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ON THE RIGIDITY OF ONE-DIMENSIONAL SYSTEMS OF CONTRACTION SIMILITUDES

ANDREI V. TETENOV

Associated family of similitudes $\mathfrak{F}(S)$ for a system S of contraction similitudes in $\mathbb{R}^d$ was primarily introduced by C.Bandt and S.Graf in their work [1] as a tool for checking, whether the invariant set $K(S)$ of the system S has positive Hausdorff measure. This approach was developed in [3],[5],[2], leading to formulation of weak separation property (WSP) and it's application to graph-directed systems.

Nevertheless, thorough consideration of the associated family of similitudes allows to throw light upon some other properties of self-similar sets. If the identity map is a limit point for the associated family $\mathfrak{F}(S)$, the system S demonstrates the rigidity phenomenon. We regard it in the case when the invariant set $K(S)$ of the system S is the segment $[0,1]$. Our main result is that in this situation any continuous map $\varphi : K(S) \to K(T)$ of the attractor of the system S to the attractor of the other system of contraction similitudes T which agrees with the structure of self-similar set on the sets $K(S)$ and $K(T)$ is a linear map of $[0,1]$ to a straight line segment.

1. Definitions and notation. A compact set $K = K(S)$ in $\mathbb{R}^d$ is called an *attractor* or *invariant set* of the system $S = \{S_1, \dots, S_m\}$ of contraction similitudes in $\mathbb{R}^d$ if $K = S_1(K) \cup ... \cup S_m(K)$. Existence and uniqueness of the set K follow from theorem of Hutchinson ([4, стр. 724]).

We denote by $I = \{1, \dots, m\}$ a set of first m natural numbers , by I^k – the set of *multiindices* of length k and $I^* = \bigcup\limits_{k=1}^{\infty} I^k$. Given a system $S = \{S_1, \dots, S_m\}$ of maps of $\mathbb{R}^d$ to itself, for each multiindex $\mathbf{i} \in I^k$, $\mathbf{i} = i_1 i_2 \dots i_k$, we define $S_{\mathbf{i}} = S_{i_1} \cdot S_{i_2} \cdot \dots \cdot S_{i_k} : \mathbb{R}^d \to \mathbb{R}^d$.

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We denote by $\mathrm{Lip}(g)$ the expansion ratio of the similitude g, and by $\mathrm{fix}(g)$ we denote it's fixed point. If g is a translation we put $\mathrm{fix}(g) = \infty$.

1.1 The associated family. A family of all transformations having the form ${S_{\mathbf{i}}}^{-1} \cdot S_{\mathbf{j}} : \mathbb{R}^d \to \mathbb{R}^d$, where $\mathbf{i} \in I^*$, $\mathbf{j} \in I^*$ and $i_1 \neq j_1$ we call, following [1, p.996], the *associated family* $\mathfrak{F}(S)$ of similitudes of the system S .

1.2 Structure-preserving homeomorphisms. Let $S = \{S_1, ..., S_m\}$ and $T = \{T_1, ..., T_m\}$ be two systems of contraction maps of complete metric spaces X_1 and X_2 to themselves. Let $K(S) \subset X_1$ and $K(T) \subset X_2$ be the attractors of these two systems . A continuous map $\varphi : K(S) \to K(T)$ is said to be *structure-preserving*, if

$$\forall x \in K(S) \ \ \forall i \in I \ \ \varphi(S_i(x) = T_i(\varphi(x)) \tag{1}$$

It follows immediately from this definition, that for any multiindex $\mathbf{j} \in I^*$, and for any $x \in [0,1]$ the relation $\varphi(S_{\mathbf{j}}(x) = S_{\mathbf{j}}(\varphi(x))$ is also valid. The same is also true for the elements of associated families:

Proposition 1. *Let $\varphi : K(S) \to K(T)$ – be a structure-preserving map of the attractors of systems S and T. Let $g = {S_{\mathbf{i}}}^{-1} \cdot S_{\mathbf{j}}$ and $h = {T_{\mathbf{i}}}^{-1} \cdot T_{\mathbf{j}}$ – be the elements of $\mathfrak{F}(S)$ and $\mathfrak{F}(T)$. The relation $\varphi \cdot g(x) = h \cdot \varphi(x)$ holds for any $x \in K(S) \cap g^{-1}(K(S))$.*

Two systems $S = \{S_1, ..., S_m\}$ and $T = \{T_1, ..., T_m\}$ are called *structurally equivalent*, if there is a structure-preserving homeomorphism $\varphi : K(S) \to K(T)$.

2. The main results.

The first statement belongs to the case when each of the invariant sets is [0,1]:

Theorem 1. *Let $S = \{S_1, \dots, S_m\}$, $T = \{T_1, \dots, T_m\}$ be systems of contraction similitudes in $\mathbb{R}$, invariant set of each being the segment $[0,1]$, and let $\varphi : K(S) \to K(T)$ be a structure-preserving homeomorphism for these two systems, such that $\varphi(0) = 0$ and $\varphi(1) = 1$. If* Id *is a limit point of the associated family $\mathfrak{F}(S)$ for the system S, then $\varphi(x) \equiv x$, and $S = T$.*

The scheme of the proof is the following:

Let $g_n = S_{\mathbf{i}}^{-1} S_{\mathbf{j}}$ be such a sequence of elements of the family $\mathfrak{F}(S)$, that $g_n \to \mathrm{Id}$ (and $g_n([0,1]) \cap [0,1] \neq \emptyset$), and let $h_n = T_{\mathbf{i}}^{-1} T_{\mathbf{j}}$ be a sequence of elements of $\mathfrak{F}(S)$, which are conjugate to g_n via the homeomorphism φ in the sense of Proposition 1 . We prove that this pair of conjugate sequences may be modified to satisfy the following conditions:

1. Each map g_n, h_n preserves orientation on $\mathbb{R}$.

2. Either (A): all similitudes g_n are translations $g_n(x) = x + \Delta_n$, where $\Delta_n > 0$,

or (B): the expansion ratios $p_n = \mathrm{Lip}(g_n) \neq 1$, $g_n(0) \geq 0$ and the sequence of points $\mathrm{fix}(g_n)$ converges to infinity;

3. Either (I): all similitudes h_n are translations $h_n(x) = x + \delta_n, \delta_n > 0$,

or (II): the expansion ratios $q_n = \mathrm{Lip}(h_n) \neq 1$, $g_n(0) \geq 0$ and the sequence of points $\mathrm{fix}(h_n)$ converges to infinity;

4. In the case (,II) there is $\lim\limits_{n\to\infty} \log_{p_n} q_n = \alpha$;

5. For any n, $g_n(0) > 0, h_n(0) > 0$.

Let $\Gamma(\varphi) \subset [0,1] \times [0,1]$ be the graph of the homeomorphism φ. Take some $\varepsilon \in (0, 1/4)$.

For n sufficiently large, the points $(g_n^k(0), h_n^k(0))$, where k runs from 0 to $M_n = max\{k : g_n^k(0) \leq 1\}$, form an $\varepsilon-$net in the set $\Gamma(\varphi)$. At the same time these points belong to an arc γ, specified by an equation $y = \psi(x), x \in [0,1]$, where:

$\psi(x) = \dfrac{\Delta_n}{\delta_n} x$ in the case (AI);

$\psi(x) = L(e^{\lambda x} - 1)$, where $\lambda = \dfrac{\log q_n}{\Delta_n}$, $L = -\operatorname{fix}(h_n)$ in the case (AII);

$\psi(x) = L'\left(1 - \left(1 - \dfrac{x}{L}\right)^{\lambda}\right)$, $\lambda = \dfrac{\log q_n}{\log p_n}$, $L = -\operatorname{fix}(g_n)$, $L' = -\operatorname{fix}(h_n)$ in the case (BII).

In each of these three cases we proof that for n sufficiently large, Hausdorff distance between γ and a segment $C = \{y = x, x \in [0,1]\}$, is not greater than ε. Since ε may be arbitrarily small, γ and C(and therefore $\Gamma(\varphi)$ and C) coincide, hence $\varphi(x) \equiv x$.

The second statement treats structure-preserving homeomorphisms $\varphi : K(S) \to K(T)$ in the case, when $K(S) = [0,1]$, and $K(T)$ is a Jordan arc.

Theorem 2. *Let $S = \{S_1, \dots, S_m\}$ be a system of contraction similitudes in $\mathbb{R}$, whose attractor is segment $[0,1]$. Let Jordan arc $\gamma \in \mathbb{R}^d$ with endpoints a_0, a_1 is the attractor of a system $T = \{T_1, ..., T_m\}$ of contraction similitudes in $\mathbb{R}^d$. Let $\varphi : [0,1] \to \gamma$ be a structure-preserving homeomorphism for the systems S and T.*

If Id is a limit point for the family $\mathfrak{F}(S)$, then the arc γ is a straight line segment with endpoints a_0, a_1.

The method of the proof of this theorem is similar to the previous one. Assuming, that the affine hull of γ is $\mathbb{R}^d$, we build such sequence $g_n \in \mathfrak{F}(S)$, that for it and for the sequence $h_n \in \mathfrak{F}(T)$, conjugate to g_n via homeomorphism φ, the following conditions are satisfied:

1. g_n and h_n preserve orientation in [0,1] and γ correspondingly;
2. $g_n(0) > 0$;
3. a sequence of distances Δ_n from the point a_0 to the closest proper strictly invariant affine codimension 2 subspace V_n of similitude h_n converges to infinity.

Take $\varepsilon > 0$. For n sufficiently large, the points $h_n^k(a_0)$, where k runs from 0 to $M_n = max\{k : g_n^k(0) \le 1\}$ form an $\varepsilon-$net in γ. At the same time they all are contained in an arc of a loxodrome $L_n = \{h_n^t(a_0), t \in [0, M_n]\}$. By virtue of condition 3, n may be chosen so that Hausdorff distance between the arc L_n and a straight line segment $[a_0, a_1]$ is no greater than ε. Since ε may be arbitrarily small, this means that γ is equal to the segment $[a_0, a_1]$.

Using the result of Theorem 2 and slightly modifying the argument in the proof of Theorem 3, we obtain:

Theorem 3. *Let $S = \{S_1, \dots, S_m\}$ be a system of contraction similitudes in $\mathbb{R}$, whose attractor is segment $[0,1]$. Let a continuum $\gamma \in \mathbb{R}^d$ be the attractor of a system $T = \{T_1, ..., T_m\}$ of contraction similitudes in $\mathbb{R}^d$. Let $\varphi : [0,1] \to \gamma$ be a structure-preserving map of the attractors of the systems S and T.*

If the identity map Id is a limit point for the family $\mathfrak{F}(S)$, then γ is a straight line segment, and φ is a linear map.

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Andrei V. TETENOV
Gorno-Altaisk State University,
Lenkin st. 1,
649000, Gorno-Altaisk, Russia
E-mail address: atet@gasu.ru