\varepsilon.$$

Therefore (3.14)-(3.15) are true for any k. Using (**??**) we have

$$\begin{aligned}&\|x\,(\mathcal{N}(u_1) - \mathcal{N}(u_2))\|_{\mathbf{Y}} \\ &\leq C\,\|(u_1 - u_2)\|_{\mathbf{X}}\,(\|u_1\|_{\mathbf{X}}^{\sigma_1} + \|u_2\|_{\mathbf{X}}^{\sigma_1}).\end{aligned} \tag{3.16}$$

Thus in the same manner we obtain

$$\|V_{k+1} - V_k\|_{\mathbf{Z}} \le \frac{1}{2} \|V_k(\cdot) - V_{k-1}(\cdot)\|_{\mathbf{Z}}$$

and therefore estimate (3.14) is valid for any $k \ge 1$. Hence $\mathcal{R}$ is a contraction mapping and there exists a unique solution $V(\xi)$ to integral equation (3.12). We are now in a position to prove asymptotics of solutions v. We prove by induction

$$\left\| \langle t \rangle^{\gamma} \left(u_k(t) - t^{-1} V_k\left((\cdot)\, t^{-\frac{1}{2}} \right) \right) \right\|_{\mathbf{X}} < C\varepsilon, \tag{3.17}$$

where $\gamma > 0$ is small. The estimate (3.17) is true for $k = 0$ since

$$\begin{aligned} &\left\| \langle t \rangle^{\gamma} \left(v_0(t) - t^{-1} V_0\left((\cdot)\, t^{-\frac{1}{\rho}} \right) \right) \right\|_{\mathbf{X}} \\ &= \|\langle t \rangle^{\gamma} (\mathcal{G}(t) v_0 - \theta G_0(t))\|_{\mathbf{X}} \le C\varepsilon. \end{aligned} \tag{3.18}$$

We assume that (3.17) is valid for some k. Changing variables such that $\tau = zt$ and $\xi \tau^{-\frac{1}{2}} = y$ and using $\sigma_1 = \frac{3}{2}$ we have

$$\begin{aligned} &\int_0^t \tau^{\frac{2(\sigma_1+1)}{2}} \mathcal{G}_0(t-\tau)\, \tau^{-\frac{2(\sigma_1+1)}{2}} F_k\left(\cdot \tau^{-\frac{1}{2}} \right) d\tau \\ &= t^{-1} \int_0^1 \frac{dz}{z^{\frac{3}{2}} (1-z)^{\frac{1}{2}}} \int_{\mathbf{R}^+} dy G_1 \left(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{y z^{\frac{1}{2}}}{(1-z)^{\frac{1}{2}}} \right) F_k(y)\, dy \\ &= t^{-1} \left(V_0\left(x t^{-\frac{1}{2}} \right) - V_{k+1}\left(x t^{-\frac{1}{2}} \right) \right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &\left\| \langle t \rangle^{\gamma} \left(t^{-1} V_{k+1}\left(\cdot t^{-\frac{1}{2}} \right) - u_{k+1}(t) \right) \right\|_{\mathbf{X}} \\ &\le C \left\| \langle t \rangle^{\gamma} \left(t^{-1} V_{k+1}\left(\cdot t^{-\frac{1}{2}} \right) - \mathcal{G}_0(t) u_0 + \int_0^t \mathcal{G}_0(t-\tau) f_k(\tau)\, d\tau \right) \right\|_{\mathbf{X}} \\ &+ C \left\| \langle t \rangle^{\gamma} \int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)) f_k(\tau)\, d\tau \right\|_{\mathbf{X}} \\ &+ C \|\langle t \rangle^{\gamma} (\mathcal{G}(t) u_0 - \mathcal{G}_0(t) u_0)\|_{\mathbf{X}} \\ &\le C \left\| \langle t \rangle^{\gamma} \left(t^{-1} V_0\left(\cdot t^{-\frac{1}{\rho}} \right) - \mathcal{G}_0(t) u_0 \right) \right\|_{\mathbf{X}} \\ &+ \frac{C}{\theta^{\sigma_1 - 1}} \left\| \langle t \rangle^{\gamma} \int_0^t \mathcal{G}_0(t-\tau) \left(f_k(\tau) - \tau^{-\frac{2(\sigma_1+1)}{2}} F_k\left(\cdot \tau^{-\frac{1}{2}} \right) \right) d\tau \right\|_{\mathbf{X}} \\ &+ C \|\langle t \rangle^{\gamma} (\mathcal{G}(t) u_0 - \mathcal{G}_0(t) u_0)\|_{\mathbf{X}} \\ &+ C \left\| \langle t \rangle^{\gamma} \int_0^t (\mathcal{G}(t-\tau) - \mathcal{G}_0(t-\tau)) f_k(\tau)\, d\tau \right\|_{\mathbf{X}} \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From (3.18) we have

$$I_1 \le C\varepsilon.$$

By condition (3.16) and the estimate (3.17) we have

$$\left\| \langle t \rangle^{\gamma} \left(f_k(t) - \tau^{-\frac{2(\sigma_1+1)}{2}} F_k\left(\cdot t^{-\frac{1}{2}} \right) \right) \right\|_{\mathbf{Y}} \le C\varepsilon^{\sigma}$$

and

$$\int_{\mathbf{R}^+} x\left(f_k(x,\tau) - \tau^{-\frac{2(\sigma_1+1)}{2}} F_k\left(\cdot t^{-\frac{1}{2}}\right)\right) dx = 0$$

Therefore from (3.3) we get

$$I_2 = C\left\|\langle t\rangle^\gamma \int_0^t \mathcal{G}_0(t-\tau)\left(f_k(\tau) - \frac{1}{\tau^{\sigma_1}} F_k\left(\cdot\tau^{-\frac{1}{2}}\right)\right) d\tau\right\|_{\mathbf{X}}$$
$$\le C\left\|\langle t\rangle^\gamma \left(f_k(t) - \frac{\theta^{\sigma_1+1}}{\tau^{(\sigma_1+1)}} F_k\left(\cdot t^{-\frac{1}{2}}\right)\right)\right\|_{\mathbf{Y}} \le C\varepsilon.$$

By condition (3.3)-(3.4) we easily get

$$I_4 + I_5 \le \varepsilon.$$

Hence by induction (3.17) is true for any $k \ge 0$ uniformly with respect to k. Taking a limit $k \to \infty$ in (3.17) we get

$$\left\|\langle t\rangle^\gamma \left(u(t) - t^{-1}\theta V\left(\cdot t^{-\frac{1}{2}}\right)\right)\right\|_{\mathbf{X}} \le C\varepsilon\ .$$

This completes the proof of Theorem 1.

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