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BRAIDS: GENERALIZATIONS, PRESENTATIONS AND ALGORITHMIC PROPERTIES

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Abstract. Classical braid groups admit several types of presentations. The analogues of these presentations are obtained for the genralizations of braid groups. Garside algorithm for the word problem also works for the singular braid monoid.

We consider various generalizations of braids: surface braid groups [16, 17, 9], Artin - Brieskorn groups [5, 8], complex braid groups [6], braid groups in handlebodies [12, 13], singular braid monoids [2, 3], braid-permutation groups [10], virtual briads [14].

First of all we study several kinds of presentations for these generalizations of braids.

In the initial paper [1] Artin gave the canonical presentation of the braid group on n strings with $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$. In the same article he gave also a presentation of the braid group, with two generators, say σ_1 and σ, and the following relations:

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} & = \sigma^i\sigma_1\sigma^{-i}\sigma_1 \ \text{ for } \ 2 \le i \le n/2, \\ \sigma^n & = (\sigma\sigma_1)^{n-1}. \end{cases}$$

The connection with the canonical generators is given by the formulae:

$$\sigma = \sigma_1\sigma_2 \ldots \sigma_{n-1},$$

$$\sigma_{i+1} = \sigma^i\sigma_1\sigma^{-i}, \quad i = 1, \ldots n-2.$$

We call this presentation the *second Artin presentation*. It is also described in [7]

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J. S. Birman, K. H. Ko and S. J. Lee [4] introduced a presentation with generators a_{ts} with $1 \le s < t \le n$, and relations

$$\begin{cases} a_{ts}a_{rq} &= a_{rq}a_{ts} \text{ for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} &= a_{tr}a_{ts} = a_{sr}a_{tr} \text{ for } 1 \le r < s < t \le n. \end{cases}$$

The generators a_{ts} are expressed by canonical generators σ_i in the following form:

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) \text{ for } 1 \le s < t \le n.$$

For every planar graph Vlad Sergiescu [11] constructed a presentation of the classical braid group group Br_n, where n is the number of vertices of the graph, with generators corresponding to edges and relations reflecting the geometry of the graph.

We give the analogous the second Artin presentation for various generalizations of braids. For example for the complex braid group $B(2e, e, r)$ it has the generators $\tau_2, \tau, \sigma, \tau_2'$ and relations

$$\begin{cases} \tau_2\tau^i\tau_2\tau^{-i} &= \tau^i\tau_2\tau^{-i}\tau_2 \text{ for } 2 \le i \le r/2, \\ \tau^r &= (\tau\tau_2)^{r-1}, \\ \sigma\tau^i\tau_2\tau^{-i} &= \tau^i\tau_2\tau^{-i}\sigma, \text{ for } 1 \le i \le r-2, \\ \sigma\tau_2'\tau_2 &= \tau_2'\tau_2\sigma, \\ \tau_2'\tau\tau_2\tau^{-1}\tau_2' &= \tau\tau_2\tau^{-1}\tau_2'\tau\tau_2\tau^{-1}, \\ \tau\tau_2\tau^{-1}\tau_2'\tau_2\tau\tau_2\tau^{-1}\tau_2'\tau_2 &= \tau_2'\tau_2\tau\tau_2\tau^{-1}\tau_2'\tau_2\tau\tau_2\tau^{-1}, \\ \underbrace{\tau_2\sigma\tau_2'\tau_2\tau_2'\tau_2\tau_2'\dots}_{e+1 \text{ factors}} &= \underbrace{\sigma\tau_2'\tau_2\tau_2'\tau_2\tau_2'\tau_2\dots}_{e+1 \text{ factors}}. \end{cases}$$

The Birman – Ko – Lee presentation for the singular braid monoid SB_n has the generators a_{ts}, a_{ts}^{-1} for $1 \le s < t \le n$ and b_{qp} for $1 \le p < q \le n$ and relations

$$\begin{cases} a_{ts}a_{rq} &= a_{rq}a_{ts} \text{ for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} &= a_{tr}a_{ts} = a_{sr}a_{tr} \text{ for } 1 \le r < s < t \le n, \\ a_{ts}a_{ts}^{-1} &= a_{ts}^{-1}a_{ts} = 1 \text{ for } 1 \le s < t \le n, \\ a_{ts}b_{rq} &= b_{rq}a_{ts} \text{ for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}b_{ts} &= b_{ts}a_{ts} \text{ for } 1 \le s < t \le n, \\ a_{ts}b_{sr} &= b_{tr}a_{ts} \text{ for } 1 \le r < s < t \le n, \\ a_{sr}b_{tr} &= b_{ts}a_{sr} \text{ for } 1 \le r < s < t \le n, \\ a_{tr}b_{ts} &= b_{sr}a_{tr} \text{ for } 1 \le r < s < t \le n, \\ b_{ts}b_{rq} &= b_{rq}b_{ts} \text{ for } (t-r)(t-q)(s-r)(s-q) > 0. \end{cases}$$

As an example of the Sergiescu graph presentation we give the following one for the singular braid monoid.

Theorem 1. *Let Γ be a planar graph with n vertices. The singular braid monoid SB_n has the presentation $\langle X_\Gamma, R_\Gamma \rangle$ where $X_\Gamma = \{\sigma_a, \sigma_a^{-1}, x_a |\ a$ is an edge of $\Gamma\}$ and R_Γ is formed by the following six types of relations:*

- *disjointedness: if the edges a and b are disjoint, then*

$$\sigma_a\sigma_b = \sigma_b\sigma_a,\ x_ax_b = x_bx_a,\ \sigma_ax_b = x_b\sigma_a,$$

- *commutativity:*
$$\sigma_a x_a = x_a \sigma_a,$$
- *invertibility:*
$$\sigma_a \sigma_a^{-1} = \sigma_a^{-1} \sigma_a = 1,$$
- *adjacency: if the edges a and b have a common vertex, then*
$$\sigma_a \sigma_b \sigma_a = \sigma_b \sigma_a \sigma_b, \quad x_a \sigma_b \sigma_a = \sigma_b \sigma_a x_b,$$
- *nodal: if the edges a, b and c have a common vertex and are placed clockwise, then*
$$\sigma_a \sigma_b \sigma_c \sigma_a = \sigma_b \sigma_c \sigma_a \sigma_b = \sigma_c \sigma_a \sigma_b \sigma_c,$$
$$x_a \sigma_b \sigma_c \sigma_a = \sigma_a \sigma_b \sigma_c x_a, \quad \sigma_a \sigma_b x_c \sigma_a = \sigma_b x_c \sigma_a \sigma_b, \quad x_a \sigma_b x_c \sigma_a = \sigma_b x_c \sigma_a x_b,$$
- *pseudocycle: if the edges $a_1, \dots, a_n$ form an irreducible pseudocycle and if a_1 is not the starting edge nor a_n is the end edge of a reverse, then*
$$\sigma_{a_1} \dots \sigma_{a_{n-1}} = \sigma_{a_2} \dots \sigma_{a_n}, \quad x_{a_1} \sigma_{a_2} \dots \sigma_{a_{n-1}} = \sigma_{a_2} \dots \sigma_{a_{n-1}} x_{a_n}.$$

Garside's results and the existense of the greedy normal form for braids are shown to be true for the singular braid monoid in the canonical presentation. For the Birman – Ko – Lee presentation it was done by V. Chaynikov.

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