S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

*Том 3, стр. 83–85 (2006)* УДК 516.162.3
*Краткие сообщения* MSC 57M99, 57M20

# UNIQUENESS OF THE EMBEDDING OF A 2-COMPLEX INTO A 3-MANIFOLD

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Let a 2-dimensional complex $\mathcal{K}^2$ embedded into a 3-dimensional manifold $\mathcal{M}^3$ be given.

**Question .** *To which extent does $\mathcal{K}^2$ determine its regular neighbourhood in $\mathcal{M}^3$ ?*

For the restricted class of special polyhedra *Casler* [1] showed that in fact the thickening is unique (independently of the surrounding manifold).

Trying to solve the question for general 2-complexes I started to collect counterexamples to uniqueness. Together with C. Hog-Angeloni we found four potential obstructions to uniqueness which lead to restrictions on $\mathcal{M}^3$ and the regular neighbourhood of $\mathcal{K}^2$:

Assume $\mathcal{M}^3$ and $\mathcal{K}^2$ to be compact, connected and piecewise linear. $\mathcal{M}^3$ moreover should be orientable. Consider the following phenomena:

**1.:** The lens spaces $L_{5,1}$ and $L_{5,2}$ are non-homeomorphic but they have the same spine $\mathcal{K}^2$. Hence there are non-homeomorphic thickenings in a connected sum. Therefore we assume $\mathcal{M}^3$ to be prime.

**2.:** If $\mathcal{M}^3$ is a counterexample to the 3-dimensional Poincaré conjecture then a spine $\mathcal{K}^2$ might also embed into a 3-ball $\mathcal{B}^3$.

If Perelman's attempt to prove the Poincaré conjecture turns out to be correct, this phenomenon doesn't occur.

**3.:** For $\mathcal{K}^2 = S^2 \vee S^1$ there are two non-homeomorphic thickenings regarding the glueing of $S^1$ to $S^2$. One thickening has connected boundary and the other one has two boundary components.

Therefore we assume that the boundary of the regular neighbourhood of $\mathcal{K}^2$ be connected.

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Glock, J., Uniqueness of the embedding of a 2-complex into a 3-manifold.



*Communicated by S.V. Matveev March 6, 2006, published March 14, 2006.*

**4.:** Consider the square and the granny knot embedded in $S^3$. Both knots are connected sums of two copies of the trefoil knot. The knot complements are not homeomorphic but they have the same spine.

Therefore we assume that the regular neighbourhood of $\mathcal{K}^2$ doesn't contain essential annuli.

Using these assumptions we proved that the thickening of $\mathcal{K}^2$ in $\mathcal{M}^3$ is unique.

**Theorem** (Hog-Angeloni and G. [3]). *Let $\mathcal{M}^3$ be orientable, prime (and not a Poincaré counterexample) and $f : \mathcal{K}^2 \to \mathcal{M}^3$ be a p.l. embedding of a compact connected 2-complex. If the regular neighbourhood $\mathcal{N} = \mathcal{N}(f(\mathcal{K}^2))$ does not contain essential annuli and has connected boundary, then $\mathcal{N}$ is determined by $\mathcal{K}^2$.*

For the proof we used several theorems from 3-manifold theory, like the JSJ-decomposition or results of Waldhausen and Johannson.

Since the question about the uniqueness of thickenings arose from the theory of 2-complexes, it would be nice to avoid 3-manifold theory and find conditions based on 2-complexes. An even more interesting goal would be to get 3-manifold results using the theory of 2-complexes in turn.

Hence from now on I would like to study the homeomorphism type of the pair $(\mathcal{N}, \mathcal{K}^2)$ for a regular neighbourhood $\mathcal{N}$ of $\mathcal{K}^2$ in an orientable $\mathcal{M}^3$ which moreover should be independent of the particular surrounding manifold. Neither of these requirements follows from the previous ones.

This time we start with a given presentation $P = < x_1, ..., x_k \mid r_1, ..., r_l >$ of $\mathcal{K}^2$ which embeds in some 3-manifold with $k$ generators and $l$ relators: Let $P$ be a finite presentation and let $\mathcal{C}_P$ be the standard complex associated to $P$. The 1-skeleton of $\mathcal{C}_P$ is a bouquet of oriented loops which correspond to the generators. For each relator $r_i$ there exists a 2-cell in $\mathcal{C}_P$ with attaching map according to $r_i$. *Then the Whitehead graph $\mathcal{WG}(P)$ is the intersection of the boundary of the regular neighbourhood of the vertex in $\mathcal{C}_P$ (which is a 2-sphere) with $\mathcal{C}_P$ itself.* The Whitehead graph is also known as link, star or coinitial graph (with slightly different definitions regarding multiplicity of edges).

Since we started with a Whitehead graph $\mathcal{WG}(P)$ of a given presentation $P$ in a 3-manifold, the following holds: *If the Whitehead graph is uniquely embedded in $S^2$ then $\mathcal{K}^2$ has a unique thickening*, since orientable 1-handles for $P$ can be attached in at most one way. So we need to find conditions under which the embedding of $\mathcal{WG}(P)$ is unique.

For this purpose there is a useful result of Whitney: He calls a graph $G$ *$n$-separated* ($n \geq 0, n \in \mathbb{N}$) if it is the union of two subgraphs $G_1$ and $G_2$ with the following properties:

- no common edges
- the number of common vertices is $\leq n$
- each $G_i$ has a vertex not belonging to the other

A graph which is not n-separated is called *$(n+1)$-connected.* With this notation Whitney proves ([4, 5]):

*Every 3-connected, planar graph without loops and multiple edges (= simple) has a unique embedding (up to homeomorphism) in $S^2$.*

The next step will be to apply Whitney's result to Whitehead graphs of presentations: *A syllable in a cyclically written defining relator of $P$ is a sequence of two adjacent symbols $x_i^{\pm 1}, x_j^{\pm 1}$.* Since each syllable corresponds to exactly one edge in the Whitehead graph, one can easily see that $\mathcal{WG}(P)$ is simple iff the presentation $P$ is reduced and no syllable occurs more than once.

At this point many questions come up which I will study:

**1.:** In order to apply Whitney's result, $\mathcal{WG}(P)$ should be 3-connected. So we want to avoid Whitehead graphs which become disconnected after removing one or two vertices. How can cut vertices or pairs of cut vertices of $\mathcal{WG}(P)$ be detected resp. excluded in $P$?

**2.:** In the case of a presentation $P$ with 2 generators, the Whitehead graph is 3-connected if every possible syllable appears in $P$. The only 3-connected graph with 4 vertices is the complete graph $K_4$. In this context I plan to study 2-bridge knots which have a 2-generator, 1-relator group as a fundamental group.

**3.:** C. Hog-Angeloni has given syllable conditions for a presentation to decide whether the corresponding 2-dimensional complex is a spine of a 3-manifold. She uses the so called railroad systems [2]. So the question is which of these railroad systems lead to a unique thickening.

**4.:** Finally I want to know whether the Whitehead graph carries information about the decomposition of a 3-manifold like the factor splitting of the fundamental group or the JSJ-decomposition. To which extent can the structure of the fundamental group of a 2-complex be seen in the Whitehead graph ?

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