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ON UNIFORMLY CONTINUOUS OPERATORS AND SOME WEIGHT-HYPERBOLIC FUNCTION BANACH ALGEBRA

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Abstract. We consider an abelian non-unitary Banach algebra $\mathfrak{A}$, ruled by an hyperbolic weight, defined on certain space of Lebesgue measurable complex valued functions on the positive axis. Since the non-convolution Banach algebra $\mathfrak{A}$ has its own interest by its applications to the representation theory of some Lie groups, we search on various of its properties. As a Banach space, $\mathfrak{A}$ does not have the Radon-Nikodým property. So, it could be exist not representable linear bounded operators on $\mathfrak{A}$ (cf. [6]). However, we prove that the class of locally absolutely continuous bounded operators are representable and we determine their kernels.

1. Introduction

Let $\mathfrak{A} = \mathcal{L}^1_{\mathbb{C}}(\sinh(2t)\,dt)$, i.e. $\mathfrak{A}$ contains those Lebesgue complex measurable functions x on $\mathbb{R}^+$ so that $\int_0^\infty |x(t)|\sinh(2t)dt < \infty$. If $\|x\|$ denotes the former integral $(\mathfrak{A}, \|\circ\|)$ becomes a Banach space. Indeed, it is a Banach algebra if for $x, y \in \mathfrak{A}$ and for $u > 0$ we set

$$(x \odot y)(u) = \int_0^\infty x(t) \int_{|u-t|}^{u+t} y(s)dsdt. \tag{1}$$

For, if $x, y \in \mathfrak{A}$ we write $\mathfrak{R}$ for the unbounded solid of points (u, t, s) so that $u > 0, t > 0$ and $|u - t| < s < u + t$. There exist Borel functions x_0, y_0 so that $x = x_0$ a.e. and $y = y_0$ a.e.. So, it is no lose of generality in assuming that both x and y are Borel functions. The functions $\beta, \gamma : \mathfrak{R} \to \mathbb{R}^+$ so that $\beta(u, t, s) = t$ and

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$\gamma(u,t,s)=s$ are Borel functions. Thus $x\circ\beta$, $y\circ\gamma$ and also $(x\circ\beta)\cdot(y\circ\gamma)$ are Borel functions. Next, by Fubini-Tonnelli theorem we obtain

$$\begin{aligned}\iiint_{\mathfrak{R}} |(x\circ\beta)\cdot(y\circ\gamma)|\,du\times dt\times ds &= \int_0^\infty\int_0^\infty |x(t)| \int_{|u-t|}^{u+t} |y(s)|\,ds\ dt\sinh(2u)du \\ &= \int_0^\infty\int_0^\infty |x(t)y(s)| \int_{|t-s|}^{t+s} \sinh(2u)\,du\ dt\ ds \\ &= \int_0^\infty\int_0^\infty |x(t)y(s)|\sinh(2t)\sinh(2s)dt\ ds \\ &= \|x\|\,\|y\| < \infty.\end{aligned}$$

Therefore, by the Fubini-Tonelli theorem (1) defines an almost everywhere complex measurable function $x\odot y$ and $\|x\odot y\|\le\|x\|\,\|y\|$, i.e. $\mathfrak{A}$ becomes a Banach algebra. It is straightforward to see that $\mathfrak{A}$ is an abelian algebra. Moreover, $\mathfrak{A}\subseteq L^1(a,\infty)$ if $a>0$. For, if $x\in\mathfrak{A}$ and $0<a$ then

$$\int_a^\infty |x(t)|\,dt \le \|x\|/\sinh(2a)<\infty.$$

Consequently, $\mathfrak{A}$ has no unity. For, let us suppose that there is a unit μ of $\mathfrak{A}$, i.e. if $x\in\mathfrak{A}$ then

$$x(\varepsilon)=\int_0^\infty x(t)\int_{|t-\varepsilon|}^{t+\varepsilon}\mu(s)dsdt \quad a.e.\ \varepsilon>0. \tag{2}$$

So, let $0<a<b$ and let $x\in\mathfrak{A}$ so that $x(t)=0$ a.e. $0<t<a$. By (2) we have

$$\int_a^b x(t)\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\mu(s)dsdt=0 \quad a.e.\ 0<\varepsilon<a. \tag{3}$$

Observe that x and μ become locally integrable functions on $\mathbb{R}^+$. Moreover, if $0<\varepsilon<a/2$ then

$$\left| x(t)\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\mu(s)ds\right| \le \frac{|x(t)|}{2}\int_{-1}^{1}|\mu(t+\sigma\varepsilon)|\,d\sigma \le \frac{|x(t)|}{2}\int_{a/2}^{a+b}|\mu|\,.$$

But $x(t)\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}\mu(s)ds \to x(t)\mu(t)$ a.e. $a<t<b$. So, letting $\varepsilon\to 0^+$ in (3) the Lebesgue dominated convergence theorem gives $\int_a^b x(t)\mu(t)dt=0$. If we choose $x=\overline{\mu}\,|_{(a,b)}$then $\int_a^b|\mu(t)|^2\,dt=0$, i.e. $\mu=0$ a.e. $a<t<b$. Since a and b are arbitrary relation (2) becomes clearly absurd and our claim holds.

Recently, the authors considered questions concerning to the existence and the structure of bounded derivations within certain Banach algebras (cf. [2], [3]). In the frame of formal weighted convolution Banach algebras of power series the reader can see [1]. The Banach algebra $\mathfrak{A}$ is not a convolution algebra, i.e. it can not be identified with any subalgebra of the Fréchet algebra of formal power series (cf. [7]). The discrete version of $\mathfrak{A}$ is related to the study of irreducible linear representations of the Lie group $SU(2)$ (see [5], Ch. XXI, §9, p. 71). So, the analysis of the structure of this algebra has its own interest. In Section 2 we´ll describe the maximal ideal space and the Gelfand transform of $\mathfrak{A}$. In Section 3 we search on the structure of locally absolutely continuous operators on $\mathfrak{A}$, i.e. bounded operators A on $\mathfrak{A}$ so that $E \to A(\chi_E)$ is realized as the indefinite integral of a Bochner integrable $\mathfrak{A}$ - valued function on $\mathbb{R}^+$ if E is any Lebesgue measurable subset of finite measure of $\mathbb{R}^+$.

2. The maximal ideal space $\mathfrak{X}(\mathfrak{A})$

Theorem 1. *The maximal ideal space $\mathfrak{X}(\mathfrak{A})$ is homeomorphic to the one-point compactification of the strip $S = \{\upsilon \in \mathbb{C} : |Re(\upsilon)| \leq 2\}$. Moreover, for $x \in \mathfrak{A}$ and $\upsilon \in \mathfrak{X}(\mathfrak{A})$ the Gelfand transform $\mathfrak{G} : \mathfrak{A} \to C_{\mathbb{C}}(\mathfrak{X}(\mathfrak{A}))$ can be written as*

$$(4) \qquad \mathfrak{G}(x)(\upsilon) = \begin{cases} 2\int_0^\infty x(t)t\,dt & if \quad \upsilon = 0, \\ 2/\upsilon \int_0^\infty \sinh(\upsilon t)\, x(t)\,dt & if \quad |Re(\upsilon)| \leq 2,\ \upsilon \neq 0, \\ 0 & if \quad \upsilon = \infty. \end{cases}$$

Proof. Given $x \in \mathfrak{A}$ the function $\Lambda x(t) = x(t)\sinh(2t)$ is well defined a.e. $t > 0$. Hence it is defined a linear operator $\Lambda : \mathfrak{A} \to \mathcal{L}^1_{\mathbb{C}}(\mathbb{R}^+)$. It is easy to see that Λ is an isometric isomorphism between $\mathfrak{A}$ and $\mathcal{L}^1_{\mathbb{C}}(\mathbb{R}^+)$. Since $\mathcal{L}^1_{\mathbb{C}}(\mathbb{R}^+)^* \simeq \mathcal{L}^\infty_{\mathbb{C}}(\mathbb{R}^+)$ given $\mathfrak{h} \in \mathfrak{A}^*$ there is a unique $\widetilde{\omega}_{\mathfrak{h}} \in \mathcal{L}^\infty_{\mathbb{C}}(\mathbb{R}^+)$ so that $\mathfrak{h}(x) = \int_0^\infty x(a)\widetilde{\omega}_{\mathfrak{h}}(a)\sinh(2a)da$ if $x \in \mathfrak{A}$. Thus, if $\mathfrak{h} \in \mathfrak{X}(\mathfrak{A})$, $x, y \in \mathfrak{A}$ and we write $\omega_{\mathfrak{h}}(a) = \widetilde{\omega}_{\mathfrak{h}}(a)\sinh(2a)$ if $a > 0$ by Fubini´s theorem we have

$$(5) \qquad \begin{aligned} \int_0^\infty x(a)\omega_{\mathfrak{h}}(a)da \int_0^\infty y(b)\omega_{\mathfrak{h}}(b)db &= \int_0^\infty \left(\int_0^\infty x(a) \int_{|a-c|}^{a+c} y(b)\,db\,da \right) \omega_{\mathfrak{h}}(c)dc \\ &= \int_0^\infty \int_0^\infty x(a)y(b) \int_{|a-b|}^{a+b} \omega_{\mathfrak{h}}(c)dc\ db\ da \end{aligned}$$

because $\mathfrak{h}(x \odot y) = \mathfrak{h}(x)\mathfrak{h}(y)$. Let us fix $s > 0, t > 0$ and for each positive integer n let $x = \chi_{(s-1/n,s+1/n)}$, $y = \chi_{(t-1/n,t+1/n)}$. Now (5) reads

$$(6) \qquad \int_{s-1/n}^{s+1/n} \omega_{\mathfrak{h}}(a)da \int_{t-1/n}^{t+1/n} \omega_{\mathfrak{h}}(b)db = \int_{s-1/n}^{s+1/n} \int_{t-1/n}^{t+1/n} \int_{|a-b|}^{a+b} \omega_{\mathfrak{h}}(c)dc\ db\ da$$

Multiplying both sides in (6) by $4/n^2$ and noting that $\omega_{\mathfrak{h}} \in \mathcal{L}^1_{\mathbb{C}}(0,\eta)$ for all $\eta > 0$ we deduce that

$$\omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}}(t) = \int_{|s-t|}^{s+t} \omega_{\mathfrak{h}}(c)dc \text{ a.e. } s,t > 0.$$

Accordingly,

$$\omega_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}}(t) = \int_{-s}^{s} \omega_{\mathfrak{h}}(t+c)\,dc \text{ a.e. } 0 < s < t. \tag{7}$$

Thus, as $\omega_{\mathfrak{h}}$ is finite almost everywhere and $\omega_{\mathfrak{h}} \in \mathcal{L}^1_{loc}(\mathbb{R}^+)$ then it becomes a locally absolutely continuous function on $\mathbb{R}^+$ and $\lim_{s\to 0^+} \omega_{\mathfrak{h}}(s) = 0$. Now, from (7) we have that

$$\omega^{(1)}_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}}(t) = \omega_{\mathfrak{h}}(t+s) + \omega_{\mathfrak{h}}(t-s) \text{ a.e. } 0 < s < t. \tag{8}$$

By (8) we may suppose that $\omega_{\mathfrak{h}}$ is infinitely differentiable. Indeed, if $n \in \mathbb{N}$ then

$$\omega^{(n)}_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}}(t) = \omega^{(n-1)}_{\mathfrak{h}}(t+s) + (-1)^{n-1}\omega^{(n-1)}_{\mathfrak{h}}(t-s) \text{ if } 0 < s < t. \tag{9}$$

So, we can write $\omega^{(n)}_{\mathfrak{h}}(0^+) = \lim_{s\to 0^+} \omega^{(n)}_{\mathfrak{h}}(s)$ and all right derivatives of even order of $\omega_{\mathfrak{h}}$ vanish at zero. Moreover, $\omega^{(1)}_{\mathfrak{h}}(0^+) \cdot \omega_{\mathfrak{h}}(t) = 2\omega_{\mathfrak{h}}(t)$ if $t > 0$. If $\omega_{\mathfrak{h}} \neq 0$ then $\omega^{(1)}_{\mathfrak{h}}(0^+) = 2$. By means of (9) and by continuity we get

$$\omega^{(3)}_{\mathfrak{h}}(s) \cdot \omega_{\mathfrak{h}}(t) = \omega^{(2)}_{\mathfrak{h}}(t+s) + \omega^{(2)}_{\mathfrak{h}}(t-s) \text{ if } 0 < s \leq t. \tag{10}$$

So,

$$\omega^{(3)}_{\mathfrak{h}}\left(0^+\right) \cdot \omega_{\mathfrak{h}}(t) = 2\omega^{(2)}_{\mathfrak{h}}(t) \text{ if } t > 0. \tag{11}$$

Hence, if $\omega^{(3)}_{\mathfrak{h}}(0^+) = 0$ we deduce that $\omega^{(2)}_{\mathfrak{h}} \equiv 0$. Consequently, $\omega^{(1)}_{\mathfrak{h}} \equiv 2$ and so $\omega_{\mathfrak{h}}(t) = 2t$. Then, we´ll seek for the infinitely differentiable solutions $\omega_{\mathfrak{h}}$ of (11) whose all right limits at zero exist, their derivatives of even order vanish at 0^+, $\omega_{\mathfrak{h}}(0^+) = 0$ and $\omega^{(1)}_{\mathfrak{h}}(0^+) = 2$. On fixing a branch of the complex square root function, the unique solution with a prescribed non-zero complex value $\gamma = \omega^{(3)}_{\mathfrak{h}}(0^+)$ is computed as

$$\omega_{\mathfrak{h}}(t) = 2\sqrt{2/\gamma}\sinh\left(t\sqrt{\gamma/2}\right),$$

i.e. $\omega_{\mathfrak{h}}$ must be a function on $\mathbb{R}^+$ of the form $\omega_{\mathfrak{h}}(t) = 2\sinh(\upsilon t)/\upsilon$. Since the quotient

$$\left|\frac{\sinh(\upsilon t)}{\sinh(2t)}\right|^2 = \frac{\sinh^2(t\mathrm{Re}(\upsilon)) + \sin^2(t\mathrm{Im}(\upsilon))}{\sinh^2(2t)}$$

was required to be bounded on $\mathbb{R}^+$ then $|\mathrm{Re}(\upsilon)| \leq 2$. ∎

Remark 2. *We just proved that all complex homomorphisms on $\mathfrak{U}$ are of the form (4). In particular, given $x, y \in \mathfrak{U}$ we have*

$$\begin{aligned}
\mathfrak{G}(x \odot y)(0) &= 2\int_0^\infty u \int_0^\infty x(t) \int_{|u-t|}^{u+t} y(s) ds\ dt\ du \\
&= 2\int_0^\infty u \left(\int_0^u x(t) \int_{u-t}^{u+t} y(s) ds dt + \int_u^\infty x(t) \int_{t-u}^{u+t} y(s) ds dt \right) du \\
&= \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left(x(t) y(s)\ \ \left[u^2\right]_{\max\{t, s-t\}}^{s+t} \right) ds \times dt + \\
&\quad + \int_0^\infty x(t) \left(\int_0^{2t} y(s) \left[u^2\right]_{|s-t|}^{t} \right)\ ds\ dt \\
&= \int_0^\infty x(t) \left(\int_0^{2t} y(s) \left(s^2 + 2st\right) ds + \int_{2t}^\infty y(s) 4st\ ds \right) dt + \\
&\quad + \int_0^\infty x(t) \int_0^{2t} y(s) \left(2st - s^2\right) ds\ dt \\
&= 2\int_0^\infty x(t) t dt\ 2\int_0^\infty y(s) s ds = \mathfrak{G}(x)(0) \cdot \mathfrak{G}(y)(0).
\end{aligned}$$

Indeed, if $|Re(\upsilon)| \le 2$, $\upsilon \ne 0$ then

$$\begin{aligned}
\mathfrak{G}(x \odot y)(\upsilon) &= 2/\upsilon \int_0^\infty \sinh(\upsilon u) \int_0^\infty x(t) \int_{|u-t|}^{u+t} y(s) ds\ dt\ du \\
&= 2/\upsilon \int_0^\infty x(t) \int_0^{2t} y(s) \int_{|s-t|}^{t} \sinh(\upsilon u) du\ ds\ dt + \\
&\quad + 2/\upsilon \int_0^\infty x(t) \int_0^\infty y(s) \int_{\max\{s-t, t\}}^{s+t} \sinh(\upsilon u) du\ ds\ dt \\
&= 2/\upsilon \int_0^\infty x(t) \int_0^t y(s) \left(\int_{|s-t|}^{t} + \int_t^{s+t} \right) \sinh(\upsilon u) du\ ds\ dt + \\
&\quad + 2/\upsilon \int_0^\infty x(t) \int_t^{2t} y(s) \left(\int_{|s-t|}^{t} + \int_t^{s+t} \right) \sinh(\upsilon u) du\ ds\ dt + \\
&\quad + 2/\upsilon \int_0^\infty x(t) \int_{2t}^\infty y(s) \int_{s-t}^{s+t} \sinh(\upsilon u) du\ ds\ dt
\end{aligned}$$

$$
\begin{aligned}
&= 2/\upsilon \int_0^\infty \int_0^\infty x(t)y(s) \left[\frac{\cosh(\upsilon u)}{\upsilon}\right]_{|s-t|}^{s+t} ds \times dt \\
&= 2/\upsilon \int_0^\infty \sinh(\upsilon t)\, x(t)dt \; 2/\upsilon \int_0^\infty \sinh(\upsilon s)\, y(s)ds \\
&= \mathfrak{G}(x)(\upsilon) \cdot \mathfrak{G}(y)(\upsilon).
\end{aligned}
$$

3. Locally absolutely continuous operators on $\mathfrak{A}$

A Banach space $\mathbb{X}$ has the Radon-Nikodým property with respect to a finite measure space (Ω, Σ, μ) if for each $\mu-$ continuous vector measure $\Sigma \underset{\rightarrow}{G} \mathbb{X}$ of bounded variation there exists $g \in \mathcal{L}^1(\mu, \mathbb{X})$ such that $G(E) = \int_E g d\mu$ for all $E \in \Sigma$. Further, $\mathbb{X}$ is said to have the Radon-Nikodým property if it has the Radon-Nikodým property with respect to any finite measure space. In particular, if $\mathfrak{L}$ denotes the σ-algebra of measurable subsets of $\mathbb{R}^+$ the measure space $(\mathbb{R}^+, \mathfrak{L}, \sinh(2t)dt)$ has no atoms. In fact, if $E \in \mathfrak{L}$ and $\int_E \sinh(2t)dt > 0$ then E must have positive Lebesgue measure. So, there exists a Lebesgue measurable subset F of E whose measure is positive and smaller than that of E. So, $0 < \int_F \sinh(2t)dt < \int_E \sinh(2t)dt$ and our assertion follows. By ([4], Ch. II, §1, p. 61) we conclude that the Radon-Nikodým property is not fulfilled by the σ-finite measure space $\mathcal{L}^1_{\mathbb{C}}(\mathbb{R}^+, \sinh(2t)dt)$. Nevertheless, we can still characterize the class of locally absolutely continuous operators on $\mathfrak{A}$.

Theorem 3. *If $A \in \mathcal{B}(\mathfrak{A})$, A is a locally absolutely continuous operator if and only if there is an a.e. uniquely determined $\mathfrak{A}$-valued function f_A so that f_A is Bochner integrable on $(0,\tau)$ for all $\tau > 0$, the function $t \to f_A(t)/\sinh(2t)$ is essentially bounded and*

$$
Ax(s) = \int_0^\infty x(t) f_A(t) s dt
$$

for all $x \in \mathfrak{A}$ and almost all $s > 0$.

Proof. Let f be a Bochner integrable $\mathfrak{A}$ - valued function on $(0,\tau)$ for all $\tau > 0$. If we fix $\tau > 0$, f and $\|f\|$ become Bochner measurables and $\|f\| \in \mathcal{L}^1(0,\tau)$ (cf. [4], Ch. II, §2, Th. 2). Therefore, the function $(t,s) \to f(t)s\sinh(2s)$ belongs to $\mathcal{L}^1_{\mathbb{C}}((0,\tau) \times \mathbb{R}^+)$ and so it is Lebesgue measurable. Further, if the function $t \to \|f(t)\| / \sinh(2t)$ is essentially bounded on $\mathbb{R}^+$ and $x \in \mathfrak{A}$ we apply the Fubini-Tonelli theorem to get

$$
\begin{aligned}
(14) \quad \int_0^\infty \int_0^\infty |x(t)f(t)s|\, dt \sinh(2s) ds &= \int_0^\infty |x(t)| \int_0^\infty |f(t)s| \sinh(2s)\, ds dt \\
&\leq \|x\| \operatorname{esssup}_{t>0} \|f(t)\| / \sinh(2t) < \infty.
\end{aligned}
$$

Thus $\int\limits_0^\infty |x(t)f(t)s|\,dt < \infty$ a.e. $s > 0$, i.e. the function $s \to \int\limits_0^\infty x(t)f(t)sdt$ is defined and finite almost everywhere on $\mathbb{R}^+$. So, writting $(Ax)\,s = \int\limits_0^\infty x(t)f(t)sdt$ for $x \in \mathfrak{A}$ and for almost all $s > 0$ then $Ax \in \mathfrak{A}$. Indeed, $A \in \mathcal{B}(\mathfrak{A})$, by (14) is $\|A\| \le \operatorname{esssup}_{t>0} \|f(t)\| / \sinh(2t)$ and if E is a Lebesgue measurable subset of finite measure of $\mathbb{R}^+$ then

$$A(\chi_E) = \int_E f(t)dt, \tag{15}$$

i.e. A is a locally absolutely continuous operator on $\mathfrak{A}$ and the conditions are sufficient. Assume conversely that A is a locally absolutely continuous linear operator on $\mathfrak{A}$ and let $f : \mathbb{R}^+ \to \mathfrak{A}$ be a Bochner integrable function that satisfies (15) whenever E is a measurable subset of finite measure of $\mathbb{R}^+$. In particular, there exists a sequence of simple functions (f_υ) such that

$$\int\limits_0^\tau \|f(t) - f_\upsilon(t)\|\,dt \to 0 \tag{16}$$

and $A\left(\chi_{(0,\tau)}\right) = \lim_{\upsilon\to\infty} \int\limits_0^\tau f_\upsilon(t)dt$ for each $\tau > 0$. By (16) and as

$$\int\limits_0^\tau \|f(t) - f_\upsilon(t)\|\,dt = \int\limits_0^\infty \int\limits_0^\tau |f(t)s - f_\upsilon(t)s|\,dt \sinh(2s)\,ds$$

there is an increasing sequence (υ_j) so that $\int\limits_0^\tau \left|f(t)s - f_{\upsilon_j}(t)s\right| dt \to 0$ a.e. $s > 0$. Let´s write $f_\upsilon(t) = \sum\limits_{h=1}^{h(\upsilon)} x_\upsilon^h \chi_{E_\upsilon^h}(t)$, with $\upsilon, h(\upsilon) \in \mathbb{N}$, $t \in \mathbb{R}^+$, $x_\upsilon^h \in \mathfrak{A}$ and where the E_υ^h´s are measurable subsets of $\mathbb{R}^+$ of finite Lebesgue measure for all pairs υ, h ´s. Hence, if $\tau > 0$ and m denotes the Lebesgue measure on $\mathbb{R}^+$ we get

$$A\left(\chi_{(0,\tau)}\right) = \lim_{j\to\infty} \int\limits_0^\tau f_{\upsilon_j}(t)dt = \lim_{j\to\infty} \sum_{h=1}^{h(\upsilon_j)} x_{\upsilon_j}^h m\left(E_{\upsilon_j}^h\right)$$

in $\mathfrak{A}$. Thus there is a subsequence (υ_{j_l}) of (υ_j) so that

$$A\left(\chi_{(0,\tau)}\right) s = \lim_{l\to\infty} \sum_{h=1}^{h\left(\upsilon_{j_l}\right)} x_{\upsilon_{j_l}}^h(s) m\left(E_{\upsilon_{j_l}}^h\right) = \lim_{l\to\infty} \int\limits_0^\tau f_{\upsilon_{j_l}}(t)sdt \text{ a.e. } s > 0. \tag{17}$$

Since (16) holds if we replace υ by υ_{j_l} we conclude that $A\left(\chi_{(0,\tau)}\right) s = \int\limits_0^\tau f(t)sdt$ a.e. $s > 0$. The $\mathfrak{A}$ - valued function $\tau \to A\left(\chi_{(0,\tau)}\right)$ becomes almost derivable and

$$\frac{\partial A\left(\chi_{(0,\tau)}\right)}{\partial \tau} = f(\tau) \text{ a.e. } \tau > 0$$

(see [4], Ch. II, §2, Th. 9). So, if $\lim_{n\to\infty} \left\|f(\tau) - nA(\chi_{(\tau,\tau+1/n)})\right\| = 0$ we can choose an increasing sequence (n_k) that depends on τ so that

$$f(\tau)s = \lim_{k\to\infty} n_k A(\chi_{(\tau,\tau+1/n_k)})s \quad \text{a.e. } s > 0.$$

Thus, due to Fatou´s lemma we obtain

$$\begin{aligned}
(18)\qquad \|f(\tau)\| &= \int_0^\infty \left| \frac{\partial A\left(\chi_{(0,\tau)}\right)}{\partial \tau}(s) \right| \sinh(2s)\, ds \\
&= \int_0^\infty \lim_{k\to\infty} n_k \left| A(\chi_{(\tau,\tau+1/n_k)})s \right| \sinh(2s) ds \\
&\le \underline{\lim}_{k\to\infty} n_k \int_0^\infty \left| A(\chi_{(\tau,\tau+1/n_k)})s \right| \sinh(2s) ds \\
&= \underline{\lim}_{k\to\infty} n_k \left\| A(\chi_{(\tau,\tau+1/n_k)}) \right\| \le \|A\| \sinh(2\tau) \ \text{ a.e. } \tau > 0,
\end{aligned}$$

i.e. the function $t \to f(t)/\sinh(2t)$ becomes essentially bounded on $\mathbb{R}^+$. Finally, the class $S_{\mathbb{C}}(\mathbb{R}^+)$ of complex valued simple functions on $\mathbb{R}^+$ is dense in $\mathfrak{A}$. For, if $x \in \mathfrak{A}$ so that $x \ge 0$ a.e. on $\mathbb{R}^+$ there is an increasing sequence $\{x_n\}$ in $S_{\mathbb{R}^+}(\mathbb{R}^+)$ bounded above by x so that $x = \lim_{n\to\infty} x_n$ almost-everywhere on $\mathbb{R}^+$. So, $0 \le x_n(t)\sinh(2t) \le x(t)\sinh(2t)$ a.e. $t > 0$ and by the dominated convergence theorem is

$$(19)\qquad \lim_{n\to\infty} \|x - x_n\| = \lim_{n\to\infty} \int_0^\infty (x(t) - x_n(t)) \sinh(2t)\, dt = 0.$$

The general case follows by considering the positive and negative parts of $\operatorname{Re}(x)$ and $\operatorname{Im}(x)$. Therefore, given $x \in \mathfrak{A}$ let $\{x_n\} \subseteq S_{\mathbb{R}^+}(\mathbb{R}^+)$ so that (19) holds. As above, we can deduce that each function $(t,s) \to (x_n(t) - x(t)) f(t) s \sinh(2s)$ is Lebesgue measurable on $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, if $n \in \mathbb{N}$ by (18) we have

$$\begin{aligned}
\left\| A(x_n) - \int_0^\infty x(t) f(t) dt \right\| &= \left\| \int_0^\infty (x_n(t) - x(t)) f(t) dt \right\| \\
&= \int_0^\infty \left| \int_0^\infty (x_n(t) - x(t)) f(t) s dt \right| \sinh(2s) ds \\
&\le \int_0^\infty \int_0^\infty |f(t)s|\, ds \sinh(2s) ds\, |x_n(t) - x(t)|\, dt \\
&= \int_0^\infty \|f(t)\|\, |x_n(t) - x(t)|\, dt \le \|A\| \, \|x - x_n\|,
\end{aligned}$$

and by letting $n \to \infty$ is $Ax = \int_0^\infty x(t) f(t) dt$ as we claimed. ∎

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