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RECOGNIZABILITY BY SPECTRUM OF THE GROUP $L_2(7)$ IN THE CLASS OF ALL GROUPS

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Abstract. It is proved that the finite simple group $L_2(7)$ is characterized up to isomorphism by its set of element orders in the class of all groups. This gives the answer to the question 16.57 from "Kourovka Notebook".

Introduction

The *spectrum* of a periodic group G is the set $\omega(G)$ consisting of element orders of G. A group G is *recognizable* by spectrum in the class of all groups if every group with spectrum coinciding with the spectrum of G is isomorphic to G.

Our goal is to give a positive answer to the question 16.57 from [1] (see also [2]) on recognizability by spectrum of the simple group $L_2(7)$.

Theorem. *If the spectrum of a group G is equal to $\{1, 2, 3, 4, 7\}$ then $G \simeq L_2(7)$.*

For finite groups this result is proved in [3].

1. Notation and known results

If H is a subgroup of group G, $x, y \in G$, X, Y are subsets of G, then $x^y = y^{-1}xy, X^y = \{y^{-1}xy | x \in X\}, [x, y] = x^{-1}x^y, x^Y = \{x^y | y \in Y\}, X^Y = \{x^y | x \in X, y \in Y\}$, $N_H(X) = \{g \in H | X^g = X\}$, $\langle X \rangle$ is a subgroup generated by X, $[X, Y] = \langle [x, y] | x \in X, y \in Y \rangle$, $C_H(X) = \{h \in H | [h, x] = 1 \text{ for all } x \in X\}, Z(G) = C_G(G)$. If X is a group then $X' = [X, X]$. If p is a prime then $O_p(G)$ is defined as a product of all normal p-subgroups of G, and $O_{p,q}(G)$ as a full pre-image in G of group $O_q(G/O_p(G))$. Let $\Phi(G)$ be the Frattini subgroup, $SL_2(q)$ ($L_2(q)$) be (projective) special linear group of dimension 2 over a field of order q.

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We need the following known facts which will be referred to as propositions with corresponding numbers.

1. (Shunkov's theorem [4]). Periodic group containing an involution with a finite centralizer is locally finite.

2. (W. Shi [3]). If F is a finite group and $\omega(F) = \{1, 2, 3, 4, 7\}$, then $F \simeq L_2(7)$.

3. (J. Thompson [5]). Finite group which admits an automorphism of a prime order without fixed points is nilpotent.

4. (Theorem 1 of [6]). Let G be a group whose element orders are at most 4. Then G is locally finite, and one of the following holds:

a) G is a group of exponent 3 or 4.

b) G contains a normal elementary abelian 3-subgroup N and G/N is isomorphic to a subgroup of the quaternion group of order 8.

c) G contains a normal elementary abelian 2-subgroup N, and G/N isomorphic to S_3.

d) G contains a normal 2-subgroup N of index 3 and nilpotency class 2.

5. (Lemmas 4 and 5 from [7]). Let G be a group with $\omega(G) = \{1, 2, 3, 4, 7\}$. Then the subgroup generated by all involutions of G is not a 2-group.

2. Preliminary lemmas

The following results (Lemmas 1-5) can be easily verified with the help of the coset enumeration algorithm realized, for example, in GAP system [9].

Lemma 1. *Let $H_i = \langle x, y | R_i \rangle$ be a group with two generators defined by the set $R_i, i = 1, 2 \dots, 11$, of relations recorded in i-ths line of Table 1. Then the order of H_i equals to the number h_i from this table.*

Table 1

i	R_i	h_i
1	$x^3 = y^2 = (xy)^7 = (yy^x)^7 = 1$	1092
2	$x^3 = y^2 = (xy)^7 = (yy^x)^4 = 1$	168
3	$x^3 = y^2 = (xy)^7 = (yy^x)^3 = 1$	1
4	$x^3 = y^2 = (xy)^4 = 1$	24
5	$x^3 = y^2 = (xy)^3 = 1$	12
6	$x^4 = y^2 = (xy)^4 = ((xy)^2y)^3 = 1$	36
7	$x^4 = y^2 = (xy)^4 = ((xy)^2y)^4 = 1$	64
8	$x^4 = y^2 = (xy)^7 = (yy^x)^4 = (x^2y)^4 = 1$	2
9	$x^4 = y^2 = (xy)^7 = (yy^x)^4 = (x^2y)^3 = 1$	168
10	$x^4 = y^2 = (xy)^7 = (yy^x)^3 = (x^2y)^3 = 1$	1
11	$x^4 = y^2 = (xy)^7 = (yy^x)^3 = (x^2y)^4 = 1$	2

Lemma 2. *Let $X(m, n) = \langle x, y, z | x^3 = y^2 = (xy)^7 = (yy^x)^4 = z^2 = (z^x z)^2 = zz^x z^{x^2} = (tz)^2 = (zu)^2 = (u^x z)^2 = (yz)^m = (xyz)^n = 1 \rangle$, where $m, n \in \{3, 4, 7\}, u = y^{(xy)^2 x}, t = u^y$. Then $|X(4, 7) : H| = 64, |X(4, 4) : H| = 8$ and $X(m, n) = 1$ for other m and n.*

Note that, by Lemma 1, H is a homomorphic image of group $L_2(7)$, therefore all the groups $X(m, n)$ from Lemma 2 are finite.

Lemma 3. *Let* $X(m,n) = \langle x,y,z | x^2 = y^2 = z^3 = (xy)^7 = (xz)^2 = (yz)^m = (xyz)^n = 1\rangle$*. Then* $X(3,3) = X(3,4) = 1, |X(4,3)| = 6, |X(4,4)| = 68880$.

Lemma 4. *Let* $X(m,n) = \langle x,y,z | x^2 = y^2 = z^3 = (xy)^7 = (xz)^2 = (yz)^4 = (xyz)^7 = (xz^y)^m = (xz^{yxy})^n = 1\rangle$*. Then* $|X(4,4)| = 14, X(4,3) = X(3,3) = X(3,4) = 1$.

Lemma 5. *Let* $X(m,n) = \langle x,y,z | x^2 = y^2 = z^3 = (xy)^7 = (xz)^3 = (yz)^3 = (xyz)^m = ((xy)^2xz)^n = 1\rangle$*. Then* $|X(3,3)| = 12, X(3,4) = X(4,3) = X(4,4) = 1$.

The following remark follows from description of subgroups in $L_2(q)$ (see section II.8 in [8]).

Lemma 6. *If* M *is a proper subgroup in* $L_2(7)$*, then the order of* M *is not divisible by* 14*. If* M *is a proper subgroup in* $L_2(13)$*, then the order of* M *is not divisible by* 21.

Lemma 7. *The following isomorphisms hold.*
1. $L_2(13) \simeq \langle x,y | x^3 = y^2 = (xy)^7 = (yy^x)^7 = 1\rangle$.
2. $L_2(7) \simeq \langle x,y | x^3 = y^2 = (xy)^7 = (yy^x)^4 = 1\rangle$.
3. $L_2(7) \simeq \langle x,y | x^4 = y^2 = (xy)^7 = (yy^x)^4 = (x^2y)^3 = 1\rangle$.
4. $S_4 \simeq \langle x,y | x^3 = y^2 = (xy)^4 = 1\rangle$.
5. $A_4 \simeq \langle x,y | x^3 = y^2 = (xy)^3 = 1\rangle$.

Proof. Let $a = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2(13)$. If x and y are images of a and b in $L_2(13)$, then x and y satisfy relations of item 1. Since the order of $\langle a,b\rangle$ obviously is divisible by 21, Lemma 6 implies that $\langle x,y\rangle \simeq L_2(13)$. Since order of $L_2(13)$ equals 1092, item 1 is true by Lemma 1.

Let x and y be images of elements a,b in $L_2(7)$, where $a = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2(7)$. Then x,y satisfy relations of item 2 and arguments similar to these in previous paragraph show trueness of item 2.

Item 3 can be proved in the same way with replacing a by $\begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix} \in SL_2(7)$.

Items 5 and 6 are well known.

Lemma 8. *Let* $a = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in SL_2(7), x$ *and* y *be images of* a *and* b*, respectively, under natural homomorphism* $SL_2(7)$ *onto* $L_2(7) = SL_2(7)/Z(SL_2(7))$*. Then orders of elements* x, y, xy *and* yy^x *are equal respectively to* $3, 2, 7$ *and* 4*; in particular,* x *and* y *generate* $L_2(7)$*. If* $t = y^{(xy)^3}, u = y^{(xy)^2y}$*, then* $x^t = x^{-1}, u^t = u, [u,u^x] = 1, u^{x^2} = uu^x$ *and* $u^{xt} = uu^x$*. Specifically,* $U = \langle u, u^x\rangle$ *is elementary abelian subgroup of order* $4, R = \langle t,x\rangle \simeq S_3, R$ *normalizes* U *and* $X = UR \simeq S_4$*. In adition,* $S = \langle t,u,u^x\rangle$ *is a Sylow* 2*-subgroup of* $L_2(7)$ *and* $z \in Z(S)$*.*

Proof is a straightforward calculation.

3. Proof of the main result

Let G be a group whose spectrum $\omega(G)$ is equal to $\{1,2,3,4,7\} = \omega(L_2(7))$.

Lemma 9. *If* G *contains a subgroup* H *isomorphic to* $L_2(7)$ *then* $G = H$.

Proof. Suppose the contrary. Choose elements x and y of H in accordance with Lemma 8 and use all the other notations of that lemma.

We shall show that $C = C_G(U) \nleqslant H$. Indeed, C is X- invariant 2-group. By Proposition 4, C is elementary abelian 2-group. If $C \leqslant H$, then $C = U$ is a coinciding with its centralizer in $C_G(u)$ elementary group of order 4. Since $C_G(u)$ is a locally finite 2-group of exponent 4, $C_G(u)$ is finite. By Proposition 1, G is locally finite. By Proposition 2, $G \simeq L_2(7)$ contradicting to assumption. Thus, $|C| \geqslant 16$ and therefore $C_C(t)$ contains an element z not belonging to H. As C is x-invariant and x acts on C without fixed points, $zz^x z^{x^2} = 1$ and $[z,u] = [z^x,u] = 1$. Hence $K = \langle x,y,z\rangle$ is a homomorphic image of one of the groups $X(m,n) = \langle x,y,z | x^3 = y^2 = (xy)^7 = (yy^x)^4 = z^2 = (z^x z)^2 = zz^x z^{x^2} = (tz)^2 = (zu)^2 = (u^x z)^2 = (yz)^m = (xyz)^n = 1\rangle$, where $m,n \in \{3,4,7\}, t = y^{(xy)^3}, u = y^{(xy)^2 x}$.

By Lemma 2, K is finite. By Proposition 2, $K = H$ contradicting the choice of z. The lemma is proved.

Suppose that G does not contain a subgroup isomorphic to $L_2(7)$.

Lemma 10. *If the product of any two involutions of G is a $\{2,3\}$-element then the subgroup generated by all involutions of G is a $\{2,3\}$-group.*

Proof. Suppose the contrary. Let I is the set of all involutions of G and $H = \langle I\rangle$. Every element z of H can be expressed in the shape $z = i_1 i_2 \ldots i_r$, where $i_j \in I$. Suppose r is the least for which the order of z is equal to 7. Then $r \geqslant 3$ and $x = i_1 i_2 \ldots i_{r-1}$ is an element of order 3 or 4. Suppose $y = i_r, K = \langle x,y\rangle$. By assumption, $(yy^x)^4 = 1$ or $(yy^x)^3 = 1$. If x is a 3-element, then, by Lemmas 1 and 7, K is a homomorphic image of $L_2(7)$. Since $L_2(7)$ is simple, $K \simeq L_2(7)$. By Lemma 9, $G \simeq L_2(7)$ contradicting the assumption.

If x is a 4-element, then x^2 is an involution and hence $(yy^x)^r = (x^2 y)^s = 1$, where $r,s \in \{3,4\}$. Again, by Lemmas 1 an 7, K is a homomorphic image of $L_2(7)$ and as earlier we face a contradiction.

Lemma 11. *G contains two involutions whose product is an element of order 7.*

Proof. Suppose the contrary. By Lemma 10, the subgroup I generated by all involutions of group G is a normal $\{2,3\}$-subgroup in G. By assumption, G contains an element x of order 7. By Proposition 5, I is not a 2-group and hence contains elements y and z of orders 2 and 3, respectively. The subgroup $N = \langle \langle y,z\rangle^{\langle x\rangle}\rangle$ is a finitely generated subgroup from I. By Proposition 4, N is a finite $\langle x\rangle$-invariant subgroup. By Proposition 3, N is nilpotent and hence the order of yz is equal to six contradicting the assumption.

Lemma 12. *The product of any 3-element and an involution is a $\{2,3\}$-element.*

Proof. Suppose the contrary. Let x be element of order 3, y be involution of G whose product is an element of order 7. Then $(yy^x)^m = 1$ for $m \in \{3,4,7\}$. By Lemmas 1 and 7, $\langle x,y\rangle \simeq L_2(7)$ contradicting Lemma 9. The lemma is proved.

Let us finish the proof of Theorem. By Lemma 11, G contains involutions x,y, whose product is of order 7. In addition, by Lemma 12, the product of any 3-element and an involution is a $\{2,3\}$-element.

Suppose first that G contains an element z of order 3, such that $(xz)^3 = (yz)^3 = 1$. Since x and $(xy)^2 x$ are involutions, by Lemma 12, $(xyz)^m = ((xy)^2 xz)^n = 1$ for some m,n equal to 3 or 4. This contradicts Lemma 5.

Thus we may assume that there exists an element z of order 3 for which $(xz)^4 = 1$. If the order of xz equals 4 then, by Lemma 7, $\langle x, z\rangle \simeq S_4$, hence $\langle x, z\rangle$ contains an element z_1 of order 3 for which $(xz_1)^2 = 1$, hence, replacing, when needed, z_1 with z we may assume that $(xz)^2 = 1$.

By Lemma 12, yz is a $\{2, 3\}$-element, hence $(yz)^3 = 1$ or $(yz)^4 = 1$. If yz is of order 3 then, by Lemma 12, xyz is a $\{2, 3\}$-element and, by Lemma 3, $\langle x, y, z\rangle = 1$ which is not true. Therefore $(yz)^4 = 1$. If yz is an element of order 2 then $z^y = z^x = z^{-1}$, hence $z^{xy} = z$ and xyz is of order 21 which is not true. It follows that yz is an element of order 4 and, in addition, $K = \langle x, y, z\rangle$ satisfies the equality $\omega(K) = \omega(G)$. By Lemma 12, xz^y and xz^{yxy} are $\{2, 3\}$-elements, hence K is finite by Lemmas 3 and 4. By Proposition 2, $K \simeq L_2(7)$ contradicting the assumption. Theorem is proved.

Notice that all computations used in the proof have been realized with aid of GAP [9] and checked with the help of a special algorithm devised by the second author.

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