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# ISOSPECTRAL FLAT ORIENTABLE 2-ORBIFOLDS

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The spectrum of the Laplace — Beltrami operator $\Delta = -\text{div grad}$, or the Laplacian for short, on a compact flat two-dimensional orbifold is studied. Two orbifolds $V$ and $V'$ are said to be *isospectral*, if the spectra of their Laplacians on orbifolds $V$ and $V'$ coincide.

There is a well-known problem about isospectrality of Riemannian manifolds: whether isospectral manifolds are isometric. Kac [1] famously framed it in the question, "Can one hear the shape of a drum?"The answer is negative in general (see [2] for more details). For example, Milnor in [3] constructed a pair of 16-dimensional isospectral non-isometric tori. The basic definitions and results concerning spectral theory of the Laplacian defined on a Riemannian manifolds and orbifolds could be found, for instance, in Buser [4] and Chiang [5].

In this paper we give an answer to this problem for compact flat orientable two-dimensional orbifolds $V_1 = S^2(2,2,2,2)$, $V_2 = S^2(2,4,4)$, $V_3 = S^2(2,3,6)$, $V_4 = S^2(3,3,3)$, which underlying space is the two-dimensional sphere $S^2$ and the singular set consists of finitely many elliptic points of given orders.

To describe the spectrum of an orbifold $V$ we use the trace function

$$\mathrm{tr}(H_V) = \int_V H_V(x,x,t)\,dV,$$

where $H_V(x,y,t)$ is a fundamental solution to the heat equation on $V$, and $dV$ is the volume element. We apply the method given in [6] to derive the following theorems.

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**Theorem 1.** *The trace function of every compact flat orientable* 2*-orbifold* $V_i$, $i = 1, \dots, 4$, *can be calculated explicitly:*

$$\operatorname{tr}(H_{V_i}) = \frac{\operatorname{area}(V_i)}{4\pi t} \sum_{\ell \in \Lambda} e^{-|\ell|^2/4t} + \lambda_i, \quad i = 1, \dots, 4.$$

*Here* $\Lambda$ *is a lattice of the flat torus which is a finite-fold covering of* $V_i$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{4}$, $\lambda_3 = \frac{5}{6}$, $\lambda_4 = \frac{2}{3}$.

**Theorem 2.** *Any two compact flat orientable* 2*-orbifolds are isospectral if and only if they are isometric.*

In order to prove Theorem 1, we use a relationship between the fundamental solution to the heat equation on orbifold $V$ and on regular covering of $V$ (see [7]).

Theorem 1 combined with the following statements implies Theorem 2:

(i) The spectrum of $V$ uniquely determines the trace function $\operatorname{tr}(H_V)$, and, conversely, the trace function $\operatorname{tr}(H_V)$ uniquely determines the spectrum of $V$.

(ii) The trace function $\operatorname{tr}(H_V)$ determines a compact flat orientable 2-orbifold $V$ up to isometry.

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