S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 3, стр. 62–66 (2006) УДК 514.13

Краткие сообщения MSC 57M25, 57N10

IDEAL TURAEV–VIRO INVARIANTS

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ABSTRACT. Turaev–Viro invariants are defined via state sum polynomials associated to special spines of a 3-manifold. Its evaluation at solutions of certain polynomial equations yields a homeomorphism invariant of the manifold, called a *numerical Turaev–Viro invariant.* The coset of the state sum modulo the ideal generated by the equations also is a homeomorphism invariant of compact 3-manifolds, called an *ideal Turaev–Viro invariant.* Ideal Turaev–Viro invariants are at least as strong as numerical ones, without the need to compute any explicit solution of the equations. We computed various ideal Turaev–Viro invariants for closed orientable irreducible manifolds of complexity up to 9. This is an outline of [5].

1. DEFINITION AND COMPUTATION OF IDEAL TURAEV–VIRO INVARIANTS

Let P be a special 2-polyhedron [9] with a choice of orientation for each 2-stratum. Let $\mathcal{C}(P)$ be the set of its 2-strata, $\mathcal{E}(P)$ the set of its true edges and $\mathcal{V}(P)$ the set of its true vertices. Let $\mathcal{F}$ be a finite set with an involution "$-$", and let $\mathcal{G}$ be another finite set. A *$\mathcal{F},\mathcal{G}$-colouring* of P is any pair (φ,ψ) of maps $\varphi\colon \mathcal{C}(P)\to\mathcal{F}$, $\psi\colon \mathcal{E}(P)\to\mathcal{G}$, assigning "colours" to 2-strata and true edges. For an oriented 2-stratum of colour $f\in\mathcal{F}$, the oppositely oriented 2-stratum shall have the colour $-f$. Let $\Phi_{\mathcal{F},\mathcal{G}}(P)$ be the set of all $\mathcal{F},\mathcal{G}$-colourings of P.

The *weight* of $f\in\mathcal{F}$ is the symbol $w(f)$. At a true vertex v of P, six 2-strata and four true edges meet (counted with multiplicities). Let $(\varphi,\psi)\in\Phi_{\mathcal{F},\mathcal{G}}(P)$ assign to the 2-strata and true edges in the neighbourhood of v the coulours $a,\dots,f\in\mathcal{F}$ and $A,\dots,D\in\mathcal{G}$, respectively, as depicted in Figure 1 (orientations of 2-strata are indicated by arrows). The *$6j4k$-symbol* $v^{\varphi,\psi}$ of v is the symbol $\left|\begin{smallmatrix} a & b & c\\ f & e & d\end{smallmatrix}\right|^{A,B}_{C,D}$.

King S.A., Ideal Turaev–Viro invariants.

Partially supported by INTAS project CALCOMET-GT ref. 03-51-3663.

Communicated by A.D.Mednykh February 27, 2006, published March 1, 2006.

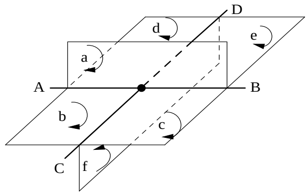

FIGURE 1. A true vertex v with coloured neighbourhood

We shall work with equivalence classes of colour weights and $6j4k$-symbols according to the following *symmetry assumptions*: For all $f \in \mathcal{F}$ let $w(f) = w(-f)$, and for all $a, b, c, d, e, f \in \mathcal{F}$ and all $A, B, C, D \in \mathcal{G}$ let $\left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|^{A,B}_{C,D} = \left|\begin{smallmatrix} b & c & a \\ -e & -d & f \end{smallmatrix}\right|^{C,A}_{B,D} = \left|\begin{smallmatrix} a & -d & -e \\ -f & -c & -b \end{smallmatrix}\right|^{A,B}_{D,C}$. We make no notational difference between a colour weight respectively a $6j4k$-symbol and its equivalence class. Let R be the polynomial ring over some field $\mathbb{F}$ whose variables are the equivalence classes of colour weights and $6j4k$-symbols. Then, the *Turaev–Viro state sum* of P of type (m, n),

$$TV_{m,n}(P) := \sum_{(\varphi,\psi)\in\Phi_{\mathcal{F},\mathcal{G}}(P)} \left(\prod_{C\in\mathcal{C}(P)} w(\phi(C)) \right) \cdot \left(\prod_{v\in\mathcal{V}(P)} v^{\varphi,\psi} \right) \in R,$$

only depends on the homeomorphism type of P.

The *Turaev–Viro ideal* $I_{m,n} \subset R$ of type (m, n) is generated by

$$\sum_{A\in\mathcal{G}} \left|\begin{smallmatrix} j_1 & j_2 & j_3 \\ j_9 & j_8 & j_7 \end{smallmatrix}\right|^{k_1,k_2}_{k_3,A} \cdot \left|\begin{smallmatrix} j_4 & j_5 & j_6 \\ -j_9 & -j_8 & -j_7 \end{smallmatrix}\right|^{k_4,k_5}_{k_6,A} -$$
$$\sum_{A_1,A_2,A_3\in\mathcal{G}} \sum_{j\in\mathcal{F}} w(j) \cdot \left|\begin{smallmatrix} j & j_1 & j_2 \\ j_7 & -j_5 & -j_4 \end{smallmatrix}\right|^{A_1,A_2}_{k_1,k_4} \cdot \left|\begin{smallmatrix} j & j_2 & j_3 \\ j_9 & -j_6 & -j_5 \end{smallmatrix}\right|^{A_2,A_3}_{k_3,k_6} \cdot \left|\begin{smallmatrix} j & j_3 & j_1 \\ -j_8 & -j_4 & -j_6 \end{smallmatrix}\right|^{A_3,A_1}_{k_2,k_5},$$

for $j_1, \ldots, j_9 \in \mathcal{F}$, $k_1, \ldots, k_6 \in \mathcal{G}$. Let $tv_{m,n}(P) = TV_{m,n}(P) + I_{m,n} \in R/I_{m,n}$ be the coset of the Turaev–Viro state sum with respect to the Turaev–Viro ideal. The following theorem is based on the Matveev–Piergallini theorem (see, e.g., [9]).

Theorem 1. *If P is any special spine of a compact 3-manifold M with at least two true vertices, then $tv_{m,n}(P)$ only depends on the homeomorphism type of M.* □

We call $tv_{m,n}(M) = tv_{m,n}(P)$ an *ideal Turaev–Viro invariant* of type (m, n). For x in the affine zero variety $\mathfrak{v}(I_{m,n})$ of $I_{m,n}$ over the algebraic closure $\hat{\mathbb{F}}$ of $\mathbb{F}$, the evaluation $TV_{m,n}(P)(x) \in \hat{\mathbb{F}}$ of the state sum is a homeomorphism invariant of M, called *numerical Turaev–Viro invariant* associated to $tv_{m,n}(\cdot)$. Many examples of numerical Turaev–Viro invariants arise from the theory of Quantum Groups [10]. By definition, if an ideal Turaev–Viro invariant coincides on two compact 3-manifolds then *all* associated numerical Turaev–Viro invariants coincide. Let $\sqrt{I_{m,n}}$ be the radical of $I_{m,n}$. We call $\widehat{tv}_{m,n}(M) = TV_{m,n}(P) + \sqrt{I_{m,n}} \in R/\sqrt{I_{m,n}}$ the *universal numerical Turaev–Viro invariant* of M associated to $tv_{m,n}$. This name is justified by the following application of Hilbert's Nullstellensatz.

Theorem 2. *$\widehat{tv}_{m,n}(\cdot)$ coincides on two compact 3-manifolds if and only if all numerical Turaev–Viro invariants associated to $tv_{m,n}(\cdot)$ coincide.* □

Some of the Turaev–Viro ideals studied in this paper are not radical. Therefore we expect that, in general, an ideal Turaev–Viro invariant can distinguish strictly more manifolds than all its associated numerical Turaev–Viro invariants together.

A potential application of ideal Turaev–Viro invariants concerns the minimal number $\tilde{c}(M)$ of true vertices of a special spine of a compact 3-manifold M. Let $\deg_w(p)$ ($\deg_{6j}(p)$) be the total degree of $p \in R$ in the colour weights (the $6j4k$-symbols). For any $A \subset R$, let $\deg_w(A) = \min\{\deg_w(p)\colon\ p \in A\}$ and $\deg_{6j}(A) = \min\{\deg_{6j} w(p)\colon\ p \in A\}$.

Lemma 1. *For closed* 3*-manifolds M with $\tilde{c}(M) > 1$ holds*

$$\tilde{c}(M) \geq \max\left\{\deg_w\left(tv_{m,n}(M)\right) - 1, \deg_{6j}\left(tv_{m,n}(M)\right)\right\}. \quad \square$$

How to compute ideal Turaev–Viro invariants? Our input data are lists of special spines of compact 3-manifolds, encoded according to [9, Sec. 7.1]. By a maple [8] program we compute the Turaev–Viro state sums of the special spines. In order to compute ideal Turaev–Viro invariants, we need to compare cosets with respect to ideals in a polynomial ring over a field. This is a standard application of the theory of Gröbner bases (e.g., [3]). Once we have computed a Gröbner base of $I_{m,n}$ with respect to any monomial order on R, we can compare $tv_{m,n}(M_1)$ with $tv_{m,n}(M_2)$ for any pair M_1, M_2 of 3-manifolds by computing the normal form of the Turaev–Viro state sums. For the computation of Gröbner bases and normal forms, we used Singular [4]. For computing $\sqrt{I_{m,n}}$ and the associated universal numerical Turaev–Viro invariant, we used the primdec.lib library of Singular [2]. Even for small number of colours, we have to deal with huge polynomial systems over many variables. Therefore we added various simplifying assumptions (see [5] for details).

2. Examples

In all examples, we chose $\mathbb{F} = \mathbb{Q}$ and we provide R with some degree reverse lexicographic order. The generators of $I_{m,n}$ and Gröbner bases can be found online [7]. Our first example $\widetilde{tv}_{2,1}$ is of type $(2, 1)$ with trivial involution on $\mathcal{F}$. We applied simplifying assumtions that also hold for Matveev's ϵ-invariant [9]. The Turaev–Viro ideal in this setting is generated by 12 polynomials, and one obtains a Gröbner base formed by 22 polynomials. We also constructed an ideal Turaev–Viro invariant of type $(2, 2)$ We obtained two ideal Turaev–Viro invariants of type $(3, 1)$ with non-trivial involution. One, obtained under a very slight simplification, is denoted by $tv^+_{3,1}(\cdot)$. The other, subject to simplifying assumptions similar to the ones used in the case of $\widetilde{tv}_{2,1}(\cdot)$, is denoted by $\widetilde{tv}^+_{3,1}(\cdot)$ and is far easier to compute as $tv^+_{3,1}(\cdot)$. The following list of statements is result of our computations.

(1) The Turaev–Viro ideals of $\widetilde{tv}_{2,1}(\cdot)$, $\widetilde{tv}^+_{3,1}(\cdot)$ and $tv^+_{3,1}(\cdot)$ are not radical.
(2) On the 1900 closed orientable irreducible 3-manifolds of complexity ≤ 9, the ϵ-invariant atteins 35, $\widetilde{tv}_{2,1}(\cdot)$ 134, and $\widetilde{tv}^+_{3,1}(\cdot)$ 242 different values, whereas homology atteins 272 different values.
(3) Using the combination of homology and $\widetilde{tv}_{3,1}(\cdot)$ one can distinguish 764 homeomorphism types of closed irreducible orientable 3-manifolds of complexity ≤ 9.

(4) On closed irreducible orientable 3-manifolds of complexity ≤ 9, $\widetilde{tv}^+_{3,1}(\cdot)$ and $tv^+_{3,1}(\cdot)$ are equivalent invariants. On closed irreducible orientable 3-manifolds of complexity ≤ 6, $\widetilde{tv}_{2,1}(\cdot)$ and $\widetilde{tv}_{2,2}(\cdot)$ are equivalent invariants.
(5) On the closed irreducible orientable 3-manifolds that we considered, the ideal Turaev–Viro invariants $\widetilde{tv}_{2,1}(\cdot)$, $\widetilde{tv}^+_{3,1}(\cdot)$ and $tv^+_{3,1}(\cdot)$ are equivalent to their associated universal numerical Turaev–Viro invariant.
(6) The lower bound for the complexity stated in 1 is trivial in all examples that we computed.
(7) Ideal Turaev–Viro invariants are, in general, not multiplicative under connected sum of compact 3-manifolds. □

Statement (4) is surprising, because one would expect that one obtains a stronger invariant if one avoids to impose simplifying assumptions. But this is not necessarily the case. Statement (5) is even more surprising, because by Statement (1) the Turaev–Viro ideals are not radical. Are there compact 3-manifolds M_1, M_2 that can be distinguished by some ideal Turaev–Viro invariant $tv(\cdot)$ but not by $\widehat{tv}(\cdot)$? Note that $\widetilde{tv}_{2,1}$ is stronger than the ϵ-invariant; but the ϵ-invariant is not the only numerical Turaev-Viro invariant associated to $\widetilde{tv}_{2,1}$ (see [9, Sec. 8.1]). Statement (7) is a bad news if one wants to construct a Topological Quantum Field Theory. But it is a good news if one aims to construct invariants that potentially detect counterexamples for the Andrews-Curtis conjecture. Namely, by a result of Bobtcheva and Quinn [1], an invariant for Andrews-Curtis moves descending from a multiplicative invariant of 4-thickenings of special 2-polyhedra only depends on homology if the Euler characteristic of the 2-complex under consideration is at least 1. But a non-multiplicative ideal Turaev–Viro invariant for Andrews-Curtis moves [6] is potentially more useful.

Acknowledgement. I'm grateful to S. V. Matveev for providing me with lists of special spines of 3-manifolds. I thank I. Bobtcheva and M. Brickenstein for interesting discussions. A part of the computations in this paper (as indicated in the text) were performed by using maple$^{\mathrm{TM}}$ [8]. The other part of the computations in this paper were performed by using Singular [4].

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