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STRATEGIES TOWARDS A DISPROOF OF THE GENERAL ANDREWS–CURTIS CONJECTURE

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The general Andrews Curtis Conjecture (AC) asks, whether simply h–equivalent 2–complexes K_1, K_2 can be transformed into each other using only cells up to dimension 3. In addition, one may require a subcomplex $K_0 \subseteq K_1 \cap K_2$ to remain fixed throughout the deformation (relative case). For an introduction into the problem and for terminology we refer to ([4], Ch. I and Ch. XII). We give a summary on several approaches:

I) There exist potential counterexamples in the relative case modelled on the fact that one gets simple homotopies of 2–complexes, if corresponding relators differ by commutators of relations. Here are examples of such 2–complexes:

$$\frac{R : a = [t(a), a^m]}{T : [t(a), a^m]^m} \text{ versus } \frac{S : a = 1}{T : [t(a), a^m]^m}, \quad m \geq 2;$$

$$\frac{R : a = [t^2(a), t(a)]}{T : [[t^3(a), t^2(a)], [t^2(a), t(a)]]} \text{ versus } \frac{S : a = 1}{T : [[t^3(a), t^2(a)], [t^2(a), t(a)]]}.$$

Here $[x, y] = xyx^{-1}y^{-1}$ denotes a commutator and $x(y)$ is an abbreviation for xyx^{-1}; K_1 resp. K_2 are the standard complexes for $\langle a, t \mid R, T\rangle$ and $\langle a, t \mid S, T\rangle$ respectively; K_0 is the standard complex for $\langle a, t \mid T\rangle$; $\pi_1 = \mathbb{Z}$ is generated by t. (These examples become Q–equivalent if T is not required to remain unchanged.)

It is easy to see that R cannot be transformed into $S^{\pm 1}$ by a consequence of T. But we also have to take into account 2–dimensional expansions introducing additional generators $a_1, \ldots, a_n$. As it is possible to lump the multiplication of consequences of T to the end or the beginning of a relative Q–transformation of the expanded presentations, we would get a Q–transformation

(1) $\langle a, t, a_1, \cdots, a_n \mid R \cdot f, a_1 \cdot f_1, \cdots, a_n \cdot f_n\rangle \rightarrow \langle a, t, a_1, \cdots, a_n \mid a, a_1, \cdots, a_n\rangle$

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with certain conjugate products f and f_i of T. In particular, we would get an isomorphism for the groups defined in (1) where the defining relator T is suppressed.

(2) *The goal of this approach is to show that for* ***no*** *choice of the f and the f_i in $\langle a, t, a_1, \cdots, a_n \mid R \cdot f, a_1 \cdot f_1, \cdots, a_n \cdot f_n \rangle$ the generator a becomes trivial.*

We have large classes of modifications f, f_i for which we have exhibited this nontriviality (and no counterexample). Our techniques involve representations into semidirect products, locally indicable groups, Fox–calculus and commutator calculus. Given the f and f_i, the nontriviality of t–translates of a is inductively established by improving a given representation taking into account higher and higher commutators. A brief survey on most of these ideas can be found in [6].

Note that it is essential to *first* choose the modifications f, f_i and *then* work with (a series of) reductions mod higher commutators, as by [5] simple homotopy equivalent K_1, K_2 don't survive a test for Q–inequivalence in a solvable quotient, see II). By a result of A. Kühn this also holds in the relative case.

II) Recently Borovik, Lubotzky and Myasnikov have shown that for the original case of contractible 2–complexes (AC), any test for Q–inequivalence which projects the free group of the generators into a finite test group, must fail [1]. Their result has consequences for F. Quinn's approach via TQFT [11], as all (potential) Q–invariants which ultimately can be read off from the images of the defining relators after such a projection must coincide with those of the trivial presentation. This fact generalizes criteria to exclude certain test groups which were established by W. Browning and Hog–Angeloni/Metzler [5], see [10] and [3]. As M. Bridson has announced a result on lower bounds for the number of elementary Q–transformations which are necessary to transform Q–equivalent presentations into each other [2], we nevertheless pursue the idea to look for a series of test quotients for which such lower bounds become arbitrarily large, thereby disproving (AC).

III) Together with us and with Carsten Cleve, Timo Stey [12] succeeded in defining for every prime a reduced Turaev–Viro type invariant for 3–deformations of 2–complexes. These invariants are easier to compute than those of Quinn, but most probably they have similar properties; hence we don't expect that already a single of them actually distinguishes between different Q^{**}–classes. Simultaneously these reductions nevertheless might do so, see II). These considerations are related to work of Simon King, see the abstract of his talk in this volume. As a general reference for Turaev–Viro type invariants see Matveev [8].

IV) Alexander Kühn and Wolfgang Metzler plan to continue the study of characterizing Andrews–Curtis classes by their Q–stabilizer groups as defined in [9]: Such Q–symmetries of presentations — unlike the McCool stabilizers in the automorphism group of a free group — in general involve decision problems of identities and the second homotopy of presentations. But if we restrict the stabilizers to modifications by Peiffer identities, these decision questions don't show up, and the restricted stabilizer groups still characterize Q–classes.

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