S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 3, стр. 428–440 (2006) УДК 519.172.2
MSC 05C15

PLANAR GRAPHS WITHOUT TRIANGLES ADJACENT TO CYCLES OF LENGTH FROM 3 TO 9 ARE 3-COLORABLE

O.V. BORODIN, A.N. GLEBOV, T.R. JENSEN, A. RASPAUD

Abstract. Planar graphs without triangles adjacent to cycles of length from 3 to 9 are proved to be 3-colorable, which extends Grötzsch's theorem. We conjecture that planar graphs without 3-cycles adjacent to cycles of length 3 or 5 are 3-colorable.

1. Introduction

In 1976, Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colorable (see [6]). Erdős (see [8]) suggested the following relaxation of this problem: does there exist a constant C such that the absence in a planar graph of cycles of length from 4 to C guarantees its 3-colorability? Abbott and Zhou [1] proved that such a C exists with $C \leq 11$. This result was later improved to $C \leq 10$ by Borodin [2] and to $C \leq 9$ by Borodin [3] (see also [6, p. 43–44]) and Sanders and Zhao [7]. In this paper we give a common extension of the Grötzsch 3-Color Theorem [5] and the above result:

Theorem 1. *Every planar graph without a triangle adjacent to a cycle of length from 3 to 9 is 3-colorable.*

We would also like to pose the following Novosibirsk 3-Color Conjecture (Nsk3CC):

Conjecture 2. *Every planar graph without 3-cycles adjacent to cycles of length 3 or 5 are 3-colorable.*

O.V. Borodin, A.N. Glebov, T.R. Jensen, A.Raspaud, Planar graphs without triangles adjacent to cycles of length from 3 to 9 are 3-colorable.

© 2006 O.V. Borodin, A.N. Glebov, T.R. Jensen, A. Raspaud.

The first and second authors were supported by the grants: DFG Ha 202/107-1 from The German Research Council, and grants 05-01-00816, 06-01-00694 and 05-01-00395 of the Russian Foundation for Basic Research. The fourth author was supported by the NATO Collaborative Research Grant n° 97-1519.

Received December 14, 2006, published December 23, 2006.

Nsk3CC is stronger than both Steinberg's conjecture and the recent conjecture by Borodin and Raspaud in [4] that a plane graph having neither a 5-cycle nor two adjacent triangles is 3-colorable. It is obvious that Nsk3CC if true would imply Grötzsch's theorem.

Let G be a plane graph, and let $V(G)$, $E(G)$ and $F(G)$ be its sets of vertices, edges and faces, respectively. We consider only simple graphs. A (proper) 3-*coloring* of G is a mapping $f : V(G) \longrightarrow \{1, 2, 3\}$ such that $f(x) \neq f(y)$ whenever x and y are adjacent in G. Denote the degree of a vertex v by $d(v)$ and the size of a face f (bridges are counted twice) by $r(f)$; a k-vertex is one of degree k. We write $\geq k$-vertex for a vertex of degree at least k, etc. Similar notation is used for faces; triangle is a synonym for 3-cycle.

Let C be a cycle in G; by $Int(C)$ and $Ext(C)$ we denote the subgraphs of G spanned by the vertices lying (strictly) inside and outside of C, respectively, and put $\overline{Int(C)} = G - Ext(C)$, $\overline{Ext(C)} = G - Int(C)$. A cycle C is *separating* if $Int(C) \neq \emptyset \neq Ext(C)$, i.e., G has at least one vertex both inside and outside C. By a *chord* in a cycle C we mean an edge joining two nonconsecutive vertices of C.

To formulate the main result to be proved instead of Theorem 1, we need a few more special definitions. A vertex or edge is *triangular* if it belongs to a triangle, whereas a cycle is *triangular* if it shares an edge with a triangle. Observe that a triangle is (by assumption) not triangular. Triangular vertices of degree 2 are called *special* vertices. A triangle incident with a 2-vertex is *special.*

A 6-cycle C in G is *bad* if $\overline{Int(C)}$ has an induced subgraph G_C each internal face of which has size 4. In other words, the area inside a bad 6-cycle has a partition into 4-faces of G_C. Similarly, a triangular 12-cycle C in G is *bad* if $\overline{Int(C)}$ has a triangular 10-cycle C' such that the area between C and C' is partitioned into 4-faces in a certain induced subgraph of G. The 4- and triangular 10-faces in a bad partition are called 4- and 10-*cells* of G respectively.

A chord of a cycle C is *triangular* if it joines two vertices at distance 2 along C. Observe that if a triangular cycle $C = v_1v_2v_3\dots$ has a triangular chord, say v_1v_3, then there is a shorter triangular cycle $C' = v_1v_3\dots$. We say that a cycle is *reduced* if it has no triangular chord. A reduced cycle C is *good* if C is not bad and either $|C| \leq 6$, or C is triangular and $|C| \leq 12$.

Instead of Theorem 1, it was easier for us to prove the following stronger fact:

Theorem 3. *Suppose G is a plane graph without triangles adjacent to cycles of length from 3 to 9; then*

(i) G is 3-colorable, and

(ii) if D is a good cycle with only internal chords and only special vertices in $Ext(D)$, then every 3-coloring of D can be extended to a 3-coloring of G.

2. Some properties of the minimal counterexample to Theorem 3

Suppose a graph G is a counterexample to Theorem 3 which has a minimal total number of vertices and edges. If G is actually a counterexample to Statement (i) of Theorem 3, then we first make it into a minimal counterexample to Statement (ii), as follows.

Since G is color-4-critical, it is 2-connected and $d(v) \geq 3$ for each $v \in V(G)$. By an immediate consequence of Euler's formula, G has a face f of size at most 5. W.l.o.g., we can assume that f is the infinite face of G. Since G is 2-connected, it follows that the boundary D of f is a cycle. As D has no chords, it has a 3-coloring φ. As G is not 3-colorable, φ cannot be extended to G.

So, we can assume that a graph G, its cycle $D = d_1 d_2 \dots d_{|D|}$, and a 3-coloring φ of D yield a minimal counterexample to (ii) of Theorem 3. (This implies that each plane graph without triangles adjacent to cycles of length from 3 to 9 having less than $|V(G)| + |E(G)|$ vertices and edges in total satisfies both (i) and (ii) of Theorem 3.)

We now prove some structural properties of G.

(0) G has no triangular cycles of length at most 9.

(1) If $v \in Int(D)$ then D does not become bad in $G - v$.

Indeed, since $G - v$ is a subgraph of G, any bad partition of $\overline{Int(C)}$ (into 4-cells and possibly a triangular 10-cell) in $G - v$ is also a bad partition of $\overline{Int(C)}$ in G, contrary to the assumption that D is good in G.

(2) Every special triangle T is adjacent to D, and if T exists then it is the only triangle adjacent to D.

Suppose $v \in T$ is a special vertex. If $T = D$, we color $G - v$ using the minimality of G and then color v; a contradiction. Otherwise, $v \notin D$; then deleting v leaves D good, and we use the minimality of G again.

In what follows, we assume that if a special triangle T_0 exists then it lies outside D. Thus, the length of the outside face f_∞ of G is $|D| + 1$ if T_0 exists and $|D|$ otherwise.

(3) If G has a reduced cycle C of length at most 12 sharing an edge xz with a triangle $T = xyz$, then $y \notin C$ and y cannot be adjacent to a vertex of C other than x, z.

This follows immediately from (0) and the absence of triangular chords in C.

(4) G is 2-connected.

Otherwise, the minimality of G implies that the block B_D of G that contains D can be colored according to φ due to (ii) of Theorem 3, while every other component or block of G, by (i). This yields an extension of φ to G; a contradiction.

(5) If $v \in Int(D)$ then $d(v) \geq 3$.

Indeed, if v is an internal 2-vertex, then D is good in $G - v$ provided that D was such in G.

(6) G has no separating good cycle S other than D.

Suppose S is a shortest such cycle; in particular, S is reduced. If $|S| \leq 6$ then we first extend φ to $G - Int(S)$ (this is possible because D cannot become bad due to (1)). Then we delete all the vertices and chords outside S and extend the 3-coloring of S induced by φ to the remaining graph, which has fewer vertices than G.

Now suppose that $10 \leq |S| \leq 12$. We first try to extend φ to $G - Int(S)$, which is smaller than G. In view of (1) again, if deleting some vertex inside S preserves at least one triangle adjacent to D, we are done. Otherwise, there is only one vertex v inside S, and v belongs to all triangles adjacent to D. By (3), v has degree 2, contrary to (5).

So, we have got a coloring φ' on S induced by φ, and we want to extend it inside S. If S has an external chord d, then deleting d leaves a smaller graph G^- due to the minimality of G on $|V|+|E|$. Since S is reduced in G, it remains triangular in G^-, and we are done.

Hence, the only obstacle for extending φ' inside S is that deleting any vertex outside S destroys all the triangles adjacent to S. By (3) again, such a vertex must be unique and special in G. But then $S=D$; a contradiction.

(7) If a good cycle C in G has a chord d, then $|C|\in\{6,12\}$, d is external, and d cuts off a 4-cycle from C.

Clearly, a good cycle C has no chords if $|C|\leq 5$ due to $!C^{\nabla}_{\leq 9}$. For $|C|=6$, the only possible chord in C is external and splits C into two 4-cycles. If $10\leq|C|\leq 11$ then no chord can cut off a 3-cycle from C since C is reduced; by (0) neither can it cut off a ≥ 4-cycle from the part of C that is adjacent to the triangle that makes C triangular. Suppose $|C|=12$; then, similarly, a chord should be external and split C into a 4-cycle and a triangular 10-cycle.

(8) If D is a good cycle, then there is no 2-path xyz joining two different and nonconsecutive vertices x, z of D through $y\in Int(D)$.

Suppose D is split by such a path xyz into cycles D' and D'', where $4\leq|D'|\leq|D''|$ (and $|D'|+|D''|=|D|+4$). If $|D|\leq 6$ then $|D'|=4$ while $|D''|=|D|=6$ due to $d(y)\geq 3$ combined with (6) and (7). However, now due to (7) an edge going out of y not to x or z must split D'' into two 4-cycles, making D bad.

Suppose D is large. If $|D'|=4$ then again combining $d(y)\geq 3$ with (6) and (7) implies that $|D''|=|D|=12$, where D'' in turn must be split into a 4-cycle and a triangular 10-cycle by an edge going out of y inside D''. This violates the assumption that D is good. For $|D'|=5$ we have $|D''|\leq 11$, which is impossible due to the same reasons. For $|D'|\geq 6$ we must have $|D''|\leq 10$, which implies that $|D'|=6$, $|D''|=10$. By (7), a chord from y splits D' into two 4-cycles. But then D is bad; a contradiction.

(9) If D is a good cycle, then there is no 3-path $P=wxyz$ joining two different and nonconsecutive vertices w, z of D through $x,y\in Int(D)$ provided that the 2-path $P'=wxy$ is incident with an ≤ 5-face f, unless $r(f)=5$ and f is incident with all the vertices of P and another vertex of D.

The argument in proving (8) is applied with obvious changes; P splits $G-Ext(D)$ into two cycles, D' and D'', where $5\leq|D'|\leq|D''|$. If $|D'|=5$, then due to (6) and (7), D' must form a 5-face f'. Due to (5) applied to x, it follows that $f'=f$, and we are done.

So, suppose $|D'|\geq 6$. By $d(x)\geq 3$ and the presence of f, each of D', D'' either is separating or contains a chord. Due to (6) combined with (7), none of them can be good.

If $|D|\leq 6$ then $|D'|=|D''|=6$, so that each of D', D'' is bad. Hence D is a bad 6-cycle; a contradiction. Suppose D is triangular. Since $|D'|+|D''|=|D|+2|P|=|D|+6\leq 18$, it follows that $|D'|\leq 9$, i.e., $|D'|$ is nontriangular. Then $|D''|$ is triangular. But $|D''|\leq|D|\leq 12$, hence $|D''|$ is a bad 12-cycle. This implies that $|D'|=6$, so that D' is also bad. By the definition of badness, it follows that D is bad; a contradiction.

Observe that (7–9) implies that D has no chords at all.

(10) Remark. In what follows, we shall make G into smaller graphs by deleting and identifying vertices, combined with inserting edges. In doing so, we should be sure that we do not (a) identify two vertices of D, because then D is not a cycle anymore, or (b) create an edge between two vertices of D colored the same, for otherwise our precoloring φ of D would be spoiled. On the other hand, to stay within our assumptions on G, we should not create: (c) loops, (d) multiple edges, (e) triangles, and (f) triangular cycles of length at most 9. Observe that, assuming (e), the only way to violate (f) is to identify two vertices on a triangular path. The last possible obstacle is (g) making D into a bad cycle.

(11) G has no 4-cycles other than D.

By (6), G has no separating 4-cycle. Of course, G has no 4-cycle with just one edge inside. So suppose $f = wxyz$ is a face inside D.

First observe that identifying w with y (or x with z) within f cannot violate (10a); namely, suppose $w, y \in D$. By (0), w and y are not consecutive along D. Then by (6), (8), neither of x, z can be internal. It follows from (7) that the only obstacle for (10a) is the trivial case of $G = D = wxyz$.

Next suppose (10b) is an obstacle for identifying x with z. W.l.o.g., $x \in D$, $z \notin D$, and there is an edge zd_i such that $d_i \in D$, where d_i is not adjacent to x along D. By (0), (6) and (8) apllied to the path yzd_i it follows that y must be internal, which contradicts (9).

Now observe that identifying x with z cannot creat loops, multiple edges, or 3-cycles. The first claim follows from $!C^{\nabla}_{\leq 9}$. The other two imply a 2-path joining two vertices in a 4- or 5-cycle through its internal vertex, which contradicts (6), (8). So, none of (10c–e) is an obstacle for this operation.

Next suppose (10f) is an obstacle, i.e. we have created a triangular $\leq$ 9-cycle $C = wv_1 \dots v_k$, $k \leq 8$, by identifying w with y. We pick a shortest such C, which implies that C has no triangular chord. Then G has reduced triangular cycles $C_x = yxwv_1 \dots v_k$ and $C_z = yzwv_1 \dots v_k$, each of length at most 11. W.l.o.g., we can assume that z lies (nonstrictly) inside C_x. However, z cannot actually coincide with one of v_i's, $1 \leq i \leq k$, by (7). Hence, z is strictly inside the good cycle C_x, which contradicts (6), (8).

Finally, suppose collapsing the 4-face f by identifying x with z makes D into a bad cycle in the graph G^* obtained, i.e., (10g) is an obstacle. Let S^* be a bad partition of G^*. If $x * z$ is a vertex inside a cell of S^*, then S^* is also a bad partition of D in G, a contradiction.

So suppose $x * z$ is a vertex of S^* in G^*. By C_y denote a cell of S^* that contains y nonstrictly inside. (Clearly, if y is strictly inside a cell of S^*, then C_y is uniquely defined; otherwise, the edge xy, which is not a chord in any cell by (0) being checked two paragraphs ago, belongs to the boundary of two adjacent cells, and we can take any of them as C_y.) A cell C_w is defined similarly. We make the following convention: if y and w lie nonstrictly inside the boundary of a certain cell of S^*, we take $C_y = C_w$.

Observe that if y, along with edge xy, lies on the boundary of C_y, then the cycle C_y is also a cycle of G. (The same is true for w and C_w.) This implies that if $C_y = C_w$ then, again, S^* is a bad partition of G, a contradiction.

So suppose $C_y \neq C_w$. In particular, this implies that in G^* no Jordan curve joining w with y can have all its internal points strictly inside a certain cell in S^*.

Now if both y and w are on the boundaries of their cells, then S^* augmented by the 4-face $wxyz$ yields a partition S of G. Clearly, S shows that D is bad in G, a contradiction. Indeed, if $|D| = 6$ then both S^* and S consist of 4-cycles only, whereas $|D| = 12$ implies that both S^* and S have one (and the same) triangular 10-cycle each, while the other their cycles have length 4.

If $y \in Int(C_y)$ while w is on the boundary of C_w, then in G we have a cycle $C'_y = xyzy_1 \dots y_k$ which is by 2 longer than the cycle $C_y = xy_1 \dots y_k$ in G^*. Since $d(y) \geq 3$ in G by (5), y has a neighbour t nonstrictly inside C'_y such that $t \notin \{x, z\}$. It follows from (7), (8) applied to t that the 6- or triangular 12-cycle C'_y is bad. Combining a bad partition of C'_y with cycle $wxyz$ and all the cells of S^* except C_y yields a bad partition S of G, as desired. More specifically: if $|C_y| = 10$, then $|D| = 12$ and C_y was the bad 10-cell of S^* in G^*, while in G the bad 10-cell of S is inside the bad 12-cycle C'_y. Suppose $|C_y| = 4$; then the 10-cell exists if and only if $|D| = 12$, and it is the same both in S^* and in S. Finally, if $|D| = 6$ then necessarily $|C_y| = 4$, and the argument is even simpler.

It remains to assume that $y \in Int(C_y)$, $w \in Int(C_w)$. Then, similarly, we have bad cycles C'_y, C'_w, and the bad partition S of D in G can be combined from their bad partitions (in G) augmented by $wxyz$ and $S^* - \{C_w, C_y\}$. This completes the proof of (11).

(11′) G has no bad cycles.

Since any bad cycle has at least one 4-cell in its interior, it follows from (11).

(12) G has neither nonfacial 6-cycles nor triangular nonfacial reduced 12-cycles other than D.

Such a cycle C must be good due to (11′), and it cannot have a chord due to (7). It follows that C is separating, contrary to (6).

(13) G has no nontriangular faces of size at least 6 other than D.

The proof proceeds along the lines of proving (11). Suppose $f = wxyz \dots$ is such a face inside D. If all vertices of f lie on D, then G has no vertices inside D by (7), and we are done. So suppose $y \notin D$.

First observe that identifying x with z within f cannot create loops, multiple edges, or 3-cycles. The first follows since f is nontriangular, the second from (11), and the third yields a separating 5-cycle other than D.

If we have created a triangular cycle $C = xv_1 \dots v_k$, $k \leq 8$, by identifying x with z, then G should have a separating triangular ≤ 11-cycle $C' = zyxv_1 \dots v_k$, which is impossible. Now consider (10a); namely, suppose $x, z \in D$. This contradicts (6), (8).

The next case to consider is (10b). Then w.l.o.g. $x \in D$, $y, z \notin D$, and there is an edge zd_i such that $d_i \in D$ and zd_i creates a chord under contraction. Recall that by (11′), we have no bad cycles anymore.

Following the argument in proving (9), assume that f is inside C'. Then $d(y) \geq 3$ implies that $|C''| \geq 7$ and, moreover, $|C''| \geq 13$ if C'' is triangular. Clearly, $|C'| \geq 6$, and, moreover, $|C'| \geq 13$ if C' is triangular. This implies that $|D| \geq 7$ and, moreover, $|D| \geq 13$ if D is triangular; a contradiction.

Finally, the proof that (10g) cannot be an obstacle becomes easier than in (11). Indeed, suppose identifying x with z makes D bad. Due to (11), this means that the graph G^* obtained has a 4-cycle $xy_1y_2y_3$. Then we have a 6-cycle $C = zyxy_1y_2y_3$ in G. (Here, $y \neq y_2$ since f is not triangular, while $y = y_1$ would imply a 4-cycle $= zy_1y_2y_3$ in G.) By (12), C must be facial, which yields $d(y) = 2$, contrary to (5).

(14) A vertex $v \in D$ of degree 2 cannot be incident with an internal 5-face $uvwxy$, unless $\varphi(u) = \varphi(w)$.

Suppose $\varphi(u) \neq \varphi(w)$. Excluding the trivial case $|D| = |G| = 5$ and using (6), (7) and (8) implies that $x, y \notin D$. By (11′), we have $|D| = 6$ or 12. Indeed, otherwise we delete v and extend φ first to x, y (which is possible due to (6), (8)), then to $G - v$, and finally to G; a contradiction.

We now add an edge uw to $G - v$. This can create neither multiple edges by (0), nor a triangle due to (11). Neither can this operation create a triangular ≤ 9-cycle $uwx_1 \dots x_k$, for otherwise the ≤ 10-cycle $uvwx_1 \dots x_k$ and vertices x, y contradict (5–7). Of course, the currently outside face D' cannot be bad, because $|D'| = 5$ or 11. This completes the proof of (14).

Let $f = v_1 \dots v_5$ be a 5-face, where all v_i's except possibly v_1 are internal 3-vertices. By w_i denote the vertex adjacent to v_i, $2 \leq i \leq 5$, and not incident with f. Then f is *weak* if it is surrounded by three 5-faces $f_i' = w_iv_iv_{i+1}w_{i+1}x_i$, $2 \leq i \leq 4$.

(15) Graph G cannot have a weak internal 5-face.

The proof of (15) is split into steps (16–18) below. Let $G^*{=}G - \{v_2, \dots, v_5\}$, and let f^* be the new face created by this deletion. By *α_i-operation* we mean the identification of w_i with w_{i+1}, $2 \leq i \leq 4$, inside f^*.

(16) An α_i-operation is always possible to perform, unless w_i and w_{i+1} are already precolored the same color.

The argument as in proving (11) and (13) shows that the only obstacle for identifying w_i with w_{i+1} is that $w_i, w_{i+1} \in D$. Moreover, x_i should also belong to D. The statement (16) now follows from (14).

By *β-insertion* we mean adding an edge w_2w_5 inside f^*.

(17) A β-insertion is possible, unless $|D| = 6$ or 12 and D is split by w_2 and w_5 into two paths: one having length 3, the other also having length 3 or being a triangular 9-path, respectively.

We check the list of obstacles (10) for this operation. A loop implies a 4-cycle $w_2v_2v_1v_5w_5$; here $w_2 = w_5$. If the edge w_2w_5 already exists in G, we have a separating 5-cycle $w_2v_2v_1v_5w_5$, which is impossible. Creating a triangle w_2w_5z implies a 6-cycle $w_2v_2v_1v_5w_5z$ in G; here $z \neq v_1$ by (0). If we have created a triangular (reduced) ≤ 9-cycle by adding w_2w_5, then G has a good separating ≤ 12-cycle, which contradicts (6) and (12): this cycle is simple by (0), and has no chords.

Clearly, we do not identify two vertices of D by inserting w_2w_5 into f^*. Creating a chord in D means that $w_2, w_5 \in D$. Then D must be split into two paths, of length at least three each (to avoid separating ≤ 6-cycles). If one of these paths is triangular, then its length should be at least 9 in order to avoid separating triangular ≤ 12-cycles. All this is only possible if $|D| = 6$ or 12, with the only possible splitting into paths being 3+3 or 3+9, respectively. But this is precisely what was stated.

Finally, we check that making D bad by inserting w_2w_5 happens precisely in the situation described in (17). Clearly, $|D| = 6$ or 12. In the bad partition obtained, the boundary of every 4-cell must go through w_2w_5 by (11). Consider the two maximal (on inclusion) among the new 4-cycles and triangular 10-cycles obtained. One of them has the outside boundary being a 3-path P_1, while the other, a 3- or a triangular 9-path P_2. Since G has no 4-cycles different from D, it follows that $w_2, w_5 \in D$ and, moreover, the whole P_1 should go along D. Then $|P_2| = 3$ or 9 depending on whether $|D| = 6$ or 12, respectively, which completes the proof of (17).

By γ-*contraction* we mean identifying v_1 with w_4 inside f^*.

(18) If β-insertion is impossible then γ-contraction becomes possible.

Again, we check possible obstacles for identifying v_1 with w_4 one by one, making use of the fact that D is split by w_2, w_5 into a 3-path P_1 and either a 3- or a triangular 9-path P_2.

Observe that in our $G^*{=}G - \{v_2, \dots, v_5\}$, we cannot reach w_2 or w_5 from v_1 in fewer than 3 steps and from w_4 in fewer than 2 steps. Therefore, we do not create loops, multiple edges or triangles. Creating a triangular (reduced) ≤ 9-cycle implies a triangular (reduced) ≤ 9-path from v_1 to w_4. Its triangular segment between v_1 and one of w_2, w_5, as well as that between w_4 and w_5, must contain at least 8 edges due to (0). If this is the case, then our forbidding ≤ 9-cycle has to reach w_4 from w_2 in just one step (edge), which is impossible by the observation above.

So, there should be a short triangular segment between w_4 and w_2. It cannot contain less then 7 edges, since G has no separating ≤ 11-cycle $w_2v_2v_3v_4w_4\dots$. But then our triangular ≤ 9-cycle has at most two edges to reach v_1 from w_2, which is impossible.

Clearly, $v_1 \notin D$ due to (6), (8). Creating a chord in D now means that v_1 is adjacent to D, but then the 7-cycle $P_1w_2v_2vv_1v_5w_5$ must be split into a 4- and a 5-cycle, which is impossible.

It remains to verify that D cannot degenerate into a bad cycle by identifying v_1 with w_4. Clearly, $|D| = 6$ or 12. At the very least, we should be able to create a 4-cycle (going through $v_1 * w_4$); moreover, it should go through one of w_2, w_5. But again, this contradicts the observation in the second paragraph of proving (18). This completes the proof of (18).

To prove (15), it remains to show how to color the deleted vertices $v_2, \dots, v_5$. Suppose $c(w_2) = 1$, $c(w_5) = 2$ by the β-insertion, while $c(w_3) = c(w_4) = \alpha$ due to the α-operation. If $c(v_1) = 1$, we color v_5, v_4, v_3, v_2 in this order. By symmetry, suppose $c(v_1) = 3$. We put $c(v_2) = 2$, $c(v_5) = 1$. Suppose $\alpha \ne 1$; we put $c(v_3) = 1$, and then color v_4.

Now suppose that the γ-contraction yields $c(v_1) = c(w_4) = \gamma$, while $c(w_2) = c(w_3) = \alpha$ due to the α-operation. If $\alpha = \gamma$, we color v_5, v_4, v_3, v_2 in this order. Otherwise, we color v_2, v_3, v_5, v_4 in this order. This completes the proofs of (15).

3. Discharging

The rest of our proof consists in showing that the properties (1–18) of G are incompatible with each other. In fact, we prove the nonexistence of the graph G_2 obtained from G by deleting the special 2-vertex if the latter exists. Accordingly, all the concepts below, like f_∞, the degrees of vertices of D, etc., are from now on referred to G_2 rather than to G. In particular, $D = d_1 d_2 \dots d_{|D|}$ is the boundary of the outside face f_∞ of G_2. A face f is *large* if $r(f) > 5$. Observe that if a large face in G_2 is not triangular then by (13) it is adjacent to the special triangle in G.

Euler's formula $|V(G_2)| - |E(G_2)| + |F(G_2)| = 2$ for G_2 may be rewritten as

$$\sum_{v \in V(G_2)} (d(v) - 4) + \sum_{f \in F(G_2)} (r(f) - 4) = -8.$$

We set the *initial charge* of every vertex v of G_2 to be $ch(v) = d(v) - 4$, of every face $f \neq f_\infty$ to be $ch(f) = r(f) - 4$, and set $ch(f_\infty) = r(f_\infty) + 4$. Clearly,

$$\sum_{x \in V(G_2) \cup F(G_2)} ch(v) = 0.$$

We now use the discharging procedure, leading to the *final charge* ch^*, defined by applying the following rules:

R0. Each internal 3-face receives $1/3$ from each incident vertex.

R1. (a) Each internal triangular 3-vertex v receives $2/3$ from each incident large face.

(b) Each internal nontriangular 3-vertex receives $1/3$ from each incident face.

(c) If an internal 4-vertex is incident with two triangles, then it receives $1/3$ from each incident large face.

(d) Suppose that an internal 4-vertex v is incident with faces $f_1, \dots, f_4$, in a cyclic order, among which only f_1 is a triangle. Then v receives $1/3$ from f_3 if $r(f_3) > 5$; otherwise (i.e., when $r(f_3) = 5$), vertex v gets $1/6$ from each of f_2, f_4.

Before stating Rules 2 and 3, we recall that the degrees of vertices of D are assumed to be those in G_2; i.e. the special edges are discounted.

R2. Every 2-vertex (in D) receives $5/3$ or $4/3$ from f_∞ and either $1/3$ or $2/3$, respectively, from the other (internal) incident face f. More specifically: let $d(v_1) > 2$, $d(v_2) = \dots d(v_k) = 2$, $d(v_{k+1}) > 2$, where $k \geq 2$. Then each of v_2, v_k receives $5/3$ from f_∞ and $1/3$ from f, while each v_i, where $2 < i < k$, receives $4/3$ from f_∞ and $2/3$ from f.

R3. Suppose $v \in D$, where $d(v) \geq 3$. Then:

(a) If v is a triangular 3-vertex, then v receives $1/3$ from the incident large face and 1 from f_∞.

(b) If v is a nontriangular 3-vertex, then v receives $1/3$ from each large incident internal face; furthermore, f_∞ gives v charge 1, $2/3$, or $1/3$ depending on whether v is incident with no, one, or two large internal faces, respectively.

(c) If v is an ≥ 4-vertex then it receives nothing from the incident faces, unless $d(v) = 4$, in which case v gets $2/3$ or $1/3$ from f_∞ if v is incident with two or one triangle, respectively.

R4. An internal 5-face f receives $1/6$ from each incident large internal face f^* through each of their mutually incident internal 3-vertices.

R5. The charge remaining on the vertices and faces of G_2 after applying R1-R4 is transferred to f_∞.

Since the above procedure preserves the total charge, we have:

$$\sum_{x \in V(G_2) \cup F(G_2)} ch^*(x) = 0.$$

The rest of the proof consists in showing that $ch^*(x) \geq 0$ whenever $x \in V(G_2) \cup F(G_2)$, and that $ch^*(f_\infty) > 0$, with an obvious final contradiction.

4. Checking that all new charges are nonnegative and $ch^*(f_\infty) > 0$

(19) If $v \in V(G_2)$ then $ch^*(v) \geq 0$.

If $d(v) = 2$ then v cannot be incident with an internal ≤ 4-face by (5), (7), and (11). By R2, v always gets 2 in total (either as $5/3+1/3$ or as $4/3+1/3$) from f_∞ and the internal face incident with v, so that $ch^*(v) = 2 - 4 + 2 = 0$.

Suppose $d(v) = 3$. If v is not incident with a 3-face, then $ch^*(v) = 3-4+3\times 1/3 = 0$ by R1b if $v \notin D$; otherwise, $ch^*(v) = 0$ by R3b (in three options). If v is incident with a 3-face, then $ch^*(v) = -1 - 1/3 + 2 \times 2/3 = 0$ by R0 and R1a if $v \notin D$; otherwise, $ch^*(v) = -1 - 1/3 + 1/3 + 1 = 0$ by R0 and R3a.

Suppose $d(v) = 4$. If $v \in D$ then $ch^*(v) \geq 0$ by R0 and R3c. If $v \notin D$, we have three cases to consider: If v is not incident with a 3-face, then $ch^*(v) = ch(v) = 0$. If v is incident with only one 3-face, then v receives $1/3$ (in total) by R1d and sends away $1/3$ by R0, so that $ch^*(v) = 0$. If v is incident with two 3-faces, then v twice receives $1/3$ due to R1c and sends away $1/3$ twice by R0, so that $ch^*(v) = 0$.

Finally, if $d(v) \geq 5$ then v sends away $1/3$ at most $\lfloor d(v)/2 \rfloor$ times according to R0, so that $ch^*(v) \geq d(v) - 4 - d(v)/6 = (5d(v) - 24)/6 > 0$.

(20) If $f \in F(G_2)$, $r(f) \neq 5$, and $f \neq f_\infty$, then $ch^*(f) \geq 0$.

If $r(f) = 3$ then $ch^*(f) = 3 - 4 + 3 \times 1/3 = 0$ by R0. Due to (11), $r(f) \neq 4$.

Suppose $r(f) > 5$. Due to (13), f is triangular in G, which implies that $r(f) \geq 10$. Recall that at the beginning f has charge $r(f) - 4$; then f sends at most $2/3$ to each incident internal vertex v by R1, or to and through v by R1b combined with R4. By R2, R3, f sends at most $2/3$ to each incident vertex of D. For $r(f) \geq 12$ this already implies $ch^*(f) = r(f) - 4 - r(f) \times 2/3 = (r(f) - 12)/3 \geq 0$.

Suppose $r(f) = 11$. If at least one incident vertex gets at most $1/3$ from f, then $ch^*(f) \geq 7 - 1/3 - 10 \times 2/3 = 0$. Similarly, if at least two incident vertices get $1/2$ each from f by R1b combined with R4, then $ch^*(f) \geq 7 - 2 \times 1/2 - 9 \times 2/3 = 0$. So we are done unless f gives $2/3$ to ten incident vertices and at least $1/2$ to the eleventh.

Suppose f is in trouble. Then first observe that f is incident with internal 3-vertices only: otherwise an incident vertex in D having degree at least 3 gets at most $1/3$ from f by R3 and nothing by R4. (Clearly, if f is incident with a 2-vertex (in D) then it should be incident with an ≥ 3-vertex in D.) So, we assume that all vertices incident with f are internal. If one of them has degree at least 4, we are done again, since it takes at most $1/3$ by R1b and nothing by R4.

By (13), f is adjacent to at least one triangle, and hence to a large face. On the other hand, due to parity, f is incident with a nontriangular 3-vertex, which according to our assumptions above should take $2/3$ or $1/2$ by R3b and R4, and for this reason should be incident with a 5-face that is adjacent to f.

Consider a maximal sequence S_5 of 5-faces adjacent to f. From both sides, S_5 ends in a large face (it cannot finish with a 3-face). Then the two extreme vertices of S_5 along the boundary of f are different due to (4), so that each takes $1/2$ from f, and we are done.

Now suppose $r(f) = 10$. Potentially, f might send away $10 \times 2/3$, which is by $2/3$ greater than its initial charge. However, we now show that in fact f sends at most 6 in total. The argument is similar to that given for $r(f) = 11$ but more complicated. Clearly, if at least one incident vertex gets nothing from f, then f already saves $2/3$ and $ch^*(f) \geq 6 - 9 \times 2/3 = 0$. So from now on we assume that each incident vertex takes at least $1/6$ from f; in particular, each incident vertex has degree at most 4. Similarly, we are done if there are at least two vertices taking at most $1/3$ from f.

First observe that at least one incident vertex has degree > 3 or belongs to D. Indeed, if f is incident with ten internal 3-vertices, we delete all of them and color the remaining graph according to φ. Now each of the vertices of the bondary 10-cycle of f (which is a cycle indeed by (0)) has two admissible colors, so we can get a desired coloring of the whole G, a contradiction.

Case 1. No vertex incident with f belongs to D.

Let the boundary of f be the cycle $v_1 \dots v_{10}$. By above, we may assume that precisely one incident vertex, v_1, has degree 4 (but still takes $1/3$ or $1/6$ from f by R1d), while each of the other nine has degree 3. We have three options for v_1 described in R1c and R1d; in all of them v_1 is triangular.

First suppose v_1 is not incident with a 5-face. Then by parity, among the nine vertices at f other than v_1 there is a nontriangular (3-)vertex. It must be incident with a 5-face, for otherwise it also saves $1/3$ for f. Again by means of the S_5 argument above, we find two 3-vertices saving $1/6$ each.

It remains to assume that v_1 is incident with a 5-face f' and a triangle f''. Clearly, f is adjacent to both f' and f'', and still to a large face. Observe that v_1 gets $1/6$ from f and thus saves $1/2$ for f; another saver, of $1/6$, is the vertex at the end of the S_5 chain of 5-faces adjacent to f that starts at f'. In total, f saves $1/2+1/6=2/3$, and we are home.

Case 2. f has a vertex in common with D.

Recall that R4 does not take away anything from f through vertices of D. Thus, each common $\geq$ 3-vertex v of f and D takes at most $1/3$ from f by R1–3. If f is incident with a 2-vertex then it is also incident with two $\geq$ 3-vertices of D. On the other hand, if $d(v) = 3$ then f should have a vertex more in common with D. Finally, if $d(v) \geq 4$ then f saves $2/3$ at v alone.

(21) If $f \in F(G_2)$, $r(f) = 5$, and $f \neq f_\infty$, then $ch^*(f) \geq 0$.

Recall that f has initial charge 1, gives $1/3$ to each incident internal 3-vertex by R1b and to each incident 2-vertex by R2. In turn, f gets $1/6$ from each adjacent large internal face through a common internal 3-vertex by R4.

If f is incident with a 2-vertex $d_2 \in D$, then both d_1 and d_3 are $\geq$ 3-vertices of D due to (6), (7) and (8). None of d_1, d_3 takes anything from f, so that $ch^*(f) \geq 0$ in this case.

To be in trouble, f must be incident with at least four internal 3-vertices. Moreover, each large internal face adjacent to f along an edge whose both end vertices are internal and have degree 3 gives $2 \times 1/6 = 1/3$ to f by R4.

If $d(v_5) > 3$ or $v_5 \in D$, this implies that $f = v_1 \dots v_5$ must be adjacent to at least four 5-faces: those incident with edges v_1v_2, v_2v_3, v_3v_4, and with one of v_1v_5, v_4v_5, for otherwise f receives $1/3$ in total by R4. However, this contradicts (15).

Similarly, if f is incident with all five internal 3-vertices, then f gets the total of $2/3$ from two adjacent large faces due to (15), which implies that $ch^*(f) \geq 0$.

(22) $ch^*(f_\infty) > 0$.

By R1–R4, no vertex of D gets more than $5/3$ from f_∞. If $r(f_\infty) \leq 6$ then $ch^*(f_\infty) = |D| + 4 - |D| \times 5/3 = 2(6 - |D|)/3 \geq 0$. Moreover, since D cannot entirely consist of 2-vertices, and since each ≥ 3-vertex of D takes at most 1 from f_∞, it follows that $ch^*(f_\infty) > 0$ in this case.

Now suppose $r(f_\infty) \geq 10$. In fact, f_∞ is a triangular (in G) outside face in G_2 of size at most 12. Recall that $ch(f_\infty) = |D| + 4$ and observe that $|D| + 4 - |D| \times 4/3 = (12 - |D|)/3 \geq 0$ for $r(f_\infty) \leq 12$. So, it will be suffice to show that f_∞ sends each incident vertex at most $4/3$ on the average, and in turn f_∞ receives some positive charge by R5. To this end, we make a local redistribution of charges given by f_∞ to the vertices of D.

If a 2-vertex d_2 takes $5/3$ from f_∞ by R2, we instead give d_2 only $4/3$ and share the remaining $1/3$ evenly among those of d_1, d_3 that have degree at least 3. So, if both have degree at least 3, then each gets additional $1/6$ from f_∞; if only one, then it gets the whole $1/3$. (These two options are the only admissible by R2.)

After this averaging each 2-vertex gets precisely $4/3$ from f_∞. We next show that each ≥ 3-vertex d_3 of D now takes at most $4/3$, and that at least one of the vertices and faces of G_2 gives some positive charge to f_∞.

If $d(d_3) > 3$ then d_3 remains with at most 1 after averaging due to R3, because no 2-vertex belongs to a triangle. So suppose $d(d_3) = 3$.

If d_3 is not triangular then the only way for it to remain with $4/3$ is to get 1 from f_∞ by R3a and twice by $1/6$ in the course of averaging, for no 5-face can be incident with two 2-vertices in a row due to (6), (7) and (8).

If d_3 is triangular then the only way for it to remain with $4/3$ is to get 1 from f_∞ by R3a and $1/3$ from the neighbour 2-vertex incident with a large internal face (recall that no 2-vertex can be incident with a triangle).

So, we are done unless $|D| = 12$ and D has only these two types of 3-vertices, called *hungry*. If there is at least one such a hungry nontriangular 3-vertex d_3, i.e., incident with two internal 5-faces and getting $1/6$ from two 2-vertices, then all the twelve vertices of D cannot be the same because D is triangular in G.

Clearly, our hungry triangular and nontriangular 3-vertices cannot co-exist. This implies that all ≥ 3-vertices of D are triangular hungry 3-vertices. Moreover, they are split into pairs by their common triangles, and these triangles are separated from each other along D by at least two 2-vertices.

Since $G_2 \neq D$, we have an internal triangle adjacent to D and to a large internal face, f. Clearly, $r(f) \geq 10$. Since f is incident with at least four vertices of D taking $1/3$ from f each by R2 and R3a, it follows that $ch^*(f) \geq r(f) - 4 - 4 \times 1/3 - (r(f) - 4) \times 2/3 = (r(f) - 8)/3 > 0$.

Thus, after applying R1–R4 at least one vertex or face of G_2 remains with a positive charge, while the intermediate charge of f_∞ is nonnegative. This means that after applying R5, we have $ch^*(f_\infty) > 0$.

This completes the proof of (22) and, due to the remark at the end of Section 3, that of Theorem 3.

Acknowledgement

The first author is thankful to the Hamburg University for inviting him as a Visiting Professor in January-July of 2002, which made his contribution to this paper possible, and special thanks are due to Reinhard Diestel and Tommy Jensen for their hospitality in Hamburg.

References

[1] H.L. Abbott and B. Zhou, *On small faces in 4-critical graphs.* Ars. Combin., **32** (1991), 203–207.

[2] O.V.Borodin, *To the paper of H.L.Abbott and B.Zhou on 4-critical planar graphs.* Ars Combinatoria, **43** (1996), 191–192

[3] O.V.Borodin, *Structural properties of plane graphs without adjacent triangles and an application to 3-colorings.* J. of Graph Theory, **21**:2 (1996), 183–186.

[4] O.V.Borodin and A.Raspaud, *A Sufficient Condition for Planar Graphs to be 3-Colorable.* J. of Combin. Theory B, **88** (2003), 17–27.

[5] H.Grötzsch, *Ein Dreifarbenzatz für dreikreisfreie Netze auf der Kugel.* Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe, **8** (1959), 109–120.

[6] T.R. Jensen and B. Toft, *Graph coloring problems.* Wiley Interscience (1995).

[7] D.P. Sanders and Y. Zhao, *A note on the three color problem.* Graphs and Combinatorics, **11** (1995), 91–94.

[8] R. Steinberg, *The state of the three color problem. Quo Vadis, Graph Theory?* J. Gimbel, J.W. Kennedy & L.V. Quintas (eds.), Annals of Discrete Math., **55** (1993), 211–248.

Oleg Borodin
Institute of Mathematics,
Novosibirsk, 630090, Russia
E-mail address: brdnoleg@math.nsc.ru

Aleksey Glebov
Institute of Mathematics,
Novosibirsk, 630090, Russia
E-mail address: angle@math.nsc.ru

Tommy Jensen
Alpen-Adria Universität Klagenfurt,
Institut für Mathematik,
9020 Klagenfurt, Austria
E-mail address: tjensen@uni-klu.ac.at

Andre Raspaud
LaBRI, Université Bordeaux I,
33405 Talence Cedex, France
E-mail address: raspaud@labri.fr