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ON H.WEYL AND H.MINKOWSKI POLYNOMIALS

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Abstract. We introduce certain polynomials, so-called H.Weyl and H.Minkowski polynomials, which have a geometric origin. The location of roots of these polynomials is studied.

> Erasm Darwin, the nephew of the great scientist Charles Darwin, believed that sometimes one should perform the most unusual experiments. They usually yield no results but when they do ... . So once he played trumpet in front of tulips for the whole day. The experiment yielded no results.

1. H. WEYL AND H.MINKOWSKI POLYNOMIALS.

Let $\mathcal{M}$ be a smooth manifold of dimension n:

$$\dim \mathcal{M} = n,$$

which is embedded injectively into the Euclidean space of a higher dimension, say $n+p$, $p>0$. We identify $\mathcal{M}$ with the image of this embedding, so we consider $\mathcal{M}$ as a subset of $\mathbb{R}^{n+p}$:

$$\mathcal{M} \subset \mathbb{R}^{n+p}.$$

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For $x \in \mathcal{M}$, let $\mathcal{N}_x$ be the normal subspace to $\mathcal{M}$ at the point x. $\mathcal{N}_x$ is an affine subspace of the ambient space $\mathbb{R}^{n+p}$,

$$\dim \mathcal{N}_x = p.$$

For $t > 0$, let

$$D_x(t) = \{y \in \mathcal{N}_x : \operatorname{dist}(y, x) \leq t\}, \tag{1.1}$$

where $\operatorname{dist}(y, x)$ is the Euclidean distance between x and y. If the manifold $\mathcal{M}$ is compact, and $t > 0$ is small enough, then

$$D_{x_1}(t) \cap D_{x_2}(t) = \emptyset \quad \text{for} \;\; x_1 \in \mathcal{M}, x_2 \in \mathcal{M}, x_1 \neq x_2. \tag{1.2}$$

Definition 1.1. *The set*

$$\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t) \stackrel{\text{def}}{=} \bigcup_{x \in \mathcal{M}} D_x(t) \tag{1.3}$$

is said to be the tube neighborhood *of the manifold $\mathcal{M}$, or* the tube *around $\mathcal{M}$. The number t is said to be* the radius *of this tube.*

Is it clear that for manifolds $\mathcal{M}$ *without boundary*,

$$\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t) = \{x \in \mathbb{R}^{n+p} : \operatorname{dist}(x, \mathcal{M}) \leq t\}, \tag{1.4}$$

where $\operatorname{dist}(x, \mathcal{M})$ is the Euclidean distance from x to $\mathcal{M}$. Thus, for manifolds without boundary, the equality (1.4) could also be taken as a definition of the tube $\mathfrak{T}_{\mathcal{M}}(t)$. However, for manifolds $\mathcal{M}$ *with boundary* the sets $\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t)$ defined by (1.3) and (1.4) do not coincide. In this, more general, case the tube around $\mathcal{M}$ should be defined by (1.3), but not by (1.4). Hermann Weyl, [69], obtained the following result, which is the starting point of our work:

Theorem [H.Weyl]. *Let $\mathcal{M}$ be a smooth compact manifold, without boundary or with boundary, of the dimension n: $\dim \mathcal{M} = n$, which is embedded in the Euclidean space $\mathbb{R}^{n+p}$, $p \geq 1$.*

I. *If $t > 0$ is small enough*[1]*, than the $(n+p)$ - dimensional volume Vol_{n+p} of the tube $\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t)$ around $\mathcal{M}$, considered as a function of the radius t of this tube, is a polynomial of the form*

$$\operatorname{Vol}_{n+p}(\mathfrak{T}_{\mathcal{M}}^{\mathbb{R}^{n+p}}(t)) = \omega_p \, t^p \Big(\sum_{l=0}^{[\frac{n}{2}]} w_{2l,p}(\mathcal{M}) \cdot t^{2l} \Big), \tag{1.5}$$

where

$$\omega_p = \frac{\pi^{p/2}}{\Gamma(\frac{p}{2} + 1)} \tag{1.6}$$

is is the p-dimensional volume of the unit p - dimensional ball.

[1] If the condition (1.2) is satisfied.

II. *The coefficients* $w_{2l,p}(\mathcal{M})$ *depend on* p *as:*

$$w_{2l,p}(\mathcal{M}) = \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} \cdot k_{2l}(\mathcal{M})\,, \quad 0 \le l \le \left[\tfrac{n}{2}\right]\,, \tag{1.7}$$

where the values $k_{2l}(\mathcal{M})$, $0 \le l \le [\frac{n}{2}]$, may be expressed only in terms of the intrinsic metric[2] of the manifold $\mathcal{M}$. *In particular, the constant term* $w_{0,p}(\mathcal{M}) = k_0(\mathcal{M})$ *is the* n*-dimensional volume of* $\mathcal{M}$*:*

$$k_0(\mathcal{M}) = \mathrm{Vol}_n\,(\mathcal{M}). \tag{1.8}$$

H. Weyl, [69], have expressed the coefficients $k_{2l}(\mathcal{M})$ as integrals of certain rather complicated curvature functions of the manifold $\mathcal{M}$.

Remark 1.1. *In the case when* $\mathcal{M}$ *is compact without boundary and even dimensional, say* $n = 2m$*, the top coefficient* $k_{2m}(\mathcal{M})$ *is especially interesting:*

$$k_{2m}(\mathcal{M}) = (2\pi)^m \chi(\mathcal{M}), \tag{1.9}$$

where $\chi(\mathcal{M})$ *is the Euler characteristic of* $\mathcal{M}$*.*

Definition 1.2. *Let* $\mathcal{M}$ *be a smooth manifold, without boundary or with boundary, of the dimension* n*:* $\dim \mathcal{M} = n$*, which is embedded in the Euclidean space* $\mathbb{R}^{n+p}$*,* $p \ge 1$*, and* $\mathfrak{T}^{\mathbb{R}^{n+p}}_{\mathcal{M}}(t)$ *is the tube of the radius* t *around* $\mathcal{M}$*, (1.1).*

The polynomial $W^p_{\mathcal{M}}(t)$ *which appears in the expression* (1.5) *for the volume* $\mathrm{Vol}_{n+p}\left(\mathfrak{T}^{\mathbb{R}^{n+p}}_{\mathcal{M}}(t)\right)$ *of this tube:*

$$\mathrm{Vol}_{n+p}\,(\mathfrak{T}^{\mathbb{R}^{n+p}}_{\mathcal{M}}(t)) = \omega_p t^p \cdot W^p_{\mathcal{M}}(t) \qquad \text{for small positive } t, \tag{1.10}$$

is said to be the H. Weyl polynomial of the index p for the manifold $\mathcal{M}$.

Remark 1.2. *The radius* t *of the tube is a positive number, so the formula* (1.10) *is meaningful for positive* t *only. However the polynomial* $W^p_{\mathcal{M}}$ *is determined uniquely by its restriction on any fixed interval* $[0, \varepsilon]$*,* $\varepsilon > 0$*, and we may and will consider this polynomial for* every complex t*.*

Definition 1.3. *Let* $\mathcal{M}$ *be a smooth manifold of the dimension* n*:* $\dim \mathcal{M} = n$*, which is embedded in the Euclidean space* $\mathbb{R}^{n+p}$*,* $p \ge 1$*, and let* $W^p_{\mathcal{M}}$ *be the Weyl polynomial of* $\mathcal{M}$ (*defined by* (1.2), (1.10)).

The coefficients $k_{2l}(\mathcal{M})$*,* $0 \le l \le [n/2]$ *which are* defined *in terms of the Weyl polynomial* $W^p_{\mathcal{M}}$ *by the equality*

$$W^p_{\mathcal{M}}(t) \overset{\text{def}}{=} \sum_{l=0}^{[\frac{n}{2}]} \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\, k_{2l}(\mathcal{M}) \cdot t^{2l}\,, \tag{1.11}$$

are said to be the Weyl coefficients of the manifold $\mathcal{M}$*.*

Remark 1.3. *Often, the factor in* (1.11) *appears in a 'decoded' form:*

$$\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} = \frac{1}{(p+2)(p+4)\,\cdots\,(p+2l)}\,. \tag{1.12}$$

[2]That is the metric which is induced on manifold $\mathcal{M}$ from the ambient space R^{n+p}.

Remark 1.4. *Defining the Weyl polynomials $W^p_{\mathcal{M}}$ of the manifold $\mathcal{M}$ by* (1.10)*, we assumed that $\mathcal{M}$ is already embedded into $\mathbb{R}^{n+p}$. The tube around $\mathcal{M}$ and its volume are primary in this definition. So, in fact we defined the notion of the Weyl polynomial not for the manifold $\mathcal{M}$ itself but for manifold $\mathcal{M}$ which is already embedded in an ambient space. Moreover, we assume implicitly that from the very beginning the manifold $\mathcal{M}$ carries a 'natural' Riemannian metric, and that this 'original' Riemannian metric coincides with the metric on $\mathcal{M}$ induced from the ambient space $\mathbb{R}^{n+p}$. (In other words, we assume that the imbedding is isometrical.) However, in this approach the 'original' metric does not play an 'explicite' role in the definition* (1.1)-(1.10)-(1.11) *of the Weyl polynomial $W^p_{\mathcal{M}}$ and the Weyl coefficients $k_{2l}(\mathcal{M})$.*

There is another approach to define the Weyl coefficients and the Weyl polynomials, which does not require an actual embedding $\mathcal{M}$ into the ambient space. Starting from the given Riemannian metric on $\mathcal{M}$, the Weyl coefficients $k_{2l}(\mathcal{M})$ can be introduced formally, by means of the Hermann Weyl expressions for $k_{2l}(\mathcal{M})$ in terms of the given metric on $\mathcal{M}$. Then the Weyl polynomials $W^p_{\mathcal{M}}(t)$ can be defined by means of the expression (1.11)*. In this approach, the intrinsic metric of $\mathcal{M}$ is primary, but not the tubes around $\mathcal{M}$ and their volumes.*

If the codimension p of $\mathcal{M}$ equals one[3], $\dim \mathcal{M} = n$, the Weyl polynomial is of the form:

$$\mathrm{Vol}_{n+1}(\mathfrak{T}^{\mathbb{R}^{n+1}}_{\mathcal{M}}(t)) = 2t \cdot W^1_{\mathcal{M}}(t)\,, \quad W^1_{\mathcal{M}}(t) = \sum_{l=0}^{[\frac{n}{2}]} w_{2l}(\mathcal{M}) \cdot t^{2l}, \tag{1.13}$$

where

$$w_{2l}(\mathcal{M}) = \frac{2^{-l}\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+l+1)}\, k_{2l}(\mathcal{M})\,, \quad 0 \le l \le [\tfrac{n}{2}]\,. \tag{1.14}$$

In (1.13) the 'shortened' notation is used: $w_{2l}(\mathcal{M})$ instead of $w_{2l,1}(\mathcal{M})$. The factor $2t$ is the one-dimensional volume of the one-dimensional ball of radius t, that is the length of the interval $[-t, t]$.

If the *hypersurface* $\mathcal{M}$ is orientable[4], then the tube $\mathfrak{T}_{\mathcal{M}}(t)$ can be decomposed into the union of two *half-tubes*, say, $\mathfrak{T}^+_{\mathcal{M}}(t)$ and $\mathfrak{T}^-_{\mathcal{M}}(t)$. The half-tubes $\mathfrak{T}^+_{\mathcal{M}}(t)$ and $\mathfrak{T}^-_{\mathcal{M}}(t)$ are the parts of the tube $\mathfrak{T}_{\mathcal{M}}(t)$ which are situated on the distinct sides of $\mathcal{M}$. In particular, if the hypersurface $\mathcal{M}$ is the boundary of a set $V : \mathcal{M} = \partial V$, then

$$\mathfrak{T}^+_{\mathcal{M}}(t) = \mathfrak{T}_{\mathcal{M}}(t) \setminus V, \quad \mathfrak{T}^-_{\mathcal{M}}(t) = \mathfrak{T}_{\mathcal{M}}(t) \cap V\,. \tag{1.15}$$

The $(n+1)-$ *dimensional* volumes $\mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\mathcal{M}}(t))$ and $\mathrm{Vol}_{n+1}(\mathfrak{T}^-_{\mathcal{M}}(t))$ of the half-tubes also are polynomials of t. These polynomials are of the form[5]:

$$\mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\mathcal{M}}(t)) = t\, W^+_{\mathcal{M}}(t), \quad \mathrm{Vol}_{n+1}(\mathfrak{T}^-_{\mathcal{M}}(t)) = t\, W^-_{\mathcal{M}}(t)\,, \tag{1.16}$$

[3]In other words, $\mathcal{M}$ is a hypersurface in $\mathbb{R}^{n+1}$.

[4]The orientation of the hypersurface $\mathcal{M}$ can be specified by means of the continuous vector field of unit normals on $\mathcal{M}$. The half-tubes $\mathfrak{T}^+_{\mathcal{M}}(t)$ and $\mathfrak{T}^-_{\mathcal{M}}(t)$ are the parts of the tube $\mathfrak{T}_{\mathcal{M}}(t)$ corresponding to the 'positive' и 'negative' directions of these normals.

[5]The equalities (1.16), (1.17) is one of the results of the theory of tubes around manifolds. See [26], [8],[1]

where:

$$W^+_{\mathcal{M}}(t) = \sum_{l=0}^{[\frac{n}{2}]} w_{2l}(\mathcal{M}) \cdot t^{2l} + t \sum_{l=0}^{[\frac{n+1}{2}]-1} w_{2l+1}(\mathcal{M}) \cdot t^{2l}, \tag{1.17a}$$

$$W^-_{\mathcal{M}}(t) = \sum_{l=0}^{[\frac{n}{2}]} w_{2l}(\mathcal{M}) \cdot t^{2l} - t \sum_{l=0}^{[\frac{n+1}{2}]-1} w_{2l+1}(\mathcal{M}) \cdot t^{2l}, \tag{1.17b}$$

and the coefficients $w_{2l}(\mathcal{M})$ are the same that in (1.13)-(1.14). Unlike the coefficients $w_{2l}(\mathcal{M})$, the coefficients $w_{2l+1}(\mathcal{M})$ depend not only on the 'intrinsic' metric of the manifold $\mathcal{M}$, but also on how $\mathcal{M}$ is embedded to $\mathbb{R}^{n+1}$. It is remarkable that when the volumes of the half-tubes are summed:

$$2\, W_{\mathcal{M}}(t) = W^+_{\mathcal{M}}(t) + W^-_{\mathcal{M}}(t),$$

the dependence on the way of embedding disappears. As it is seen from (1.17), $W^-_{\mathcal{M}}(t) = W^+_{\mathcal{M}}(-t)$, hence

$$2\, W_{\mathcal{M}}(t) = W^+_{\mathcal{M}}(t) + W^+_{\mathcal{M}}(-t). \tag{1.18}$$

Remark also that the volumes of the half-tubes can be expressed only in the terms of the polynomial $W^+_{\mathcal{M}}$:

$$\mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\mathcal{M}}(t)) = t\, W^+_{\mathcal{M}}(\ t\ ) \quad \text{for small positive } t\,. \tag{1.19a}$$

$$\mathrm{Vol}_{n+1}(\mathfrak{T}^-_{\mathcal{M}}(t)) = t\, W^+_{\mathcal{M}}(-t) \quad \text{for small positive } t\,. \tag{1.19b}$$

The theory of the tubes around manifolds is presented in [26], and to some extent in [8], Chapter 6, and in [1], Chapter 10. The comments of V.Arnold [6] to the Russian translations of the paper [69] by H.Weyl are very rich in content.

In the event that the hypersurface $\mathcal{M}$ is the boundary of a convex set V: $\mathcal{M} = \partial V$, the Weyl polynomial $W^1_{\mathcal{M}}$ can be expressed in terms of polynomials considered in the theory of convex sets.

In the theory of convex sets the following fact, which was discovered by Hermann Minkowski, [41, 42], is of principal importance: *Let V_1 and V_2 be compact convex sets in $\mathbb{R}^n$. For positive numbers t_1, t_2, let us form the 'linear combination' $t_1V_1 + t_2V_2$ of the sets V_1 u V_2 (in the sense commonly accepted in the theory of convex sets). Then the n-dimensional Euclidean volume $\mathrm{Vol}_n(t_1V_1 + t_2V_2)$ of this linear combination, considered as a function of the variables t_1, t_2, is a homogeneous polynomial of degree n. (It may be equal zero identically.)* Choosing V as V_1 and the unit ball B^n of $\mathbb{R}^n$ as V_2, we conclude :

Let V be a compact convex set in $\mathbb{R}^n$, B^n be the unit ball of $\mathbb{R}^n$. Then n-dimensional volume $\mathrm{Vol}_n(V + tB^n)$, considered as a function of the variable $t \in [0, \infty)$, is a polynomial of degree n.

Definition 1.4. *Let V, $V \subset \mathbb{R}^n$, be a compact convex set. The polynomial which expresses the n-dimensional volume of the linear combination $V + tB^n$ as a function of the variable $t \in [0, \infty)$ is said to be* the Minkowski polynomial of the set *V and is denoted by $M^{\mathbb{R}^n}_V(t)$:*

$$M^{\mathbb{R}^n}_V(t) = \mathrm{Vol}_n(V + tB^n)\,, \quad (t \in [0, \infty)). \tag{1.20}$$

The coefficient of Minkowski polynomial are denoted by $m_k^{\mathbb{R}^n}(V)$*:*

$$M_V^{\mathbb{R}^n}(t) = \sum_{0\leq k\leq n} m_k^{\mathbb{R}^n}(V)t^k. \tag{1.21}$$

If there is no need to emphasize that the ambient space is $\mathbb{R}^n$*, the shortened notation* $M_V(t)$*,* $m_k(V)$ *for the Minkowski polynomial and its coefficients will be used.*

Of course,

$$M_V^{\mathbb{R}^n}(t) = \mathrm{Vol}_n(\mathfrak{V}_V^{\mathbb{R}^n}(t)),$$

where $\mathfrak{V}_V^{\mathbb{R}^n}(t))$ is t-neighborhood of the set V with respect to $\mathbb{R}^n$:

$$\mathfrak{V}_V^{\mathbb{R}^n}(t) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, V) \leq t\}. \tag{1.22}$$

It is evident that

$$m_0(V) = \mathrm{Vol}_n(V), \quad \text{and} \quad m_n(V) = \mathrm{Vol}_n(B^n). \tag{1.23}$$

If the boundary ∂V of a convex set V is smooth, then the $(n-1)$-dimensional volume ('the area') of the boundary ∂V can be expressed as

$$m_1(V) = \mathrm{Vol}_{n-1}(\partial V)\,. \tag{1.24}$$

For a convex set V, whose boundary ∂V may be non-smooth, the formula (1.24) serves as a *definition* of the *'area'* of ∂V. (See [10], **31**; [41], § 24; [66], **6.4**.) Let us emphasize that the Minkowski polynomial is defined for an *arbitrary* compact convex set V, without any extra assumptions. The boundary of V may be non-smooth, and the interior of V may be empty. In particular, the Minkowski polynomial is defined for any convex polytope.

Definition 1.5. *Let* V*,* $V \subset \mathbb{R}^n$*, be a convex set.* V *is said to be* solid *if the interior of* V *is not empty, and* non-solid *if the interior of* V *is not empty.*

Definition 1.6. *The* n *- dimensional closed convex surface* $\mathcal{M}$ *is the boundary* ∂V *of a solid compact convex set* V*:*

$$\mathcal{M} = \partial V, \quad V \subset \mathbb{R}^{n+1}\,. \tag{1.25}$$

The set V *is said to be the* generating set for the surface $\mathcal{M}$.

Lemma 1.1. *If the closed* n *- dimensional convex surface* $\mathcal{M}$ *is also a smooth manifold, then the Weyl polynomial* $W^1_{\mathcal{M}}$ *of the surface* $\mathcal{M}$ *and the Minkowski polynomial* $M_V^{\mathbb{R}^{n+1}}$ *of its generating set* V *are related in the following way:*

$$2t\,W^1_{\mathcal{M}}(t) = M_V^{\mathbb{R}^{n+1}}(t) - M_V^{\mathbb{R}^{n+1}}(-t). \tag{1.26}$$

Proof of Lemma 1.1. We assign the positive orientation to the vector field of exterior normals on ∂V. Let $\mathfrak{T}^+_{\partial V}(t)$ is the 'exterior' half-tube around ∂V. For positive t,

$$V + tB^{n+1} = V \cup \mathfrak{T}^+_{\partial V}(t),$$

Moreover the set V and $\mathfrak{T}^+_{\partial V}(t)$ do not intersect. Therefore,

$$\mathrm{Vol}_{n+1}(V + tB^{n+1}) = \mathrm{Vol}_{n+1}(V) + \mathrm{Vol}_{n+1}(\mathfrak{T}^+_{\partial V}(t)).$$

Hence,

$$M_V(t) = M_V(0) + t\,W^+_{\mathcal{M}}(t), \quad \mathcal{M} = \partial V,$$

where $W^{+}_{\mathcal{M}}$ is a polynomial defined in (1.16) (with n replaced by $n+1$: now $\dim V = n+1$). Then also

$$M_V(-t) = M_V(0) - t\,W^{+}_{\mathcal{M}}(-t).$$

Thus, (1.19),

$$M_V(t) - M_V(-t) = \mathrm{Vol}_{n+1}(\mathfrak{T}^{+}_{\partial V}(t)) + \mathrm{Vol}_{n+1}(\mathfrak{T}^{-}_{\partial V}(t))\,,$$

or

$$M_V(t) - M_V(-t) = t\,(W^{+}_{\mathcal{M}}(t) + W^{+}_{\mathcal{M}}(-t)).$$

The equality (1.26) follows from the last equality and from (1.18). Q.E.D.

Since the Minkowski polynomial is defined for an arbitrary compact convex set, the formula (1.26) can serve as a *definition* of the Weyl polynomial of an *arbitrary* closed convex surface, smooth or non-smooth. Even more, we can define the Weyl polynomial for the 'improper convex surface ∂V', where V is a non-solid compact convex set.

Definition 1.7. *Let V, $V \subset \mathbb{R}^{n+1}$, be a compact convex set. The boundary ∂V of the set V is said to be the* boundary surface *of V. The boundary surface of V is said to be* proper *if V is solid, and* improper *if V is non-solid.*

The following improper closed convex surface plays a role in what follow:

Definition 1.8. *Let V, $V \subset \mathbb{R}^n$, be a compact convex set which is solid* with respect to $\mathbb{R}^n$. *We identify $\mathbb{R}^n$ with it image $\mathbb{R}^n \times 0$ by the 'canonical' embedding*[6] *$\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, and the set V with the set $V \times 0$ considered as a subset of $\mathbb{R}^{n+1}$: $V \times 0 \subset \mathbb{R}^{n+1}$. The set $V \times 0$, considered as a subset of $\mathbb{R}^{n+1}$, is said to be* the squeezed cylinder with the base *V.*

Remark 1.5. *The set $V \times 0$ can be interpreted as a ' cylinder of zero hight' whose 'lateral surface' is the Cartesian product $\partial V \times [0,0]$, and whose bases, lower and upper, are the sets $V \times (-0)$ and $V \times (+0)$:*

$$\partial(V \times 0) = \big((\partial V) \times [0,0]\big) \cup \big(V \times (-0)\big) \cup \big(V \times (+0)\big)\,. \tag{1.27}$$

In other words, the boundary surface $\partial(V \times 0)$ can be considered as 'the doubly covered' set V. In particular,

$$\dim \partial(V \times 0) = n\,. \tag{1.28}$$

and the number $\mathrm{Vol}_n(V\times(-0)) + \mathrm{Vol}_n(V\times(+0)) = 2\,\mathrm{Vol}_n(V)$ can be naturally interpreted as the 'n- dimensional area' of the n- dimensional convex surface (improper) $\partial(V \times 0)$:

$$\mathrm{Vol}_n(\partial(V \times 0)) = 2\,\mathrm{Vol}_n(V)\,. \tag{1.29}$$

On the other hand, the equality (1.24), in which the squeezed cylinder $V \times 0 \subset \mathbb{R}^{n+1}$ plays the role of the set $V \subset \mathbb{R}^n$, takes the form

$$\mathrm{Vol}_n(\partial(V \times 0)) = m_1^{\mathbb{R}^{n+1}}(V \times 0)\,, \tag{1.30}$$

where $m_k^{\mathbb{R}^{n+1}}(V \times 0)$, $k = 0, 1, \ldots, n+1$, are the coefficients of the Minkowski polynomial $M^{\mathbb{R}^{n+1}}_{V\times 0}(t)$ of the squeezed cylinder $V \times 0$ with respect to the ambient space $\mathbb{R}^{n+1}$. (See (1.21).)

[6] The point $x \in \mathbb{R}^n$ is identified with the point $(x, 0) \in \mathbb{R}^{n+1}$.

In section 11 we prove the following statement, which appears as Lemma 11.1 there:

Lemma 1.2. *Let V be a compact convex set in $\mathbb{R}^n$, and*

$$M_V^{\mathbb{R}^n}(t) = \sum_{0\leq k\leq n} m_k^{\mathbb{R}^n}(V)t^k \tag{1.31}$$

be the Minkowski polynomial with respect to the ambient space $\mathbb{R}^n$. Then the Minkowski polynomial $M_{V\times 0}^{\mathbb{R}^{n+1}}(t)$ with respect to the ambient space $\mathbb{R}^{n+1}$ is equal to:

$$M_{V\times 0^1}^{\mathbb{R}^{n+1}}(t) = t\sum_{0\leq k\leq n} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, m_k^{\mathbb{R}^n}(V)\, t^k\,. \tag{1.32}$$

So,

$$m_0^{\mathbb{R}^{n+1}}(V\times 0) = 0, \quad m_{k+1}^{\mathbb{R}^{n+1}}(V\times 0) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, m_k^{\mathbb{R}^n}(V),\ k=0,\,\dots,\,n\,.$$

In particular, $m_1^{\mathbb{R}^{n+1}}(V\times 0) = 2m_0^{\mathbb{R}^n}(V)$. Since $m_0^{\mathbb{R}^n}(V) = \mathrm{Vol}_n(V)$, (1.23),

$$m_1^{\mathbb{R}^{n+1}}(V\times 0) = 2\,\mathrm{Vol}_n(V)\,. \tag{1.33}$$

The equalities (1.29), (1.30) and (1.33) agree.

Remark 1.6. *Any non-solid compact convex set V can be presented as the limit* (*in the Hausdorff metric*) *of a monotonic*[7] *family $\{V_\varepsilon\}_{\varepsilon>0}$ of solid convex sets V_ε :*

$$V = \lim_{\varepsilon\to+0} V_\varepsilon.$$

Moreover, the approximating family $\{V_\varepsilon\}_{\varepsilon>0}$ of convex sets can be chosen so that the boundary $\partial(V_\varepsilon)$ of each set V_ε is a smooth surface. Thus, the improper convex surface ∂V may be presented as the limit of proper convex smooth surfaces $\partial(V_\varepsilon)$ which shrink to ∂V:

$$\partial V = \lim_{\varepsilon\to+0} \partial(V_\varepsilon).$$

Definition 1.9. *Let V, $V\subset\mathbb{R}^{n+1}$, be an arbitrary compact convex set. The Weyl polynomial $W_{\partial V}^1(t)$ of the convex surface $\mathcal{M}=\partial V$, proper or improper, is defined by the formula* (1.26). *In other words, the Weyl polynomial $t\,W_{\partial V}^1$ is defined as the odd part of the Minkowski polynomial $M_V^{\mathbb{R}^{n+1}}$:*

$$t\cdot W_{\partial V}^1(t) = {}^{\mathcal{O}}\!M_V^{\mathbb{R}^{n+1}}(t), \tag{1.34}$$

where the notions of the even part ${}^{\mathcal{E}}P$ and the odd part ${}^{\mathcal{O}}P$ of an arbitrary polynomial P are introduced in Definition 7.2 *below.*

Remark 1.7. *In the case when the set V is solid and its boundary ∂V is smooth, both definitions,* Definition 1.9 *and* Definition 1.2 *of the Weyl polynomial $W_{\partial V}^1$, are applicable to ∂V. In this case both definitions agree.*

[7] The monotonicity means that $V_{\varepsilon'}\supset V_{\varepsilon''}\supset V$ for $\varepsilon'>\varepsilon''>0$.

Remark 1.8. Why may be useful to consider improper convex surfaces and their Weyl polynomials?
As it was remarked (Remark 1.6), *every improper convex surface ∂V is a limiting object for a family of proper smooth convex surfaces $\partial(V_\varepsilon)$. It turns out that the Weyl polynomial for this improper surface is the limit of the Weyl polynomials for this 'approximating' family $\{V_\varepsilon\}_{\varepsilon>0}$ of smooth proper surfaces.*
So the Weyl polynomials for the improper surface ∂V may be useful in the study of the limiting behavior of the family of the Weyl polynomials for the proper surfaces $\partial(V_\varepsilon)$ shrinking to the improper surface ∂V. In particular, see Theorem 2.7 *formulated in the end of* Section 2, *and its proof presented in the end of* Section 11.

Let $\mathcal{M}$ be an n-dimensional closed convex surface which is not assumed to be smooth, and V is the generating convex set for $\mathcal{M}$: $\mathcal{M} = \partial V$. Let $M_V^{\mathbb{R}^{n+1}}$ be the Minkowski polynomial for V, defined by Definition 1.4. According to Definition 1.9, the Weyl polynomial $W^1_{\mathcal{M}}$ is equal to

$$W^1_{\mathcal{M}}(t) = \sum_{0\leq l\leq [\frac{n}{2}]} m_{2l+1}(V)t^{2l}, \tag{1.35}$$

in other words,

$$w_{2l}(\mathcal{M}) = m_{2l+1}(V), \quad 0 \leq l \leq [\tfrac{n}{2}], \tag{1.36}$$

where $w_{2l}(\mathcal{M})$ are the coefficients of the Weyl polynomial $W^1_{\mathcal{M}}$, (1.13), of the n-dimensional surface $\mathcal{M}$ with respect to the ambient space $\mathbb{R}^{n+1}$, and $m_k(V)$, $k = 2l+1$, are the coefficients of the Minkowski polynomial $M_V^{\mathbb{R}^{n+1}}$:

$$M_V^{\mathbb{R}^{n+1}}(t) = \mathrm{Vol}_{n+1}(V + tB^{n+1}), \quad M_V^{\mathbb{R}^{n+1}}(t) = \sum_{0\leq k\leq n+1} m_k(V)t^k\,. \tag{1.37}$$

Definition 1.10. *Given a closed n-dimensional convex surface $\mathcal{M}$, proper or not, $\mathcal{M} = \partial V$, the numbers $k_{2l}(\mathcal{M})$, $0 \leq l \leq [\frac{n}{2}]$, are* defined *as*

$$k_{2l}(\mathcal{M}) = 2^l\,\frac{\Gamma(l+\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+1)}\,m^{\mathbb{R}^{n+1}}_{2l+1}(V), \tag{1.38}$$

where $m_k^{\mathbb{R}^{n+1}}(V)$, $k = 2l+1$, are the coefficients of the Minkowski polynomial $M_V^{\mathbb{R}^{n+1}}$ for the generating set V, (1.37). The numbers $k_{2l}(\partial V)$, $0 \leq l \leq [\frac{n}{2}]$, are said to be the Weyl coefficients for the surface *$\mathcal{M}$.*

Remark 1.9. *According to* Lemma 1.2, *in the event that the (improper) convex surface $\mathcal{M}, \dim\mathcal{M} = n$, is the boundary of the squeezed cylinder (see Definition 1.8), that is if $\mathcal{M} = \partial(V\times 0)$, where $V \subset \mathbb{R}^n$, the Weyl coefficients $k_{2l}(\mathcal{M})$, $0 \leq l \leq [\frac{n}{2}]$, are:*

$$k_{2l}(\mathcal{M}) = 2^{l+1}\,\Gamma(l+1)\,m^{\mathbb{R}^n}_{2l}(V)\,, \tag{1.39}$$

where $m_k^{\mathbb{R}^n}(V)$, $k = 2l$, are the coefficients of the Minkowski polynomial $M_V^{\mathbb{R}^n}$ for the base V of the squeezed cylinder $\partial(V\times 0)$.

Remark 1.10. *In the case when convex surface $\mathcal{M}$, $\mathcal{M} = \partial V$, is smooth and 'proper', that is the set V generating the surface $\mathcal{M}$ is solid, both definitions, Definition 1.10 and Definition 1.3 of the Weyl coefficients $k_{2l}(\mathcal{M})$ are applicable. According to* (1.13)-(1.14) *and* (1.36)-(1.38), *in this case*[8] *both definitions agree.*

Note, that according to (1.24), (see also Remark 1.5),

$$k_0(\mathcal{M}) = \mathrm{Vol}_n(\mathcal{M}) \tag{1.40}$$

for every n - dimensional closed convex surface $\mathcal{M}$.

Lemma 1.3. I. *Let V, $V \subset \mathbb{R}^n$, be a solid (with respect to $\mathbb{R}^n$) compact convex set. Then the coefficients $m_k^{\mathbb{R}^n}(V)$, $0 \leq k \leq n$, of its Minkowski polynomials*[9] *are strictly positive: $m_k^{\mathbb{R}^n}(V) > 0$, $0 \leq k \leq n$.*

II. *Let $\mathcal{M}$ be a proper compact convex surface,* $\dim \mathcal{M} = n$. *Then all its Weyl coefficients $k_{2l}(\mathcal{M})$ are strictly positive*: $k_{2l}(\mathcal{M}) > 0$, $0 \leq l \leq [\frac{n}{2}]$.

III. *Let $\mathcal{M}$ be the boundary surface*[10] *of a squeezed cylinder whose base V,* $\dim V = n$, *is a compact convex set which is solid with respect to $\mathbb{R}^n$. Then all its Weyl coefficients $k_{2l}(\mathcal{M})$ are strictly positive*: $k_{2l}(\mathcal{M}) > 0$, $0 \leq l \leq [\frac{n}{2}]$.

The statement I of Lemma 1.3 is a consequence of a more general statement related to the monotonicity properties of the mixed volumes. This will be discussed later, in Section 8. The statements II and III of Lemma 1.3 are consequences of the statement I and (1.38), (1.39).

Definition 1.11. *Given a closed n - dimensional convex surface $\mathcal{M}$, the Weyl polynomial $W_{\mathcal{M}}^p$ of the index p, $p = 1, 2, 3, \dots$, for $\mathcal{M}$ is defined as*

$$W_{\mathcal{M}}^p(t) = \sum_{l=0}^{[\frac{n}{2}]} \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} k_{2l}(\mathcal{M}) \cdot t^{2l}\,, \tag{1.41}$$

where the Weyl coefficients $k_{2l}(\mathcal{M})$ are introduced in Definition 1.10.

Let us emphasize that in Definition 1.11 no assumption concerning the smoothness of the surface $\mathcal{M}$ are made. We already mentioned that the definitions of the Weyl coefficients k_{2l} for smooth manifolds and for convex surfaces agree. Therefore, Definitions 1.2 - 1.3: (1.3)-(1.10)-(1.11) of the Weyl polynomial and the Weyl coefficients for a smooth manifold and Definition 1.11 of the Weyl polynomials for a closed convex surface agree if the convex surface is also a smooth manifold.

We also define the $W_{\mathcal{M}}^{\infty}$ of the infinite index.

Definition 1.12. *Let $\mathcal{M}$,* $\dim \mathcal{M} = n$ *be either a smooth manifold, or a closed compact convex surface, and let $k_{2l}(\mathcal{M})$, $l = 0, 1, \dots, [\frac{n}{2}]$, be the Weyl coefficients of $\mathcal{M}$, defined by* Definition 1.3 *in the smooth case, and by* Definition 1.10 *in the convex case. The Weyl polynomial of the infinite index $W_{\mathcal{M}}^{\infty}$ is defined as*

$$W_{\mathcal{M}}^{\infty}(t) = \sum_{l=0}^{[\frac{n}{2}]} k_{2l}(\mathcal{M}) \cdot t^{2l}. \tag{1.42}$$

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[8] Actually, the equalities (1.14), (1.36) served as a motivation for Definition 1.10.

[9] See (1.20), (1.21).

[10] See Definition 1.8 and Remark 1.5.

Remark 1.11. *In view of* (1.12),

$$W^p_{\mathcal{M}}(\sqrt{p}t) = k_0(\mathcal{M}) + \sum_{l=1}^{[\frac{n}{2}]} \frac{p^l}{(p+2)(p+4)\,\cdots\,(p+2l)}\, k_{2l}(\mathcal{M}) \cdot t^{2l}\,.$$

Therefore, the polynomial $W^\infty_{\mathcal{M}}(t)$ can be considered as a limiting object for the family $\{W^p_{\mathcal{M}}(t)\}_{p=1,2,3,\dots}$ of the Weyl polynomials of the index p:

$$W^\infty_{\mathcal{M}}(t) = \lim_{p\to\infty} W^p_{\mathcal{M}}(\sqrt{p}t)\,. \tag{1.43}$$

Thus, the sequence $\{W^p_{\mathcal{M}}\}_{p=1,2,3,\dots}$ of the Weyl polynomials, $\deg W^p_{\mathcal{M}} = 2\left[\frac{n}{2}\right]$, *as well as the 'limiting' polynomial $W^\infty_{\mathcal{M}}$ are related to any closed n-dimensional convex surface $\mathcal{M}$.*

Weyl polynomials (and Minkowski polynomials in the convex case) reflect somehow intrinsic properties of the appropriate manifolds. On the other hand, there are known very distinguished and remarkable geometrical objects such as regular polytopes, compact matricial groups, spaces of constant curvatures, etc. Our belief is that the Weyl polynomials related to these geometric objects are of fundamental importance and possess interesting properties. These polynomials should be carefully studied. In particular, the following question is natural:

What can we say about roots of such polynomials?

2. FORMULATION OF MAIN RESULTS.

In this section we formulate the main results of this paper about location of the roots of the Minkowski and Weyl polynomials related to convex sets and surfaces. Dissipative and conservative polynomials. We introduce two classes of polynomials: dissipative polynomials and conservative polynomials. In many cases the Minkowski polynomials related to convex sets are dissipative, and the Weyl polynomials are conservative.

Definition 2.1. *The polynomial M is said to be dissipative if all roots of M are situated in the open left half plane $\{z : \operatorname{Re} z < 0\}$. The dissipative polynomials are also called the Hurwitz polynomials, or the stable polynomials.*

Definition 2.2. *The polynomial W is said to be conservative if all roots of W are purely imaginary and simple, in other words if all roots of W are contained in the imaginary axis $\{z : \operatorname{Re} z = 0\}$, and each of them is of multiplicity one.*

Theorem 2.1. *Given a closed compact convex surface $\mathcal{M}$, $\dim \mathcal{M} = n$ $\mathcal{M} = \partial V$, let $W^1_{\mathcal{M}}$ be the Weyl polynomial of index 1 related to $\mathcal{M}$, and let $M^{\mathbb{R}^{n+1}}_V$ be the Minkowski polynomial related to the set V.*

If the polynomial $M^{\mathbb{R}^{n+1}}_V$ is dissipative, then the polynomial $W^1_{\mathcal{M}}$ is conservative.

The proof of Theorem 2.1 is based on the relation (1.26). Theorem 2.1 is derived from (1.26) using Hermite-Biehler theorem. We do this in Section 7.

From (1.43) it follows that if for every p the polynomial $W^p_{\mathcal{M}}$ has only purely imaginary roots, than all the roots of the polynomial $W^\infty_{\mathcal{M}}$ are purely imaginary as well. In particular, *all the roots of the polynomial* $W^\infty_{\mathcal{M}}$ *are purely imaginary if for every* p *the polynomial* $W^p_{\mathcal{M}}$ *is conservative.*

However, what is important for us that is the converse statement:

Lemma 2.1. *If the polynomial* $W^\infty_{\mathcal{M}}$ *is conservative, then all the polynomials* $W^p_{\mathcal{M}}$, $p = 1, 2, 3, \ldots$, *are conservative as well.*

Lemma 2.1 is the consequence of some Laguerre result about the multiplier sequences. Proof of Lemma 2.1 appeares in the end of Section 6.

Keeping in mind Lemma 2.1, we will concentrate our efforts on the study of the location of the roots of the Weyl polynomial $W^\infty_{\mathcal{M}}$ of the infinite index.

The case of low dimension. In this section we discuss the Minkowski polynomials of convex sets V, $V \subset \mathbb{R}^n$, and the Weyl polynomials of closed convex surfaces $\mathcal{M}$, $\dim V = n$, for 'small' n: $n = 2, 3, 4, 5$.

Theorem 2.2. *Let* n *be one of the numbers* 2, 3, 4 *or* 5, *and let* V, $V \subset \mathbb{R}^n$, *be a solid compact convex set. Then the Minkowski polynomial* $M^{\mathbb{R}^n}_V$ *is dissipative.*

Theorem 2.3. *Let* n *be one of the numbers* 2, 3, 4 *or* 5, *and let* $\mathcal{M}$ *be closed proper*[11] *convex surface of dimension* n.

Then:

1. *The Weyl polynomial* $W^\infty_{\mathcal{M}}$ *of infinite index is conservative.*
2. *For every* $p = 1, 2, 3, \ldots$, *the Weyl polynomial* $W^p_{\mathcal{M}}$ *of index* p *is conservative.*

Theorem 2.2 and 2.3 are proved in section 10. Proving these theorems, we combine the Routh-Hurwitz criterion, which express the property of a polynomial to be dissipative in terms of its coefficients, and the Alexandrov-Fenchel inequalities, which express the logarithmic convexity property for the sequence of the cross-sectional measures of a convex set.

Selected 'regular' convex sets: balls, cubes, squeezed cylinders. For large n, the statements analogous to Theorems 2.2 and 2.3 do not hold. If n is large enough, then there exists such solid compact convex sets[12] V, $\dim V = n$, that Minkowski polynomials $M^{\mathbb{R}^{n+1}}_V$ are not dissipative, and the Weyl polynomials $W^p_{\partial V}$ are not conservative. However, for some 'regular' convex sets V, like balls and cubes, the Weyl polynomials $W^p_{\partial V}$ are conservative, and the Minkowski polynomial are dissipative in any dimension.

Let us present the collection of 'regular' convex sets and their boundary surfaces which we are dealing with further. Such sets and surfaces will be considered for every n.

[11]That is the generating set V is solid.

[12]Very flattened ellipsoids can be taken as such V. See Theorem 2.7.

◊ The unit ball B^n:

$$B^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{1 \le k \le n} |x_k|^2 \le 1\}, \tag{2.1}$$

$$\mathrm{Vol}_n(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}. \tag{2.2}$$

◊ The squeezed spherical cylinder $B^n \times 0$, $B^n \times 0 \subset \mathbb{R}^{n+1}$.

◊ The unit sphere,

$$S^n = \{x = (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{1 \le k \le n+1} |x_k|^2 = 1\},$$

in other words, the boundary surface of the unit ball: $S^n = \partial B^{n+1}$,

$$\mathrm{Vol}_n(S^n) = (n+1)\,\mathrm{Vol}_{n+1}(B^{n+1}). \tag{2.3}$$

◊ The boundary surface of the squeezed spherical cylinder $\partial(B^n \times 0)$:

$$\mathrm{Vol}_n(\partial(B^n \times 0)) = 2\,\mathrm{Vol}_n(B^n). \tag{2.4}$$

◊ The unit cube Q^n:

$$Q^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \max_{1 \le k \le n} |x_k| \le 1\}, \tag{2.5}$$

$$\mathrm{Vol}_n(Q^n) = 2^n. \tag{2.6}$$

◊ The squeezed cubic cylinder $Q^n \times 0$, $Q^n \times 0 \subset \mathbb{R}^{n+1}$.

◊ The boundary surface ∂Q^{n+1} of the unit cube:

$$\mathrm{Vol}_n(\partial Q^{n+1}) = (n+1)\,\mathrm{Vol}_{n+1}(Q^{n+1}). \tag{2.7}$$

◊ The boundary surface of the squeezed cubic cylinder $\partial(Q^n \times 0)$:

$$\mathrm{Vol}_n(\partial(Q^n \times 0)) = 2\,\mathrm{Vol}_n(Q^n). \tag{2.8}$$

The location of roots of the Minkowski and Weyl polynomials
related to the 'regular' convex sets.

Let us state the main results about location of roots of the Minkowski polynomials and the Weyl polynomials related to the above mentioned 'regular' convex sets and their surfaces.

Theorem 2.4. *For every $n = 1, 2, 3, \ldots$:*

1. *The Minkowski polynomial $M^{\mathbb{R}^n}_{B^n}$ related to the ball B^n is dissipative, moreover all its roots are negative*[13].
2. *The Minkowski polynomial $M^{\mathbb{R}^{n+1}}_{B^n \times 0}$ related to the squeezed spherical cylinder $B^n \times 0$ is of the form*[14] *$M^{\mathbb{R}^{n+1}}_{B^n \times 0}(t) = t \cdot D^{\mathbb{R}^{n+1}}_{B^n \times 0}(t)$, where the polynomial $D^{\mathbb{R}^{n+1}}_{B^n \times 0}$ is dissipative. If n is large enough, then the polynomial $M^{\mathbb{R}^{n+1}}_{B^n \times 0}$ has non-real roots.*
3. *The Minkowski polynomial $M^{\mathbb{R}^n}_{Q^n}$ related to cube Q^n is dissipative, moreover all its roots are negative.*

[13] This part of the Theorem is trivial: $M^{\mathbb{R}^n}_{B^n}(t) = (1+t)^n$

[14] The factors t appears because the set $B^n \times 0$ is not solid in $\mathbb{R}^{n+1}$.

4. *The Minkowski polynomial* $M^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ *related to the squeezed cubical cylinder* $Q^n\times 0$ *is of the form*[14] $M^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t)=t\cdot D^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t)$, *where the polynomial* $D^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ *is dissipative, moreover all roots of the polynomial* $D^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ *are negative.*

Theorem 2.5. *For every* $n=1,\,2,\,3,\,\dots$:

1. *The Weyl polynomials* $W^{\infty}_{\partial B^{n+1}}(t)$ *of infinite index, as well as the Weyl polynomials* $W^{p}_{\partial B^{n+1}}(t)$ *of arbitrary finite index* p, $p=1,\,2,\,\dots$, *related to the boundary surface of the ball* B^{n+1} *are conservative.*
2. *The Weyl polynomials* $W^{p}_{\partial(B^n\times 0)}$ *of order*[15] $p=1$, $p=2$ *and* $p=4$ *related to the boundary surface of the squeezed spherical cylinder* $B^n\times 0$ *are conservative.*
3. *The Weyl polynomials* $W^{\infty}_{\partial Q^{n+1}}(t)$ *of infinite index, as well as the Weyl polynomials* $W^{p}_{\partial Q^{n+1}}(t)$ *of arbitrary finite index* p, $p=1,\,2,\,\dots$, *related to the boundary surface of the cube* Q^{n+1} *are conservative.*
4. *The Weyl polynomials* $W^{\infty}_{\partial(Q^n\times 0)}(t)$ *of infinite index, as well as the Weyl polynomials* $W^{p}_{\partial(Q^n\times 0)}(t)$ *of arbitrary finite index* p, $p=1,\,2,\,\dots$, *related to the boundary surface of the squeezed cubic cylinder* $Q^n\times 0$ *are conservative.*

Remark 2.1. *The roots of the Weyl polynomial* $W^{1}_{\partial B^{n+1}}$ *can be found explicitly. Indeed*

$$W^{1}_{\partial B^{n+1}}(it)=\mathrm{Vol}_{n+1}(B^{n+1})\frac{1}{2it}\big((1+it)^{n+1}-(1-it)^{n+1}\big)\,.$$

Changing variable

$$t\to\varphi:\ 1+it=|1+it|e^{i\varphi},t=\operatorname{tg}\varphi\,,\quad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}\,,$$

we reduce the equation $W^{1}_{\partial B^{n+1}}(it)=0$ *to the equation*

$$\frac{\sin{(n+1)\varphi}}{\sin\varphi}=0,\quad -\frac{\pi}{2}<\varphi<\frac{\pi}{2}\,.$$

The roots of the latter equation are:

$$\varphi_k=\frac{k\pi}{n+1},\qquad -\Big[\frac{n}{2}\Big]\leq k\leq\Big[\frac{n}{2}\Big],\quad k\neq 0\,.$$

So, the roots t_k *of the equation* $W^{1}_{\partial B^{n+1}}(it)=0$ *are*

$$t_k=\operatorname{tg}\tfrac{k\pi}{n+1},\qquad -\big[\tfrac{n}{2}\big]\leq k\leq\big[\tfrac{n}{2}\big],\quad k\neq 0\,.$$

In particular, the polynomial $W^{1}_{S^n}$ *is conservative.*

Negative results:

Theorem 2.6. *Given an integer* p, $p\geq 5$. *If* n *is large enough:* $n\geq N(p)$, *then the Weyl polynomial* $W^{p}_{\partial(B^n\times 0)}$ *is not conservative: some of its roots do not belong to the imaginary axis.*

For an integer $q:q\geq 1$, let $E_{n,\,q,\,\varepsilon}$ be the $n+q$-dimensional ellipsoid:

$$E_{n,\,q,\,\varepsilon}=\{(x_1,\,x_2,\,\dots\,,\,x_n,\,\dots\,,\,x_{n+q})\in\mathbb{R}^{n+q}:\sum_{0\leq j\leq n+q}(x_j/a_j)^2\leq 1\},\tag{2.9a}$$

[15] The case $p=3$ remains open.

where

$$a_j = 1 \;\text{ for }\; 1 \le j \le n, \quad a_j = \varepsilon \;\text{ for }\; n+1 \le j \le n+q\,. \tag{2.9b}$$

Theorem 2.7.

1. *Given an integer* q : $5 \le q < \infty$. *If* n *is large enough:* $n \ge N(q)$, *and* ε *is small enough:* $0 < \varepsilon \le \varepsilon(n, q)$, *then the Minkowski polynomial* $M^{\mathbb{R}^{n+q}}_{E_{n,\,q,\,\varepsilon}}$ *is not dissipative: some of its roots are situated in the open right-half plane.*
2. *Given an integer* p *and an odd integer* q: $p \ge 1,\, q \ge 1,\, p+q \ge 6$. *If* n *is large enough:* $n \ge N(p,q)$, *and* ε *is small enough:* $0 < \varepsilon \le \varepsilon(n, p, q)$, *then the Weyl polynomial* $W^{p}_{E_{n,\,q,\,\varepsilon}}$ *is not conservative: some of its roots do not belong to the imaginary axis.*

Proof of Theorem 2.7 is presented in Section 11.

3. THE EXPLICIT EXPRESSIONS FOR THE MINKOWSKI AND WEYL POLYNOMIALS RELATED TO THE 'REGULAR' CONVEX SETS.

Hereafter, we use the following identity for the Γ-function:

$$\Gamma(\zeta+1/2)\,\Gamma(\zeta+1) = \pi^{1/2}2^{-2\zeta}\Gamma(2\zeta+1)\,,\ \forall \zeta \in \mathbb{C} : 2\zeta \ne -1, -2, -3, \dots\,. \tag{3.1}$$

Let as present explicit expressions for the Minkowski polynomials related to the 'regular' convex sets: balls, cubes, squeezed cylinders, as well as the expression for the Weyl polynomials related to the boundary surfaces of these sets. The items related to balls are marked by the symbol $\odot$, the items related to cubes are marked by the symbol $\boxdot$.

$\odot$ The unit ball B^n.

Since $B^n + tB^n = (1+t)B^n$ for $t > 0$, then, according to (1.4),

$$M^{\mathbb{R}^n}_{B^n}(t) = \mathrm{Vol}_n(B^n)\cdot(1+t)^n\,, \tag{3.2}$$

or

$$M^{\mathbb{R}^n}_{B^n}(t) = \mathrm{Vol}_n(B^n)\sum_{0\le k\le n}\frac{n!}{(n-k)!}\cdot\frac{t^k}{k!}\,. \tag{3.3}$$

Thus, the coefficients of the Minkowski polynomial $M^{\mathbb{R}^n}_{B^n}$ for the ball B^n are:

$$m^{\mathbb{R}^n}_k(B^n) = \mathrm{Vol}_n(B^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{1}{k!}\,,\quad 0\le k\le n\,. \tag{3.4}$$

$\odot$ The squeezed spherical cylinder $B^n \times 0$.

The Minkowski polynomial for the squeezed spherical cylinder $B^n \times 0$ is:

$$M^{\mathbb{R}^{n+1}}_{B^n\times 0}(t) = \mathrm{Vol}_n(B^n)\cdot t\sum_{0\le k\le n}\frac{n!}{(n-k)!}\frac{\pi^{1/2}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\frac{1}{k!}\,t^k\,. \tag{3.5}$$

The expression (3.5) is derived from (3.3) and (1.31)-(1.32). (See Lemma 1.2.)

Thus, the coefficients of the Minkowski polynomial $M^{\mathbb{R}^{n+1}}_{B^n\times 0}$ for the squeezed spherical cylinder $B^n \times 0$ are:

$$(3.6) \quad m_0^{\mathbb{R}^{n+1}}(B^n\times 0)=0, \quad m_{k+1}^{\mathbb{R}^{n+1}}(B^n\times 0)= \\ =\mathrm{Vol}_n(B^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{\pi^{1/2}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\frac{1}{k!}\,,\quad 0\le k\le n\,.$$

⊙ The unit sphere $S^n=\partial B^{n+1}$.
According to (1.38) and (3.4), the Weyl coefficients of the n-dimensional sphere $S^n=\partial B^{n+1}$ are:

$$(3.7) \qquad k_{2l}(\partial B^{n+1})=\mathrm{Vol}_n(\partial B^{n+1})\cdot\frac{n!}{(n-2l)!}\cdot\frac{1}{l!}\frac{1}{2^l}\,,\quad 0\le l\le [\tfrac{n}{2}]\,.$$

Thus, the Weyl polynomials related to the n-dimensional sphere are:

$$(3.8) \quad W^p_{\partial B^{n+1}}(t)=\mathrm{Vol}_n(\partial B^{n+1})\cdot \\ \cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{l!}\cdot\Big(\frac{t^2}{2}\Big)^l,\quad p=1,2,\dots\,.$$

$$(3.9) \qquad W^\infty_{\partial B^{n+1}}(t)=\mathrm{Vol}_n(\partial B^{n+1})\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{1}{l!}\cdot\Big(\frac{t^2}{2}\Big)^l\cdot$$

⊙ The boundary surface $\partial(B^n\times 0)$ of the squeezed spherical cylinder $B^n\times 0$.
According to (1.39) and (3.4), the Weyl coefficients of the n-dimensional improper surface $\partial(B^n\times 0)$ are:

$$(3.10) \quad k_{2l}(\partial(B^n\times 0))=\mathrm{Vol}_n(\partial(B^n\times 0))\cdot\frac{n!}{(n-2l)!}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\frac{1}{2^l},\quad 0\le l\le[\tfrac{n}{2}]\,.$$

Thus, the Weyl polynomials related to the (improper) surface $\partial(B^n\times 0)$ are:

$$(3.11) \quad W^p_{\partial(B^n\times 0)}(t)=\mathrm{Vol}_n(\partial(B^n\times 0))\cdot \\ \cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(\frac{t^2}{2}\Big)^l,\quad p=1,2,\dots\,.$$

$$(3.12) \quad W^\infty_{\partial(B^{n+1}\times 0)}(t)=\mathrm{Vol}_n(\partial(B^{n+1}\times 0))\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(\frac{t^2}{2}\Big)^l\cdot$$

⊡ The unit cube Q^n.
The Minkowski polynomial $M^{\mathbb{R}^n}_{Q^n}$ is:

$$(3.13) \qquad M^{\mathbb{R}^n}_{Q^n}(t)=\mathrm{Vol}_n(Q^n)\cdot\sum_{0\le k\le n}\frac{n!}{(n-k)!}\frac{1}{\Gamma(\frac{k}{2}+1)k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k t^k\,.$$

The expression (3.13) is obtained in the following way. The n-dimensional cube Q^n is considered as the Cartesian product of the one-dimensional cubes:

$$Q^n=Q^1\times\,\cdots\,Q^1\,.$$

For $n = 1$, the Minkowski polynomial is: $M^{\mathbb{R}^1}_{Q^1}(t) = 2(1+t)$. Then the rule is used how to express the Minkowski polynomial of the Cartesian product in terms of the Minkowski polynomials for the Cartesian factors. (See details in Section 12.)

Thus, the coefficients of the Minkowski polynomial for the cube Q^n are:

$$m^{\mathbb{R}^n}_k(Q^n) = \mathrm{Vol}_n(Q^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{1}{\Gamma(\frac{k}{2}+1)k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k, \quad 0\le k\le n\,. \tag{3.14}$$

⊡ The squeezed cubic cylinder $Q^n\times 0$.
The Minkowski polynomial $M^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ is:

$$M^{\mathbb{R}^{n+1}}_{Q^n\times 0}(t) = \mathrm{Vol}_n(Q^n)\cdot t\sum_{0\le k\le n}\frac{n!}{(n-k)!}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+1)k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k t^k\,. \tag{3.15}$$

The expression (3.15) is derived from (3.13) and (1.31)-(1.32). (See Lemma 1.2.)

Thus, the coefficients of the Minkowski polynomial $M^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ for the squeezed cubic cylinder are:

$$\begin{aligned} m^{\mathbb{R}^{n+1}}_0(Q^n\times 0) = 0,\quad & m^{\mathbb{R}^{n+1}}_{k+1}(Q^n\times 0) = \\ &= \mathrm{Vol}_n(Q^n)\cdot\frac{n!}{(n-k)!}\cdot\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2}+1)}\frac{1}{k!}\Big(\frac{\sqrt{\pi}}{2}\Big)^k, \quad 0\le k\le n\,. \end{aligned} \tag{3.16}$$

⊡ The boundary surface ∂Q^{n+1} of the unit cube Q^{n+1}.
According to (1.38) and (3.14), the Weyl coefficients of the n-dimensional surface ∂Q^{n+1} are:

$$\begin{aligned} k_{2l}(\partial Q^{n+1}) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot\frac{n!}{(n-2l)!}\cdot \\ \cdot\frac{1}{\Gamma(l+\frac{1}{2}+1)}\frac{1}{2^l\,l!}\Big(\frac{\sqrt{\pi}}{2}\Big)^{2l+1}, \quad 0\le l\le [\tfrac{n}{2}]\,. \end{aligned} \tag{3.17}$$

Taking into account the identity $\Gamma(l+1+\frac{1}{2})\cdot\Gamma(l+1) = \pi^{1/2}2^{-(2l+1)}\Gamma(2l+2)$, which is the identity (3.1) for $\zeta = l+1/2$, we can transform (3.17) to the form

$$k_{2l}(\partial Q^{n+1}) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot\frac{n!}{(n-2l)!}\cdot\frac{1}{(2l+1)!}\Big(\frac{\pi}{2}\Big)^l, \quad 0\le l\le [\tfrac{n}{2}]\,. \tag{3.18}$$

Thus, the Weyl polynomials related to the n-dimensional surface ∂Q^{n+1} are:

$$\begin{aligned} W^p_{\partial Q^{n+1}}(t) = \mathrm{Vol}_n(\partial Q^{n+1})\,\cdot \\ \cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{(2l+1)!}\cdot\Big(\frac{\pi t^2}{2}\Big)^l, \quad p=1,2,\ldots\,. \end{aligned} \tag{3.19}$$

$$W^\infty_{\partial Q^{n+1}}(t) = \mathrm{Vol}_n(\partial Q^{n+1})\cdot\sum_{l=0}^{[\frac{n}{2}]}\frac{n!}{(n-2l)!}\cdot\frac{1}{(2l+1)!}\cdot\Big(\frac{\pi t^2}{2}\Big)^l\,. \tag{3.20}$$

⊡ The boundary surface $\partial(Q^n\times 0)$ of the squeezed cubic cylinder $Q^n\times 0$.
According to (1.39) and (3.14), the Weyl coefficients of the surface (improper)

$\partial(Q^n \times 0)$ are:

$$k_{2l}(\partial Q^n \times 0) = \mathrm{Vol}_n(\partial Q^n \times 0) \cdot \frac{n!}{(n-2l)!} \cdot \\ \cdot \frac{\sqrt{\pi}}{\Gamma(l+\frac{1}{2})} \frac{1}{l!\,2^l} \Big(\frac{\pi}{2}\Big)^l, \quad 0 \le l \le [\tfrac{n}{2}]\,. \tag{3.21}$$

Using the identity $\Gamma(l+1/2)\Gamma(l+1) = \sqrt{\pi}2^{-2l}\Gamma(2l+1)$, which is the identity (3.1) for $\zeta = l$, the equality (3.21) can be transformed to the form

$$k_{2l}(\partial Q^n \times 0) = \mathrm{Vol}_n(\partial Q^n \times 0) \cdot \frac{n!}{(n-2l)!} \cdot \frac{1}{(2l)!} \Big(\frac{\pi}{2}\Big)^{2l}, \quad 0 \le l \le [\tfrac{n}{2}]\,. \tag{3.22}$$

Thus, the Weyl polynomials related to the improper n-dimensional surface $\partial(Q^n \times 0)$ are:

$$W^p_{\partial(Q^n\times 0)}(t) = \mathrm{Vol}_n(\partial(Q^n \times 0)) \cdot \\ \cdot \sum_{l=0}^{[\frac{n}{2}]} \frac{n!}{(n-2l)!} \cdot \frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} \cdot \frac{1}{(2l)!} \cdot \Big(\frac{\pi t^2}{2}\Big)^l, \quad p = 1, 2, \ldots\,. \tag{3.23}$$

$$W^\infty_{\partial(Q^n\times 0)}(t) = \mathrm{Vol}_n(\partial(Q^n \times 0)) \cdot \sum_{l=0}^{[\frac{n}{2}]} \frac{n!}{(n-2l)!} \cdot \frac{1}{(2l)!} \cdot \Big(\frac{\pi t^2}{2}\Big)^l. \tag{3.24}$$

4. WEYL AND MINKOWSKI POLYNOMIALS OF ‘REGULAR’ CONVEX SETS AS RENORMALIZED JENSEN POLYNOMIALS.

To investigate directly a location of roots of the Minkowski polynomials $M^{\mathbb{R}^n}_{B^n}$, $M^{\mathbb{R}^{n+1}}_{B^n\times 0}$, $M^{\mathbb{R}^n}_{Q^n}$, $M^{\mathbb{R}^{n+1}}_{Q^n\times 0}$ and Weyl polynomials $W^p_{\partial B^{n+1}}$, $W^p_{\partial(B^n\times 0)}$, $W^\infty_{\partial Q^{n+1}}$, $W^p_{\partial(Q^n\times 0)}$ for a *finite* n is difficult. It turns out that it is much easier to investigate first a location of roots of the entire functions which are the limits of the (renormalized) Minkowski and Weyl polynomials as $n \to \infty$, and then to deduce properties of roots of the original Minkowski and Weyl polynomials from properties of these limiting entire functions.

Jensen polynomials. From the explicit expressions (3.3), (3.5), (3.13), (3.15) for the Minkowski polynomials and (3.8), (3.9), (3.11), (3.12), (3.19), (3.20), (3.23), (3.24) for the Weyl polynomials we notice that each of this expressions contains the factor $\dfrac{n!}{(n-k)!}$, which is ‘a part’ of the binomial coefficient $\binom{n}{k}$. The factorial ratio can be presented as

$$\frac{n!}{(n-k)!} = 1 \cdot \Big(1 - \frac{1}{n}\Big) \cdot \Big(1 - \frac{2}{n}\Big) \cdot \,\cdots\, \cdot \Big(1 - \frac{k-1}{n}\Big) \cdot n^k\,, \quad 1 \le k \le n\,. \tag{4.1}$$

Definition 4.1.
1. *Given a formal power series f:*

$$f(t) = \sum_{0 \le l < \infty} a_l t^l\,. \tag{4.2}$$

We associate with f the sequence of the polynomials $\mathfrak{J}_n(f;t)$, $n = 1, 2, 3, \dots$:

$$\mathfrak{J}_n(f;t) = \sum_{0\leq l\leq n} \frac{n!}{(n-l)!}\frac{1}{n^l} \cdot a_l t^l, \tag{4.3}$$

or, decoding the factor $\frac{n!}{(n-l)!}\frac{1}{n^l}$,

$$\mathfrak{J}_n(f;t) = a_0 + \sum_{1\leq l\leq n} 1\big(1-\tfrac{1}{n}\big)\big(1-\tfrac{2}{n}\big)\,\cdots\,\big(1-\tfrac{l-1}{n}\big)\cdot a_l t^l. \tag{4.4}$$

The polynomials $\mathfrak{J}_n(f;t)$ are said to be the Jensen polynomials associated with the power series f.

2. *Given a* function *f holomorphic in the disc $\{t : |t| < R\}$, where $R \leq \infty$, we associate the sequence of the Jensen polynomials with the Taylor series* (4.2) *of the function f according the rule* (4.3). *We denote these polynomials by $\mathfrak{J}_n(f;t)$ as well and call them* the Jensen polynomials associated with the function f.
3. *The factors*

$$j_{n,0} = 1, \quad j_{n,k} = 1\big(1-\tfrac{1}{n}\big),\big(1-\tfrac{2}{n}\big)\,\cdots\,\big(1-\tfrac{k-1}{n}\big),\ 1\leq k\leq n, \qquad j_{n,k}=0,\ k>n\,, \tag{4.5}$$

are said to be the Jensen multipliers.

Thus, the Jensen polynomials associated with f of the form (4.2) *can be written as:*

$$\mathfrak{J}_n(f;t) = \sum_{0\leq <\infty} j_{n,l}\cdot a_l t^l\,. \tag{4.6}$$

Since $j_{n,k} \to 1$ as k is fixed, $n\to\infty$, the following result is evident:

Lemma 4.1 (The approximation property of Jensen polynomials.)**.**
Given the power series (4.2), *then:*

1. *The sequence of the Jensen polynomials $\mathfrak{J}_n(f;t)$ converge to the series f coefficients-wise;*
2. *If moreover the radius of convergence of the power series* (4.2) *is positive, say equal to R, $0 < R \leq \infty$, then the sequence of the Jensen polynomials $\mathfrak{J}_n(f;t)$ converge to the function which is the sum of this power series locally uniformly in the disc $\{t : |t| < R\}$.*

The approximation property in not specific for the polynomials constructed from the *Jensen multipliers* $j_{n,k}$. This property holds for *any* multipliers $j_{n,k}$ which satisfy the conditions $j_{n,k}\to 1$ as k is fixed, $n\to\infty$, and are uniformly bounded: $\sup\limits_{k,n}|j_{n,k}| < \infty$. What is much more specific, that for some f, the polynomial $\mathfrak{J}_n(f;t)$ constructed from the *Jensen multipliers* $j_{n,k}$ preserve the property of f to possess only real roots. In particular:

Theorem 1 ([Jensen])**.** *Let f be a* polynomial *such that all its roots are real. Then for each n, all roots of the Jensen polynomial $\mathfrak{J}_n(f,\,t)$ are real as well.*

This result is a special case of Schur composition theorem [56]. Actually, Jensen, [Jen], obtained a more general result in which formulation f can be not only a polynomial with real roots, but also an entire function which belongs to the so called *Laguerre-Polya class of entire functions*. We return to this generalization later, is Section 5. Now we focus our attention on representation of the Minkowski and Weyl polynomials as Jensen polynomials of certain entire functions.

The relation (4) as well as the expressions (3.3), (3.5), (3.13), (3.15) for the Minkowski polynomials suggest us how the Minkowski polynomials should be renormalized so that the renormalized polynomials tend to a non-trivial limit as $n \to \infty$.

Entire functions which generate the Minkowski polynomials for balls, cubes, spherical and cubic cylinders. Let us introduce the infinite power series:

$$\mathcal{M}_{B^\infty}(t) = \sum_{0 \le k < \infty} \frac{1}{k!}\, t^k\,; \tag{4.7a}$$

$$\mathcal{M}_{B^\infty \times 0}(t) = \sum_{0 \le k < \infty} \frac{\Gamma(\frac{1}{2}+1)\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, \frac{1}{k!}\, t^k\,; \tag{4.7b}$$

$$\mathcal{M}_{Q^\infty}(t) = \sum_{0 \le k < \infty} \frac{1}{\Gamma(\frac{k}{2}+1)k!} \Big(\frac{\sqrt{\pi}}{2}\Big)^k\, t^k\,; \tag{4.7c}$$

$$\mathcal{M}_{Q^\infty \times 0}(t) = \sum_{0 \le k < \infty} \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{k+1}{2}+1)k!} \Big(\frac{\sqrt{\pi}}{2}\Big)^k\, t^k\,. \tag{4.7d}$$

The series (4.7) represent entire functions which grow not faster than exponentially. More precisely, the functions $\mathcal{M}_{B^\infty}$ and $\mathcal{M}_{B^\infty \times 0}$ grow exponentially: they are of order 1 and normal type, the functions $\mathcal{M}_{Q^\infty}$ and $\mathcal{M}_{BQ^\infty \times 0}$ grow subexponentially: they are of order 2/3 and normal type.

With each of the entire functions (4.7) we associate the sequence of polynomials which are the Jensen polynomials associated with this entire function:

$$\mathcal{M}_{B^n}(t) = \mathcal{J}_n(\mathcal{M}_{B^\infty}; t)\,, \tag{4.8a}$$

$$\mathcal{M}_{B^n \times 0}(t) = \mathcal{J}_n(\mathcal{M}_{B^\infty \times 0}; t)\,, \tag{4.8b}$$

$$\mathcal{M}_{Q^n}(t) = \mathcal{J}_n(\mathcal{M}_{Q^\infty}; t) \tag{4.8c}$$

$$\mathcal{M}_{Q^n \times 0}(t) = \mathcal{J}_n(\mathcal{M}_{Q^\infty \times 0}; t)\,, \tag{4.8d}$$

From the expressions (3.3), (3.5), (3.13), (3.15) for the Minkowski polynomials it follows that they are related to the above introduced polynomials (4.8) as:

$$M^{\mathbb{R}^n}_{B^n}(t) = \mathrm{Vol}_n(B^n) \qquad \mathcal{M}_{B^n}(nt)\,; \tag{4.9a}$$

$$M^{\mathbb{R}^{n+1}}_{B^n \times 0}(t) = \mathrm{Vol}_n(B^n)\omega_1 t\, \mathcal{M}_{B^n \times 0}(nt)\,; \tag{4.9b}$$

$$M^{\mathbb{R}^n}_{Q^n}(t) = \mathrm{Vol}_n(Q^n) \qquad \mathcal{M}_{Q^n}(nt)\,; \tag{4.9c}$$

$$M^{\mathbb{R}^{n+1}}_{Q^n \times 0}(t) = \mathrm{Vol}_n(Q^n)\omega_1 t\, \mathcal{M}_{Q^n \times 0}(nt)\,; \tag{4.9d}$$

The polynomials $\mathcal{M}_{B^n}$, $\mathcal{M}_{B^n \times 0}$, $\mathcal{M}_{Q^n}$, $\mathcal{M}_{Q^n \times 0}$ can be interpreted as *renormalized Minkowski polynomials* respectively. We take the equalities (4.9) as the definition of the renormalized Minkowski polynomials $\mathcal{M}_{B^n}$, $\mathcal{M}_{B^n \times 0}$, $\mathcal{M}_{Q^n}$, $\mathcal{M}_{Q^n \times 0}$ in terms of the 'original' Minkowski polynomials $M^{\mathbb{R}^n}_{B^n}$, $M^{\mathbb{R}^{n+1}}_{B^n \times 0}$, $M^{\mathbb{R}^n}_{Q^n}$, and $M^{\mathbb{R}^{n+1}}_{Q^n \times 0}$,.

From the approximative property of Jensen polynomials and from (4.8) it follows that

$$\begin{aligned}(4.10)\quad \mathcal{M}_{B^n}(t)\to\mathcal{M}_{B^\infty}(t),\ \ \mathcal{M}_{B^n\times 0}(t)\to\mathcal{M}_{B^\infty\times 0}(t),\ \ \mathcal{M}_{Q^n}(t)\to\mathcal{M}_{Q^\infty}(t),\\ \mathcal{M}_{Q^n\times 0}(t)\to\mathcal{M}_{Q^\infty\times 0}(t)\ \text{ as }\ n\to\infty\,.\end{aligned}$$

This explains the notation (4.7) .

We summarize the above stated consideration as the following

Theorem 4.1. *Let $\{V^n\}$ be one of the four families of convex sets: $\{B^n\}$, $\{B^n\times 0\}$, $\{Q^n\}$, $\{Q^n\times 0\}$. For each of these four families, there exists the* single *entire function*[16] $\mathcal{M}_{V^\infty}$ *such that in* every *dimension n, the renormalized Minkowski polynomials $\mathcal{M}_{V^n}$, defined by (4.9), are generated by this entire function $\mathcal{M}_{V^\infty}$ as the Jensen polynomials $\mathfrak{J}_n(\mathcal{M}_{V^\infty})$: the equalities (4.8) hold.*

Entire functions which generate the Weyl polynomials for the surfaces of balls, cubes, spherical and cubic cylinders. Let us introduce the infinite power series:

$$(4.11a)\qquad \mathcal{W}^p_{\partial B^\infty}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{l!}\cdot\Big(-\frac{t^2}{2}\Big)^l,\quad p=1,2,\dots;$$

$$(4.11b)\qquad \mathcal{W}^\infty_{\partial B^\infty}(t)=\sum_{l=0}^{\infty}\frac{1}{l!}\cdot\Big(-\frac{t^2}{2}\Big)^l;$$

$$(4.11c)\quad \mathcal{W}^p_{\partial(B^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(-\frac{t^2}{2}\Big)^l,\quad p=1,2,\dots;$$

$$(4.11d)\quad \mathcal{W}^\infty_{\partial(B^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{\Gamma(1/2)}{\Gamma(l+1/2)}\cdot\Big(-\frac{t^2}{2}\Big)^l;$$

$$(4.11e)\qquad \mathcal{W}^p_{\partial Q^\infty}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{(2l+1)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^l,\quad p=1,2,\dots;$$

$$(4.11f)\qquad \mathcal{W}^\infty_{\partial Q^\infty}(t)=\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^l;$$

$$(4.11g)\quad \mathcal{W}^p_{\partial(Q^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)}\cdot\frac{1}{(2l)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^l,\quad p=1,2,\dots;$$

$$(4.11h)\quad \mathcal{W}^\infty_{\partial(Q^\infty\times 0)}(t)=\sum_{l=0}^{\infty}\frac{1}{(2l)!}\cdot\Big(-\frac{\pi t^2}{2}\Big)^l.$$

The series (4.11) represent entire functions. The functions (4.11b) and (4.11d) are of order 2 and normal type, the functions (4.11a), (4.11c), (4.11f) and (4.11h) are of order 1 and normal type, the functions (4.11e) and (4.11g) are of order 2/3 and normal type.

[16]The symbol V^∞ means $\{B^\infty\}$, $\{B^\infty\times 0\}$, $\{Q^\infty\}$ or $\{Q^\infty\times 0\}$ respectively,

With each of the entire functions (4.11) we associate the sequence of polynomials which are the Jensen polynomials associated with this entire function:

(4.12a) $\quad \mathcal{W}^p_{\partial B^{n+1}}(t) = \mathcal{J}_{2[n/2]}(\mathcal{W}^p_{\partial B^\infty}; t)\,, \qquad 1 \le p \le \infty\,;$

(4.12b) $\quad \mathcal{W}^p_{\partial(B^{n+1}\times 0)}(t) = \mathcal{J}_{2[n/2]}(\mathcal{W}^p_{\partial(B^\infty\times 0)}; t)\,, \qquad 1 \le p \le \infty\,;$

(4.12c) $\quad \mathcal{W}^p_{\partial Q^n}(t) = \mathcal{J}_{2[n/2]}(\mathcal{W}^p_{\partial Q^\infty}; t)\,, \qquad 1 \le p \le \infty\,;$

(4.12d) $\quad \mathcal{W}^p_{\partial(Q^n\times 0)}(t) = \mathcal{J}_{2[n/2]}(\mathcal{W}^p_{\partial(Q^\infty\times 0)}; t)\,, \qquad 1 \le p \le \infty\,.$

From the expressions (3.8), (3.9), (3.11), (3.12), (3.19), (3.20), (3.23), (3.24), for the Weyl polynomials it follows that they are related to the above introduced polynomials (4.12) as:

(4.13a) $\quad W^p_{\partial B^{n+1}}(t) = \mathrm{Vol}_n(\partial B^{n+1}) \cdot \mathcal{W}^p_{\partial B^{n+1}}(int)\,;$

(4.13b) $\quad W^p_{\partial(B^n\times 0)}(t) = \mathrm{Vol}_n(\partial(B^n \times 0)) \cdot \mathcal{W}^p_{\partial(B^n\times 0)}(int)\,;$

(4.13c) $\quad W^p_{\partial Q^{n+1}}(t) = \mathrm{Vol}_n(\partial Q^{n+1}) \cdot \mathcal{W}^p_{\partial Q^{n+1}}(int)\,;$

(4.13d) $\quad W^p_{\partial(Q^n\times 0)}(t) = \mathrm{Vol}_n(\partial(Q^n \times 0)) \cdot \mathcal{W}^p_{\partial(Q^n\times 0)}(int)\,;$

The equalities (4.13) hold for all $n:\ 1 \le n < \infty,\ p:\ 1 \le p \le \infty.$

The polynomials $\mathcal{W}^p_{\partial B^{n+1}}$, $\mathcal{W}^p_{\partial(B^n\times 0)}$, $\mathcal{W}^p_{\partial Q^{n+1}}$, $\mathcal{W}^p_{\partial(Q^n\times 0)}$ can be interpreted as *renormalized Weyl polynomials*. We take the equalities (4.13) as the definition of the renormalized Weyl polynomials $\mathcal{W}^p_{\partial B^{n+1}}$, $\mathcal{W}^p_{\partial(B^n\times 0)}$, $\mathcal{W}^p_{\partial Q^{n+1}}$, $\mathcal{W}^p_{\partial(Q^n\times 0)}$ in terms of the 'original' Minkowski polynomials $W^p_{\partial B^{n+1}}$, $W^p_{\partial(B^n\times 0)}$, $W^p_{\partial Q^{n+1}}$, $W^p_{\partial(Q^n\times 0)}$.

From the approximative property of Jensen polynomials and from (4.12) it follows that for every fixed $p,\ 1 \le p \le \infty$,

(4.14) $$\mathcal{W}^p_{\partial B^{n+1}}(t) \to \mathcal{W}^p_{\partial B^\infty}(t), \quad \mathcal{W}^p_{\partial(B^n\times 0)}(t) \to \mathcal{W}^p_{\partial(B^\infty\times 0)}(t),$$
$$\mathcal{W}^p_{\partial Q^{n+1}}(t) \to \mathcal{W}^p_{\partial Q^\infty}(t), \mathcal{W}^p_{\partial(Q^n\times 0)}(t) \to \mathcal{W}^p_{\partial(Q^\infty\times 0)}(t) \quad \text{as} \quad n \to \infty\,.$$

This explains the notation (4.11).

We summarize the above stated consideration as the following

Theorem 4.2. *Let $\{\mathcal{M}^n\}$ be one of the four families of n-dimensional convex surfaces: $\{\partial B^{n+1}\}$, $\{\partial(B^n \times 0)\}$, $\{\partial Q^{n+1}\}$, $\{\partial(Q^n \times 0)\}$. For each of these four families, and for each p, $1 \le p \le \infty$, there exists the* single *entire function*[17] $\mathcal{W}^p_{\mathcal{M}^\infty}$ *such that in* every *dimension n, the renormalized Weyl polynomials $\mathcal{W}^p_{\mathcal{M}^n}$, defined by (4.13), are generated by this entire function $\mathcal{W}^p_{\mathcal{M}^\infty}$ as the Jensen polynomials $\mathcal{J}_2[n/2](\mathcal{W}^p_{\mathcal{M}^\infty})$.*

[17]The symbol $\mathcal{M}^\infty$ means $\{B^\infty\}$, $\{B^\infty \times 0\}$, $\{Q^\infty\}$ or $\{Q^\infty \times 0\}$ respectively,

5. ENTIRE FUNCTIONS OF THE HURWITZ AND OF THE LAGUERRE-POLYA CLASS. MULTIPLIERS PRESERVING LOCATION OF ROOTS.

Hurwitz class of entire functions.

Definition 5.1. *An entire function H is said to be in the* Hurwitz class, *written $H \in \mathcal{H}$, if*

1. *$H \not\equiv 0$, and roots of H have negative real part: if $H(\zeta) = 0$, then $\operatorname{Re}\zeta < 0$.*
2. *The function H is of exponential type: $\overline{\lim\limits_{|z|\to\infty}} \frac{\ln|H(z)|}{|z|} < \infty$, and its defect d_H is non-negative: $d_H \geq 0$, where*

$$2d_H = \overline{\lim_{r\to+\infty}} \frac{\ln|H(r)|}{r} - \overline{\lim_{r\to+\infty}} \frac{\ln|H(-r)|}{r}\,. \tag{5.1}$$

The following functions serve as examples of entire functions of class $\mathcal{H}$:
a). A dissipative polynomial $P(t)$.
b). An exponential $\exp\{\alpha t\}$, where $\operatorname{Re}\alpha \geq 0$.
c). The product $P(t)\cdot\exp\{\alpha t\}$: $P(t)$ is a dissipative polynomial, $\operatorname{Re}\alpha \geq 0$.

The significance of the Hurwitz class of entire functions stems from the fact that function in this class [18] are the locally uniform limits in $\mathbb{C}$ of dissipative polynomials.

Laguerre-Polya class of entire functions.

Definition 5.2. *An entire function E is said to be in the* Laguerre-Pólya class, *written $E \in \mathcal{L}\text{-}\mathcal{P}$, if E is real and can be expressed in the form*

$$E(t) = ct^n e^{-\beta t^2+\alpha t} \prod_{k=1}^{\infty} (1+t\alpha_k)\, e^{-t\alpha_k}, \tag{5.2}$$

where $c \in \mathbb{R}\setminus 0$, $\beta \geq 0$, $\alpha \in \mathbb{R}$, $\alpha_k \in \mathbb{R}$, n is non-negative integer, and $\sum_{k=1}^{\infty}\alpha_k^2 < \infty$.

Within the Laguerre-Polya class, those functions E are said to be of type I, *written $E \in \mathcal{L}\text{-}\mathcal{P}\text{-}\mathrm{I}$, which are representable in the form*

$$E(t) = ct^n e^{\alpha t} \prod_{k=1}^{\infty} (1+t\alpha_k)\,, \tag{5.3}$$

where $c \in \mathbb{R}\setminus 0$, $\alpha \geq 0$, $\alpha_k \geq 0$, n is non-negative integer, and $\sum_{k=1}^{\infty}\alpha_k < \infty$.

> The significance of the Laguerre-Polya class stems from the fact that function in this class, *and only these*, are the locally uniform limits in $\mathbb{C}$ of polynomials with only real roots. (See [37], Chapter 8; [43], Chapter II, Theorems 9.1, 9.2, 9.3.)

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[18] The full description of the class of entire functions which are the limits of dissipative polynomials can be found in [37], Chapter VIII, Theorem 4. This class (up to the change of variables $z \to iz$) is denoted by P^* there.

Lemma 5.1. *An entire function E which is of type* I *in the Laguerre-Polya class also is the Hurwitz class:*

$$\mathcal{L}\text{-}\mathcal{P}\text{-I} \subset \mathcal{H}\,.$$

PROOF. The roots of the entire function E which admit the representation (5.3) are located at the points $-(\alpha_k)^{-1}$, thus is strictly negative. From the properties of the infinite product $\prod\limits_{k=1}^{\infty}(1+t\alpha_k)$ with $\sum_{k=1}^{\infty}|\alpha_k|<\infty$, it follows that a function E which admit the representation (5.3) is of exponential type α, and $\overline{\lim\limits_{r\to+\infty}}\frac{\ln|H(\pm r)|}{r}=\pm\alpha$. Thus, the defect $d_H=\alpha\geq 0$ since $\alpha\geq 0$. ∎

Multipliers preserving the reality of roots.

Definition 5.3. *A sequence $\{\gamma_k\}_{0\leq k<\infty}$ of real numbers is a* multiplier sequence *if for every polynomial f:*

$$f(t)=\sum_{0\leq k\leq n}a_kt^k$$

with only real roots, the polynomial

$$h(t)=\sum_{0\leq k\leq n}\gamma_ka_kt^k$$

too has only real roots. (The degree n of the polynomial f can be arbitrary.)

Theorem 2 ([Polya, Schur]). *A sequence $\{\gamma_k\}_{0\leq k<\infty}$ of real numbers which are not all roots is a multiplier sequence if and only if the power series*

$$\Psi(t)=\sum_{0\leq k\leq\infty}\frac{\gamma_k}{k!}\,t^k$$

represents an entire function, and either the function $\Psi(t)$ or the function $\Psi(-t)$ is in the Lagierre-Polya class of type I.

This result was obtained in [49]. The presentation of this and related results can be found in Chapter VIII of [37], in Chapter II of [43], in [51] (Section 5), in numerous papers by Th. Craven and G. Csordas.

Theorem 5.1 ([Jensen-Craven-Csordas-Williamson]). *Let $E(t)$ be an entire function belonging to the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$, and $\{\mathcal{J}_n(E,\,t)\}_{n=1,\,2,\,3,\,\dots}$ be the sequence of the Jensen polynomials associated with the function $\mathcal{E}$.* (Definition 4.1.)

1. *Then for each n, all roots of the polynomial $\mathcal{J}_n(E,\,t)$ are real;*
2. *If $E(t)$ belongs to the subclass $\mathcal{L}$-$\mathcal{P}$-I of the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$, then for each n, all roots of the polynomial $\mathcal{J}_n(E,\,t)$ are negative;*
3. *If moreover $E(t)$ is not of the form $E(t)=p(t)\,e^{\beta t}$, where $p(t)$ is a polynomial, then for each n, all roots of the polynomial $\mathcal{J}_n(E,\,t)$ are simple.*

The statement 1 of the theorem was proved by Jensen [19], [Jen]. It is a special case of Theorem by G.Polya and I. Schur corresponding to $\Psi(t) = \left(1 + \frac{t}{n}\right)^n$. The refinement of the statement 1 which is formulated as the statement 3 was done by G.Csordas and J. Williamson in [18], where the alternative proof of the statement 1 also was done. In [18], the main Theorem formulated on p. 263, which appeared as the statement 3 of Theorem 5.1 of the present paper, was formulated not accurately. The correction was done in [16], Section 4.1 there.

Theorem 5.2. *Let H be an entire function belonging to the Hurwitz class $\mathcal{H}$, and $\{\mathcal{J}_n(H, t)\}_{n=1, 2, 3, \dots}$ be the sequence of the Jensen polynomials associated with the function $\mathcal{H}$.* (Definition 4.1.) *Then for each n, the polynomial $\{\mathcal{J}_n(H, t)$ is dissipative.*

Theorem 5.2 can be obtained as a consequence of Theorem 5.1 and Hermite-Bieler Theorem. Proof of Theorem 5.2 will be done in Section 5.

Laguerre multipliers.

Theorem 3 ([Laguerre])**.** *Let an entire function $E(t)$,*

$$E(t) = \sum_{0 \le l < \omega} \varepsilon_l t^l, \quad \omega \le \infty, \tag{5.4}$$

be in the Laguerre-Polya class: $E \in \mathcal{L}\text{-}\mathcal{P}$, and let an entire function ψ be in the Laguerre-Polya class $\mathcal{L}\text{-}\mathcal{P}$ and moreover satisfy the condition: all roots of ψ are negative.

1. *Then the power series*

$$E_\psi = \sum_{0 \le l < \omega} \varepsilon_l \psi(l) t^l \tag{5.5}$$

   converges for every t, and its sum is an entire function of the Laguerre-Polya class: $E_\psi \in \mathcal{L}\text{-}\mathcal{P}$.
2. *If moreover $E(t)$ is of type* I*: $E \in \mathcal{L}\text{-}\mathcal{P}$-*I*, then the the sum of power series* (5.5) *also is an entire function of the type* I*: $E_\psi \in \mathcal{L}\text{-}\mathcal{P}$-*I.

This theorem appeared by E. Laguerre, [35], section 18, p.117, or [34], p. 202. Laguerre himself has formulated this theorem for the function E which is a polynomial with real roots. The extended formulation, where E is a general entire function from the class $\mathcal{L}$-$\mathcal{P}$, can be found in the paper [49], p. 112, or in its reprint in [47], p.123. In [49] the extended formulation is attributed to Jensen, [Jen].

The presentation of the above mentioned results of Polya, Schur, Laguerre, Jensen, as well as of many related results, can be found in [43], Chapter II; [37], Chapter VIII,[51]; [51], Chapter 5, especially Sections 5.5, 5.6, 5.7; in numerous papers of Th. Craven and G. Csordas (See for example [14]). See also [50], Part five. The book of L. de Branges [19] is closely related to this circle of problems.

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[19] Though Jensen himself did not introduce explicitly the polynomials which are called 'the Jensen polynomials' now.

6. PROPERTIES OF ENTIRE FUNCTIONS GENERATING MINKOWSKI AND WEYL POLYNOMIALS OF 'REGULAR' CONVEX SETS AND THEIR SURFACES.

Entire functions generating the Minkowski polynomials.

Theorem 6.1. *The entire functions* (4.7) *generating the renormalised Minkowski polynomials of balls, cubes, squeezed spherical and cubic cylinders, possesses the following properties:*

1. *The function* $\mathcal{M}_{B^\infty}$ *is of type* I *of the Laguerre-Polya class;*
2. *The function* $\mathcal{M}_{B^\infty\times 0}$ *belongs to the Hurwitz class. It has infinitely roots, all but finitely many its roots are non-real;*
3. *The function* $\mathcal{M}_{Q^\infty}$ *is of type* I *of the Laguerre-Polya class;*
4. *The function* $\mathcal{M}_{Q^\infty\times 0}$ *is of type* I *of the Laguerre-Polya class.*

Lemma 6.1. *The function* $\dfrac{1}{\Gamma(t+1)}$, *where* Γ *is the Euler Gamma function, is in Laguerre-Polya class, and all its roots is negative.*

Indeed,

$$\frac{1}{\Gamma(t+1)} = e^{Ct}\prod_{1\leq k<\infty}\left(1+\frac{t}{k}\right)e^{-\frac{t}{k}}\,,$$

(C is the Euler constant, $C \approx 0.5772156\ldots$.) ∎

sc Proof of Theorem 6.1. The statement 1 is evident: $\mathcal{M}_{B^\infty}(t) = e^t$.

To obtain Statement 3, we remark that the function $\mathcal{M}_{Q^\infty}$ is of the form E_ψ, (5.5), where $E(t) = \exp\{\frac{\sqrt{\pi}}{2}t\}$, and $\psi(t) = \dfrac{1}{\Gamma(\frac{t}{2}+1)}$. Then we apply the Laguerre theorem on multipliers to these E and ψ. The needed property of ψ is formulated as Lemma 6.1.

The statement 4 can be obtained in the same way that the statement 3. One need take $E(t) = \exp\{\frac{\sqrt{\pi}}{2}t\}$, and $\psi(t) = \dfrac{\Gamma(\frac{1}{2})}{\Gamma(\frac{t+1}{2}+1)}$.

Proof of the statement 2 is more complicated. From (4.7b) it follows that

$$\mathcal{M}_{B^\infty\times 0}(t) = \sum_{0\leq k<\infty} B(\tfrac{k}{2}+1,\tfrac{1}{2})\frac{1}{k!}\,t^k = \sum_{0\leq k<\infty}\int\limits_0^1 \xi^{\frac{k}{2}}(1-\xi)^{-\frac{1}{2}}\,d\xi\,\frac{1}{k!}\,t^k\,.$$

Changing the order of summation and integration and summarizing the exponential series, we obtain the integral representation:

$$\mathcal{M}_{B^\infty\times 0}(t) = 2\int\limits_0^1 (1-\xi^2)^{-\frac{1}{2}}\xi e^{\xi t}d\xi\,. \tag{6.1}$$

The fact that the functions $\mathcal{M}_{B^\infty\times 0}$ belongs to the Hurwitz class will be derived from the integral representation (6.1). This will be done in Section 13. ∎

Entire functions generating the Weyl polynomials.

Lemma 6.2. *Let* E*:*

$$E(t) = \sum_{0\le l<\infty} a_l t^{2l} \tag{6.2}$$

be an even entire function of the class $\mathcal{L}$-$\mathcal{P}$*, and let* $p > 0$ *be a number. Then the function* $E_p(t)$ *defined by the power series*

$$E_p(t) \overset{\text{def}}{=} \sum_{1\le l<\infty} \frac{2^{-l}\Gamma\big(\frac{p}{2}+1\big)}{\Gamma\big(l+\frac{p}{2}+1\big)} \cdot a_l t^{2l}, \tag{6.3}$$

belongs to the class $\mathcal{L}$-$\mathcal{P}$ *as well.*

Proof. Lemma 6.2 is the consequence of the Laguerre theorem on multipliers. The function

$$\psi_p(t) = \frac{2^{-\frac{t}{2}}\Gamma\big(\frac{p}{2}+1\big)}{\Gamma\big(\frac{t}{2}+\frac{p}{2}+1\big)} \tag{6.4}$$

is in the Laguerre-Polya class (see Lemma 6.1), and its roots are negative. ∎

We point out, see (4.11), that the entire functions $\mathcal{W}^p_{\partial B^\infty}$, $\mathcal{W}^p_{\partial(B^\infty\times 0)}$, $\mathcal{W}^p_{\partial Q^\infty}$, $\mathcal{W}^p_{\partial(Q^\infty\times 0)}$, which generate the Weyl polynomials of the finite index p for the appropriate families of convex surfaces, can be obtained from the entire functions $\mathcal{W}^\infty_{\partial B^\infty}$, $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$, $\mathcal{W}^\infty_{\partial Q^\infty}$, $\mathcal{W}^\infty_{\partial(Q^\infty\times 0)}$, which generate the Weyl polynomials of the infinite index, by means of the transformation of the form

$$\sum_{0\le k<\infty} a_k t^k \to \sum_{0\le k<\infty} \psi_p(k)\, a_k t^k\,.$$

Theorem 6.2.

1. *The functions* $\mathcal{W}^\infty_{\partial B^\infty}$*,* $\mathcal{W}^\infty_{\partial Q^\infty}$*,* $\mathcal{W}^\infty_{\partial(Q^\infty\times 0)}$ *belong to the Laguerre-Polya class* $\mathcal{L}$-$\mathcal{P}$*.*
2. *The function* $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$ *does not belong to the Laguerre-Polya class* $\mathcal{L}$-$\mathcal{P}$*: this function has infinitely many non-real roots.*

Proof. The statement 1 is evident in view of the explicit expressions:

$$\mathcal{W}^\infty_{\partial B^\infty}(t) = \exp\{-t^2/2\}\,, \tag{6.5}$$

$$\mathcal{W}^\infty_{\partial Q^\infty}(t) = \frac{\sin\{(\pi/2)^{\frac{1}{2}}t\}}{(\pi/2)^{\frac{1}{2}}t}\,, \tag{6.6}$$

$$\mathcal{W}^\infty_{\partial(Q^\infty\times 0)} = \cos\{(\pi/2)^{\frac{1}{2}}\,t\}\,. \tag{6.7}$$

The function $\mathcal{W}^\infty_{\partial(B^\infty\times 0)}$, which appears in Statement 2, can not be expressed in terms of 'elementary' functions, but it can be expressed in terms of the Mittag-Leffler function $\mathcal{E}_{1,\frac{1}{2}}$:

$$\mathcal{W}^\infty_{\partial(B^\infty\times 0)}(t) = \sqrt{\pi}\,\mathcal{E}_{1,\frac{1}{2}}\left(-\frac{t^2}{2}\right), \tag{6.8}$$

where

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{0\leq k<\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}\,. \tag{6.9}$$

From (6.9) the integral representation can be derived:

$$\sqrt{\pi}\mathcal{E}_{1,\,\frac{1}{2}}(t) = 1 + t\int\limits_0^1 (1-\xi)^{-\frac{1}{2}} e^{t\xi}\, d\xi\,. \tag{6.10}$$

The integral representation (6.10) can be derived from the Tailor series (6.9) in the same way as the integral representation (6.1) was derived from the Taylor series (4.7b). From (6.10) the following asymptotic can be obtained:

$$\sqrt{\pi}\mathcal{E}_{1,\,\frac{1}{2}}(t) = \begin{cases} \frac{1}{2t}, & t\to\,-\infty,\\ \sqrt{\pi t}\, e^t, & t\to\,+\infty.\\ O(|t|), & t\to\,\pm\, i\infty. \end{cases} \tag{6.11}$$

From (6.11) it follows that the indicator diagram of the entire function $\mathcal{E}_{1,\,\frac{1}{2}}(t)$ of the exponential type is the interval $[0,\,1]$. Moreover, the function $\mathcal{E}_{1,\,\frac{1}{2}}(it)$ belongs to the class C, as this class was defined in [38], Lecture 17. From Theorem of Cartwright-Levinson (Theorem 1 of the Lecture 17 from [38]) it follows that the function $\mathcal{E}_{1,\,\frac{1}{2}}(t)$ has infinitely many roots, these roots have a positive density, and are located 'near' the rays $\arg t = \frac{\pi}{2}$ and $\arg t = -\frac{\pi}{2}$. From this and from (6.8) it follows that the roots of the function $\mathcal{W}^{\infty}_{\partial(B^\infty\times 0)}(t)$ are located near four rays $\arg t = \frac{\pi}{4}$, $\arg t = \frac{3\pi}{4}$, $\arg t = \frac{5\pi}{4}$, $\arg t = \frac{7\pi}{4}$. In particular, infinitely many of the roots of the function $\mathcal{W}^{\infty}_{\partial(B^\infty\times 0)}(t)$ are non-real. ∎

Remark 6.1. *Much more precise results about the Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}$ and distribution of its roots are known. See, for example,* [27], section 18.1, *or* [20].

Theorem 6.3.
1. *For every $p = 1,\,2,\,\ldots$, the functions $\mathcal{W}^p_{\partial B^\infty}$, $\mathcal{W}^p_{\partial Q^\infty}$, $\mathcal{W}^p_{\partial(Q^\infty\times 0)}$ belong to the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$.*
2. *If p is large enough, then the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ does not belong the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$: it has non-real roots.*

Proof. The statement 1 of this theorem is a consequence of the statement 1 of Theorem 6.2 and Lemma 6.2. The statement 2 of this theorem is a consequence of the statement 2 of Theorem 6.2 and the approximational property (1.43). ∎

Remark 6.2. *The fact that the function $\mathcal{W}^p_{\partial B^\infty}$ belongs to the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$, that is all its roots are real, can be established without reference to* Lemma 6.2. *The function $\mathcal{W}^p_{\partial B^\infty}$ can be expressed in terms of Bessel functions J_ν. Recall that for arbitrary ν,*

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{0\leq l<\infty} \frac{(-1)^l (t^2/4)^l}{l!\,\Gamma(\nu+l+1)}\,. \tag{6.12}$$

Comparing (6.12) *with* (4.11a)*, we see that*

$$\mathcal{W}^p_{\partial B^\infty}(t) = \Gamma\left(\frac{p}{2}+1\right)\left(\frac{t}{2}\right)^{-\frac{p}{2}} J_{\frac{p}{2}}(t)\,. \tag{6.13}$$

In particular,
for[20] $p=1$*,*

$$\mathcal{W}^1_{\partial B^\infty}(t) = \frac{\sin t}{t}, \tag{6.14}$$

for $p=2$*,*

$$\mathcal{W}^2_{\partial B^\infty}(t) = 2\,\frac{J_1(t)}{t}. \tag{6.15}$$

It is known that for every $\nu > -1$*, all roots of the Bessel function* $J_\nu(t)$ *are real (This result is due to A.Hurwitz. See, for example,* [64], Chapter XV, Section 15.27.*)*

The statement 2 of Theorem 6.3 may be strengthen essentially.

Theorem 6.4.
1. *For* $p = 1, 2, 4$*, the function* $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ *belongs the Laguerre-Polya class* $\mathcal{L}$-$\mathcal{P}$*;*
2. *For* $p:\ 5 \leq p \leq \infty$*, the function* $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ *does not belong the Laguerre-Polya class* $\mathcal{L}$-$\mathcal{P}$ *: it has infinitely many non-real roots.*

Proof. For every $p \geq 1$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ admits the integral representation

$$\mathcal{W}^p_{\partial(B^\infty\times 0)} = p\int\limits_0^1 (1-\xi^2)^{\frac{p}{2}-1}\xi\cos t\xi\,d\xi\,. \tag{6.16}$$

This integral representation can be obtained from (4.11c) in the same way that the integral representation (6.1) was obtained from (4.7b). Using the identity

$$\Gamma(l+1/2)\,\Gamma(l+1) = \Gamma(1/2)\,2^{-2l}\,\Gamma(2l+1),$$

we reduce (4.11c) to the form

$$\mathcal{W}^p_{\partial(B^\infty\times 0)} = \frac{p}{2}\sum_{0\leq l<\infty} \mathrm{B}(l+1,p/2)(-1)^l\,\frac{t^{2l}}{(2l)!}\,.$$

Then we use the integral representation for the function Beta, change the order of summation and integration and summarize the series using the Taylor expansion for $\cos z$. For every $p:\ 1 \leq p < \infty$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ can be calculated asymptotically. This calculation may be done using the integral representation (6.16), or in other way. The asymptotic expression for the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ is presented in Section 13, see (13.27), (13.28). From this expression it follows that:
1. For $p > 4$, infinitely many (actually all but finitely many) roots of the $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ are non-real. This is sufficiently for the negative result of the statement 2 of Theorem 6.4 to be obtained.

[20]Deriving (6.14) from (6.13), we used the formula $J_{\frac{1}{2}}(t) = \left(\frac{2}{\pi t}\right)^{\frac{1}{2}}\sin t$. (Concerning this formula, see, for example, [70], section 17.24.) However, (6.14) may be obtained directly from (4.11a).

2. For $p \leq 4$, all but finitely many roots of the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ are real and simple. This alone is not sufficiently for the result of the statement 1 of to be obtained. The additional reasoning should be invoked. For $p = 2$ and $p = 4$, the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ can be calculated explicitly. The case $p = 3$ remains open. Proof of the fact that for $p = 1, 2, 4$ all roots of the function $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ are real will be done in Section 13. See Lemma 13.6. ∎

Proof of Theorem 2.5. According to Theorems 6.2, 6.3 and 6.4 (Statements 1 of these theorems), each of the functions $\mathcal{W}^p_{\partial B^\infty}$, $\mathcal{W}^p_{\partial Q^\infty}$, $\mathcal{W}^p_{\partial(Q^\infty \times 0)}$ with $p : 1 \leq p \leq \infty$, and $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ with $p = 1, 2, 4$, belongs to the class of Laguerre-Polya $\mathcal{L}$-$\mathcal{P}$. By Theorem of Jensen-Csordas-Williamson, the Jensen polynomials associated with each of these entire functions, has only simple real roots. According to Theorem 4.2, the renormalized Weyl polynomials $\mathcal{W}^p_{\partial B^{n+1}}$, $\mathcal{W}^p_{\partial Q^{n+1}}$, $\mathcal{W}^p_{\partial(Q^n \times 0)}$ with $p : 1 \leq p \leq \infty$, and $\mathcal{W}^p_{\partial(B^n \times 0)}$ with $p = 1, 2, 4$ have only simple real roots. In view of renormalizing relations (4.13), the Weyl polynomials $W^p_{\partial B^{n+1}}$, $W^p_{\partial Q^{n+1}}$, $W^p_{\partial(Q^n \times 0)}$ are conservative. ∎

Proof of Theorem 2.6. According to Theorem 6.4, Statement 2, for $p : 5 \leq p \leq \infty$, each of the entire functions $\mathcal{W}^p_{\partial(B^\infty \times 0)}$ has infinitely many non-real roots. Since for fixed p, $\mathfrak{J}_n(\mathcal{W}^p_{\partial(B^\infty \times 0)}; t) \to \mathcal{W}^p_{\partial(B^\infty \times 0)}(t)$ locally uniformly in $\mathbb{C}$ as $n \to \infty$, by Hurwitz theorem, every polynomial $\mathfrak{J}_n(\mathcal{W}^p_{\partial(B^\infty \times 0)})$ with $p, n :\ p \geq 5,\ n \geq N(p)$ has non-real roots. By Theorem 4.2, the renormalized Weyl polynomials $\mathcal{W}^p_{\partial(B^n \times 0)}$ have non-real roots. In view of the renormalizing relations (4.12), the Weyl polynomial $W^p_{\partial(B^n \times 0)}$ have roots which do not belong to the imaginary axis. ∎

Proof of Lemma 2.1. This lemma is a consequence of lemma 6.2. If the polynomial $W^\infty_{\mathcal{M}}(t)$ is conservative, then the polynomial $E(t) = W^\infty_{\mathcal{M}}(it)$ is a real polynomial with only real simple roots. The function $E_p(t) = W^p_{\mathcal{M}}(it)$ is related with this $E(t)$ as well as the function $E_p(t)$ appeared in (6.3) is related to $E(t)$ from (6.2). By Lemma 6.2, all roots of E_p are real. Let us show that the roots are simple. Consider the function $E(t) + \varepsilon$, were ε is a small real number, positive or negative. Since all roots of the polynomial $E(t)$ are real and simple, all roots of the polynomial $E(t) + \varepsilon$ are real if $|\varepsilon|$ is small enough. By Lemma 6.2, all roots of the polynomial $E_p(t) + \varepsilon$ are real. But if the polynomial $E_p(t)$ have a multiple root, this root splits into a group of simple roots by the perturbation $E_p(t) \to E_p(t) + \varepsilon$, and by an appropriate choice of sign of ε, some of roots in this group will be non-real. ∎

Remark 6.3. *We apply* Lemma 2.1 *in special cases* $n = 2, 3, 4, 5$ *only. In these cases Lemma is quite elementary. Actually only the cases* $n = 4$ *and* $n = 5$ *deserve attention, the cases* $n = 2$ *and* $n = 3$ *are trivial. The cases* $n = 4$ *and* $n = 5$ *are reduced to the following elementary statement:*

Let k_0, k_2, k_4 be positive numbers. Assume that the roots of the polynomial $Q(t) = k_0 + k_2 t + k_4 t^2$ are negative and different. Then for every $p > 0$, the roots of the polynomial

$$Q^p(t) = k_0 + \frac{k_1}{(p+2)} t + \frac{k_4}{(p+2)(p+4)} t^2$$

are negative and different as well.

Indeed, the conditions posed on polynomials Q and Q^p are equivalent to the inequalities

$$k_1^2 > k_0 k_4 \quad \text{and} \quad \left(\frac{k_1}{p+2}\right)^2 > k_0 \frac{k_2}{(p+2)(p+4)}\,.$$

It is evident that the first of these inequalities implies the second. □

7. HERMITE-BIELER THEOREM AND ITS APPLICATION.

In its traditional form, Hermite-Bieler theorem gives conditions under which all roots of a polynomial belong to the upper half-plane $\{z : \operatorname{Im} z > 0\}$. We need the version of this theorem adopted to the left half-plane, and for the case of polynomials with non-negative coefficients only. Before to present such a reformulation of Hermite-Bieler theorem, we give several definitions:

Definition 7.1. *Let S_1 and S_2 be two sets which are situated on the same straight line*[21] *L of the complex plane: $S_1 \subset L$, $S_2 \subset L$, and moreover each of the sets S_1, S_2 consists of isolated points only. The sets S_1 and S_2* interlace *if between every two points of S_1 there is a point of S_2, and between every two points of S_2 there is a point of S_1.*

Definition 7.2. *Let P be a power series:*

$$P(t) = \sum_{0 \le k} p_k t^k, \tag{7.1}$$

where t is a complex variable, and the coefficients p_k are complex numbers.

The real part *$^{\mathcal{R}}P$ and the* imaginary part *$^{\mathcal{J}}P$ of P are defined as*

$$^{\mathcal{R}}P(t) = \frac{P(t) + \overline{P(\overline{t})}}{2}, \quad ^{\mathcal{J}}P(t) = \frac{P(t) - \overline{P(\overline{t})}}{2i}\,, \tag{7.2}$$

where the overline bar is used as a notation for the complex conjugation.

The even part *$^{\mathcal{E}}P$ and the* odd part *$^{\mathcal{O}}P$ of P are defined as*

$$^{\mathcal{E}}P(t) = \frac{P(t) + P(-t)}{2}, \quad ^{\mathcal{O}}P(t) = \frac{P(t) - P(-t)}{2}\,, \tag{7.3}$$

In term of coefficients,

$$^{\mathcal{R}}P(t) = \sum_{0 \le k} a_k t^k, \quad ^{\mathcal{J}}P(t) = \sum_{0 \le k} b_k t^k, \tag{7.4a}$$

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[21] In our considerations the straight line L will be either the real axis or the imaginary axis.

where

$$a_k = \frac{p_k + \overline{p_k}}{2}, \quad b_k = \frac{p_k - \overline{p_k}}{2i}\,, \tag{7.4b}$$

and

$$^{\mathcal{E}}P(t) = \sum_{0 \le l} p_{2l} t^{2l}, \quad ^{\mathcal{O}}P(t) = \sum_{0 \le l} p_{2l+1} t^{2l+1}. \tag{7.5}$$

Theorem 4 ([Hermite-Bieler]). *Let P be a polynomial, $A = {}^{\mathcal{R}}P$ and $B = {}^{\mathcal{I}}P$ be the real and imaginary parts of P, i.e.*

$$P(t) = A(t) + iB(t),$$

where A and B be a polynomials with real coefficients. In order for all roots of P to be contained within the open upper half-plane $\{z : \operatorname{Im} z > 0\}$, it is necessary and sufficient that the following three conditions be satisfied:

1. *The roots of each of the polynomials A and B are all real and simple.*
2. *The sets $\mathcal{Z}_A$ and $\mathcal{Z}_B$ of the roots of the polynomials A and B interlace.*
3. *The inequality*

$$B'(0)A(0) - A'(0)B(0) > 0 \tag{7.6}$$

holds.

Let us formulate a version of Hermite-Bieler Theorem for the left half-plane.

Lemma 7.1. *Let M be a polynomial with positive coefficients,*

$$M(t) = \sum_{0 \le k \le n} m_k t^k, \quad m_k > 0,\ 0 \le k \le n\,,$$

and let $^{\mathcal{E}}M$ and $^{\mathcal{O}}M$ be the even and the odd parts of M. In order for the polynomial M be dissipative it is necessary and sufficient that the following two condition be satisfied:

1. *The polynomials $^{\mathcal{E}}M$ and $^{\mathcal{O}}M$ are conservative.*
2. *The sets of roots of the polynomials $^{\mathcal{E}}M$ and $^{\mathcal{O}}M$ interlace.*

Lemma 7.2. *Let W,*

$$W(t) = w_0 + w_2 t^2 + w_4 t^4 \cdots + w_{2m-2} t^{2m-2} + w_{2m} t^{2m} \tag{7.7}$$

be an even polynomial with positive coefficients w_{2l}:

$$w_0 > 0,\ w_2 > 0,\ \dots\,,\ w_{2m} > 0\,.$$

In order for the polynomial W to be conservative it is necessary and sufficient that the polynomial $M = W + W'$ to be dissipative, where W' is the derivative of W:

$$W'(t) = 2 \cdot w_2 t + 4 \cdot w_4 t^3 \cdots + (2m-2) \cdot w_{2m-2} t^{2m-3} + 2m \cdot w_{2m} t^{2m-1}\,. \tag{7.8}$$

Proof of Lemma 7.1. Let

$$P(t) = M(it), \quad A(t) = (^{\mathcal{E}}M)(it), \quad B(t) = i^{-1} \cdot (^{\mathcal{O}}M)(it), \tag{7.9}$$

so

$$P(t) = A(t) + iB(t)\,.$$

A and B are polynomials with real coefficients:

$$A(t) = \sum_{0\leq l\leq\left[\frac{n}{2}\right]} (-1)^l m_{2l}t^{2l}, \quad B(t) = t \sum_{0\leq l\leq\left[\frac{n-1}{2}\right]} (-1)^l m_{2l+1}t^{2l} .$$

Moreover,

$$B'(0)A(0) - A'(0)B(0) = m_0\, m_1 \,. \tag{7.10}$$

From (7.9) it is evident that

$$\big(\text{All roots of } A \text{ are real and simple}\big) \Leftrightarrow \big(\text{The polynomial } {}^{\mathcal{E}}\!M \text{ is conservative}\big)$$
$$\big(\text{All roots of } B \text{ are real and simple}\big) \Leftrightarrow \big(\text{The polynomial } {}^{\mathcal{O}}\!M \text{ is conservative}\big)$$
$$\big(\text{All roots of } P \text{ lie in } \{z\colon \operatorname{Im} z > 0\}\big) \Leftrightarrow \big(\text{The polynomial } M \text{ is dissipative}\big)$$

and under condition that all roots of A and B are real,

$$\big(\text{The roots of } A \text{ and } B \text{ interlace}\big) \Leftrightarrow \big(\text{The roots of } {}^{\mathcal{E}}\!M \text{ and } {}^{\mathcal{O}}\!M \text{ interlace}\big)$$

Thus, Lemma 7.1 is an immediate consequence of Hermite-Bieler Theorem it the above stated form. *The inequality* (7.6) *is ensured automatically by* (7.10) *since the coefficients m_k are assumed to be positive.*

Q.E.D.

Proof of Lemma 7.2. It is clear that the polynomials W and W' are the even and the odd parts of $M = W + W'$:

$$W = {}^{\mathcal{E}}\!M, \quad W' = {}^{\mathcal{O}}\!M.$$

Let M be dissipative. Then, according to Lemma 7.1, W is conservative. Conversely, let W be conservative. According to Rolle theorem, the polynomial W' is conservative as well, and the sets of roots of W and W' interlace. By Lemma 7.1, the polynomial M is dissipative. Q.E.D.

Remark 7.1. *The claim of Lemma 7.1 remains true if to replace the assumption posed on the coefficients m_k of of M with a weaker assumption. It enough to assume that only the coefficients m_0 and m_1 are strictly positive, whereas the other coefficients $m_k,\ k = 2, 3, \dots, n,$ are real.*

Proof of Theorem 2.1. The relation (1.34) means that the polynomial $tW^1_{\partial V}(t)$ is the odd part of the Minkowski polynomial M_V. Thus, we are in the situation of Lemma 7.1. Since the polynomial M_V is dissipative, the point $z = 0$ is not a root of M, that is $m_0(V) \neq 0$. According to (1.23), this means that $\mathrm{Vol}_n(V) \neq 0$. Thus, the set V is solid. By Proposition 8.1, all the coefficients $m_k(V)$ of the polynomial M_V the are strictly positive. According to Lemma 7.1, the polynomial ${}^{\mathcal{O}}(M_V)$ is conservative. Since ${}^{\mathcal{O}}(M_V)(0) = 0$, the polynomial $t^{-1} \cdot {}^{\mathcal{O}}(M_V)(t) = W^1_{\partial V}(t)$ is conservative as well. Q.E.D.

Proof of Theorem 5.2. In the course of the proof we shall refer to some facts from the theory of entire functions which usually are formulated in literature for functions whose roots are in the upper rather then in the left half-plane. Therefore, it is convenient pass from the variable t to the variable it. Given a function $H(t)$ of the Hurwitz class $\mathcal{H}$, let $f(t) = H(it)$. Then f is an entire function of exponential type, all roots of f are in the upper half-plane, and moreover, the defect d_f of f is non-negative, where

$$2\,d_f = \overline{\lim_{r\to+\infty}}\, f(-ir) - \overline{\lim_{r\to+\infty}}\, f(-ir)\,.$$

(It is clear that $d_f = d_H$, where d_H is the same as in (5.1).) Thus the function f is in the class P as this class was defined in [37], Chapter VII, Section 4. Let

$$f(t) = A(t) + iB(t)\,,$$

where A and B be real entire functions. Combining Lemma 1 from [37], Chapter VII, Section 4 with Theorem 4 from [37], Chapter VII, Section 2, we obtain that the functions A and B possess the properties:

1. A and B are real entire functions of exponential type;
2. $A(0)B'(0) - B(0)A'(0) > 0$;
3. For every $\theta \in \mathbb{R}$, all roots of the linear combination C_θ, where $C_\theta(t) = \cos\theta A(t) + \sin\theta B(t)$, are simple and real. (The entire functions A and B are *a real pair* in the terminology of N.G.Chebotarev, [13].)

According to Hadamard's factorization theorem, the entire function C_θ is in the Laguerre-Polya class. According to Jensen-Csordas-Williamson Theorem (Theorem 5.1), for each n, all roots of the Jensen polynomial $C_{\theta,n}(t) = \mathfrak{J}_n(C_\theta; t)$ are real and simple. Thus, the real polynomials $A_n(t) = \mathfrak{J}_n(A;t)$ and $B_n(t) = \mathfrak{J}_n(B;t)$ possess the property: *For every $\theta \in \mathbb{R}$, all roots of the linear combination* $\cos\theta A_n(t) + \sin\theta B_n(t)$, *are real and simple.* (The polynomials A_n and B_n are real pair as well.) From the property of the polynomials A_n and $B_n > 0$ to be a real pair together with the property $A_n(0)B_n'(0) - B_n(0)A_n'(0)$ it follows that all roots of the polynomial $f_n(t) = A_n(t) + iB_n(t)$ are in the upper half-plane. Thus, all roots of the polynomial $H_n(t) = f_n(-it)$ are in the left half-plane. In other words, the polynomial H_n is a Hurwitz polynomial. On the other hand, from the construction it follows that $H_n(t) = \mathfrak{J}_n(H;t)\,.$ □

8. PROPERTIES OF MINKOWSKI POLYNOMIALS OF A CONVEX SET.

MOTION INVARIANCE: *Let V, $V \subset \mathbb{R}^n$, be a compact convex set, τ be a motion*[22] *of the space* $\mathbb{R}^n$, and $\tau(\mathrm{V})$ be the image of the set V under he motion τ. Then $M_{\tau(V)}(t) = M_V(t)$.

CONTINUITY: *The correspondence $V \to M_V$ between compact convex sets V in $\mathbb{R}^n$ and their Minkowski polynomials M_V is continuous* [23].

A sketch of the proof of the continuity property can be found in [10], section 29; [12], section 19.2; [54], section 5.1; [66],

[22]The motion of the space $\mathbb{R}^n$ is an affine transformation of $\mathbb{R}^n$ which preserves the Euclidean distance in $\mathbb{R}^n$.

[23]The set of compact convex sets in $\mathbb{R}^n$ is equipped by the Hausdorff metric, the set of all polynomials is equipped by the topology of the locally uniform convergence in $\mathbb{C}$.

MONOTONICITY: *Let V_1 and V_2 be compact convex sets in $\mathbb{R}^n$, and M_{V_1}, M_{V_2} be the appropriate Minkowski polynomials. If $V_1 \subset V_2$, then the coefficients $m_k(V_1)$, $m_k(V_2)$ of the polynomials M_{V_1}, M_{V_2}, defined as in* (1.21)*, satisfy the inequalities*

$$m_k(V_1) \leq m_k(V_2), \quad 0 \leq k \leq n\,. \tag{8.1}$$

Explanation. According to the definition of the *mixed volumes*,

$$m_k(V) = \frac{n!}{(n-k)!\,k!}\,\mathrm{Vol}(\underbrace{V, V, \,\dots\,, V}_{n-k};\, \underbrace{B^n, B^n, \,\dots\,, B^n}_{k}). \tag{8.2}$$

Inequalities (8.1) follow from the *monotonicity of the mixed volumes* (8.2) *with respect to V*. (Concerning the monotonicity of the mixed volumes see, for example, [10], section 29; [12], section 19.2; [66], Theorem 6.4.11; [54], section 5.1, formula (5.1.23).) Q.E.D.

Lemma 8.1. a). *For any compact convex set V, $V \subset \mathbb{R}^n$, the coefficients $m_k(V)$ of its Minkowski polynomial, defined as in* (1.21)*, are non-negative:*

$$0 \leq m_k(V), \quad 0 \leq k \leq n. \tag{8.3a}$$

(According to (1.23)*, the coefficient $m_n(V)$ is strictly positive.)*

b). *If moreover the set V is solid, then all coefficients $m_k(V)$ are strictly positive:*

$$0 < m_k(V), \quad 0 \leq k \leq n. \tag{8.3b}$$

The Weyl coefficients $k_{2l}(\partial V)$, $0 \leq l \leq \left[\frac{n-1}{2}\right]$, defined by Definition 1.10, *are strictly positive as well.*

Proof. Taking V as V_2 and an one-point subset of V as V_1 in (8.1), we obtain (8.3a). If the set V is solid, then there exist a ball $x_0 + \rho B^n$ of some positive radius ρ: $x_0 + \rho B^n \subset V$. Taking the ball $x_0 + \rho B^n$ as V_1 and V as V_2 in (8.1), we obtain the inequalities $m_k(x_0 + \rho B^n) \leq m_k(V)$, $0 \leq k \leq n$. Moreover, $m_k(x_0 + \rho B^n) = m_k(\rho B^n) = \rho^{n-k} m_k(B^n) = \rho^{n-k} \frac{n!}{k!\,(n-k)!}\,\mathrm{Vol}_n(B^n) > 0$. ∎

Remark 8.1. *The notion of the interior point of a set V depend on the space in which V is embedded. The set V, $V \subset \mathbb{R}^n$, which is non-solid with respect to the 'original' space $\mathbb{R}^n$, is solid if V is considered as be embedded in the space $\mathbb{R}^d$ of the 'right' dimension d, $d < n$. The dimension* $\dim V$ *of the set V should be taken as such d.*

Definition 8.1. *Let V, $V \subseteq \mathbb{R}^n$, be a convex set. The* dimension $\dim V$ *of V is the dimension of the smallest affine subspace of $\mathbb{R}^n$ which contains V.*

Lemma 8.2. *Let V, $V \subset \mathbb{R}^n$, be a compact convex set of the dimension d:*

$$\dim V = d, \quad 0 \leq d \leq n. \tag{8.4}$$

Then

$$m_k^{\mathbb{R}^n}(V) = 0 \;\text{ for }\; 0 \leq k < n-d; \;\; m_k^{\mathbb{R}^n}(V) > 0 \;\text{ for }\; n-d \leq k \leq n\,. \tag{8.5}$$

This lemma is a consequence of Lemma 8.1 and of the following

Lemma 8.3. *Let V, $V \subset \mathbb{R}^n$, be a convex set of dimension d, $d \leq n$, and let $M_V^{\mathbb{R}^n}(t) = \sum\limits_{0\leq k\leq n} m_k^{\mathbb{R}^n}(V)t^k$ and*[24] $M_V^{\mathbb{R}^d}(t) = \sum\limits_{0\leq k\leq d} m_k^{\mathbb{R}^d}(V)t^k$ *be the Minkovski polynomials of the set V with respect to the ambient spaces $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively. Then*

$$M_V^{\mathbb{R}^n}(t) = t^{n-d} \cdot \sum_{0\leq k\leq d} \frac{\pi^{\frac{n-d}{2}}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+n-d}{2}+1)} m_k^{\mathbb{R}^d}(V)t^k\,. \tag{8.6}$$

Lemma 8.3 appears in slightly different notation as Theorem 11.1 in Section 11, where proof is presented.

Definition 8.2. *The mixed volumes appearing in* (8.2) *are said to be* cross-sectional measures *of the set V and are denote as $v_{n-k}(V)$:*

$$\mathrm{Vol}(\underbrace{V, V, \ldots, V}_{n-k}; \underbrace{B^n, B^n, \ldots, B^n}_{k}) = v_{n-k}(V), \quad 0 \leq k \leq n\,. \tag{8.7}$$

Thus, the coefficients of the Minkovski polynomials M_V, which appear in (1.21), can be presented as

$$m_k(V) = \binom{n}{k} v_{n-k}(V), \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \text{ are binomial coefficients,} \tag{8.8}$$

and the Minkowski polynomial itself can be presented as

$$M_V(t) = \sum_{0\leq k\leq n} \binom{n}{k} v_{n-k}(V)t^k\,. \tag{8.9}$$

The following fact will be used essentially in Section 10:

ALEXANDROV–FENCHEL INEQUALITY. *Let V, $V \subset \mathbb{R}^n$, be a compact convex set. Then its cross-sectional measures $v_k(V)$ satisfy the inequalities*

$$v_k^2(V) \geq v_{k-1}(V)\, v_{k+1}(V), \quad 1 \leq k \leq n-1\,. \tag{8.10}$$

A.D. Alexandrov published two proofs of this inequality in [3] and [4]. The first of them, a combinatorial one, is carried out for the convex polyhedra. The second proof is more analytical. It uses the theory of self-adjoint elliptic operators depending on parameter. This proof is carried out for smooth convex bodies. To the general case, both proofs are generalized by limit arguments. The first proof is developed in detail in the textbook [36]. The second proof is reproduced in Busemann [11]. It has become customary to talk on 'Alexandrov-Fenchel inequality', because Fenchel [22] also stated the inequality and sketched the proof. Its detailed exposition was never published. At the end of 1978 independently Tessier in Paris and A.G.Khovanskiĭ in Moscow obtained an algebraic-geometrical proof of the Alexandrov-Fenchel inequality using the Hodge index theorem. This proof is developed in §27 of the English translation of [12] and was written by A.G. Khovanskiĭ. (In the Russian original of [12] an erroneous algebraic proof of the Alexandrov-Fenchel inequality was included which has been excluded in the English translation.) Regarding the Alexandrov-Fenchel inequality see also [12], § 20 and Section **6.3** of [54].

[24] Defining the Minkowski polynomial $M_V^{\mathbb{R}^d}$, we can assume that the smallest affine subspace of $\mathbb{R}^n$ which contains V is the space $\mathbb{R}^d$.

Definition 8.3. *A sequence $\{p_k\}_{0\leq k\leq n}$ of non-negative numbers:*

$$p_k \geq 0\,, \quad 0 \leq k \leq n, \tag{8.11}$$

is said to be logarithmic concave*, if the following inequalities hold:*

$$p_k^2 \geq p_{k-1}p_{k+1}, \quad 1 \leq k \leq n-1\,. \tag{8.12}$$

Thus, the Alexandrov-Fenchel inequalities can be formulated in the form:

> *For any convex set V, the sequence $\{v_k(V)\}_{0\leq k\leq n}$ of its cross sectional measures is logarithmic concave.*

Under the extra condition (8.11), the logarithmic concavity inequalities (8.12) for the coefficients of the polynomial

$$P(t) = \sum_{0\leq k\leq n} \binom{n}{k} p_k t^k, \tag{8.13}$$

or for the coefficients of the entire function

$$P(t) = \sum_{0\leq k<\infty} \frac{p_k}{k!}\, t^k, \tag{8.14}$$

have been considered in connection with distribution roots of P. In this setting, such (and analogous) inequalities are commonly known as Turán Inequalities (Turán-like Inequalities). Concerning Turán inequalities see, for example, [30] and [15].

Remark 8.2. *The Turán inequalities* (8.12) *for the coefficients of the polynomial* (8.13) *or entire function* (8.14) *impose some restrictions on location of roots of P. However,* these inequalities alone do not ensure *that all roots of P are located in the left half-plane $\{z : \operatorname{Re} z < 0\}$.*

For example, given $m \in \mathbb{N}$, let

$$p_k = 1 \ \text{ for } \ k = 0, 1, \ldots, m \ \text{ and } \ p_k = 0 \ \text{ for } \ k > m. \tag{8.15}$$

Such p_k satisfy the Turán inequalities (8.12). The function (8.14) corresponding to these p_k is the polynomial

$$P_m(t) = \sum_{0\leq k\leq m} \frac{t^m}{m!}\,. \tag{8.16}$$

This polynomial is a section of the exponential series. It is known that already for $m = 5$ the polynomial (8.16) has two roots located in the half-plane $\{z : \operatorname{Re} z > 0\}$. G. Szeg'o, [60], studied the limiting distribution of roots of the sequence of polynomials P_m, (8.16), as $m \to \infty$. From his results on the limiting distributions of the roots it follows that for large m the polynomial P_m not only has roots in the half-plane $\{z : \operatorname{Re} z > 0\}$, but that the total number of its roots located there has a positive density as $m \to \infty$. Regarding roots of sections of power series we address to the book [21] and to the survey [44]. For $m < n$, the polynomial (8.13) with p_k as in (8.15) takes the form

$$P_{m,n}(t) = \sum_{0\leq k\leq m} \binom{n}{k} t^k\,. \tag{8.17}$$

I.V. Ostrovskii, [45], studied the limiting distribution of roots of the sequence of the polynomials $P_{m,n}$ as $m,n \to \infty$, $m/n \to \alpha$, $\alpha \in (0,1)$. From his results it follows that for large m,n: $n/m = O(1)$, $n/(n-m) = O(1)$ the polynomial $P_{m,n}$ not only has roots in the half-plane $\operatorname{Re} z > 0$, but that the total number of its roots located there has a positive density as $m,n \to \infty$, $m/n \to \alpha \in (0,1)$.

9. ROUTH-HURWITZ CRITERION.

Of course it is desirable to obtain an information about the location of roots of the Weyl and Minkowski polynomials directly from *geometrical* considerations. At the moment we are not able to do this. The only general tool from geometry which we can use are the Alexandrov-Fenchel inequalities (8.10) for cross-sectional measures $v_k(V)$ of convex sets. Therefore one should express all polynomials which we investigate in terms of this cross-sectional measures.

As it was explained in (8.9), the expression of the Minkowski polynomial $M_V^{\mathbb{R}^n}$ for the convex set V, $V \subset \mathbb{R}^n$, in terms of the cross-sectional measures $v_k(V)$ is

$$M_V^{\mathbb{R}^n}(t) = \sum_{0 \le k \le n} \binom{n}{k} v_{n-k}(V) t^k \,. \tag{9.1}$$

Lemma 9.1. *Let $\mathcal{M}$ be a closed convex surface, $\dim \mathcal{M} = n$, and let V, $V \subset \mathbb{R}^{n+1}$, be a generating convex set: $\mathcal{M} = \partial V$.*

Then the Weyl $W_{\mathcal{M}}^{\infty}$ can be expressed as

$$W_{\mathcal{M}}^{\infty}(t) = \sum_{0 \le l \le \left[\frac{n}{2}\right]} \frac{(n+1)!}{2^l l! (n-2l)!}\, v_{n-2l}(V)\, t^{2l} \,, \tag{9.2}$$

where $v_k(V)$ are the cross-sectional measures of the generating convex set V.

Proof. The expression (9.2) is a consequence of (1.41), (1.38) and (8.8). ∎

To extract an information about the location of roots of the Minkowski polynomial $M_V^{\mathbb{R}^n}$ from (9.1), we may use the Alexandrov-Fenchel inequalities (8.10). The Alexandrov-Fenchel inequalities relate the cross-sectional measures $v_k(V)$, $v_{k-1}(V)$, $v_{k+1}(V)$. To extract such an information about the roots of the Weyl polynomial $W_{\mathcal{M}}^{\infty}$ from (9.2), we need the analogous inequalities which relate the cross-sectional measures $v_k(V)$, $v_{k-2}(V)$, $v_{k+2}(V)$.

Lemma 9.2. *Let V, $V \subset \mathbb{R}^{n+1}$, be a compact convex set. Then its cross-sectional measures satisfy the inequalities*

$$v_k^2(V) \ge v_{k-2}(V) v_{k+2}(V), \quad 2 \le k \le n-1. \tag{9.3}$$

Proof. We derive (9.3) from (8.10). Rising the inequality (8.10) to square, we obtain that $v_k^4(V) \ge v_{k-1}^2(V) v_{k+1}^2(V)$. Inequalities (8.10) with k replaced with $k-1$ and $k+1$ are:

$$v_{k-1}^2(V) \ge v_{k-2}(V) v_k(V) \text{ and } v_{k+1}^2(V) \ge v_k(V) v_{k+2}(V)$$

respectively. The inequality (9.3) is the consequence of the last three inequalities. Q.E.D.

The inequalities (8.10) and (9.3) for the coefficients of the polynomials (9.1) and (9.2) respectively is one of two general tools which will be used in the study of the location of roots of these polynomials. The second general tool is the criteria which express the properties of polynomials to be dissipative (or conservative respectively) in terms of their coefficients. Such criteria are formulated as the positivity of certain determinants constructed from the coefficients of the tested polynomials.

Theorem 5 ([Routh-Hurwitz]). *Let*

$$P(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n \tag{9.4}$$

be a polynomial with strictly positive coefficients:

$$a_0 > 0,\, a_1 > 0,\, \dots,\, a_{n-1} > 0,\, a_n > 0\,. \tag{9.5}$$

For the polynomial P to be dissipative, it is necessary and sufficient that all the determinants Δ_k, $k = 1, 2, \dots, n-1, n$, be strictly positive:

$$\Delta_1 > 0,\, \Delta_2 > 0,\, \dots,\, \Delta_{n-1} > 0,\, \Delta_n > 0, \tag{9.6}$$

where

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \tag{9.7}$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \quad \dots \quad, \Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & \dots & 0 \\ a_0 & a_2 & a_4 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & a_n \end{vmatrix}.$$

This result, as well as many relative results, can be found in [24], Chapter XV. See also [32].

Remark 9.1. *Actually, to prove that the polynomial P,* (9.4)*, of degree n with positive coefficients a_k, $k = 1, 2, \dots, n$, is dissipative, there is no need to inspect all Hurwitz determinants Δ_k, $k = 1, 2, \dots, n$, for positivity. It is enough to inspect either the determinants Δ_k only with even k, or the determinants Δ_k only with even k.* (See [24], Chapter XV, §13.)

Applying the Routh-Hurwitz criterion to investigate whether the Minkowski polynomial $M_V^{\mathbb{R}^n}$ is dissipative, we should take, according to (9.1),

$$a_k = \frac{n!}{k!(n-k)!} v_k(V)\,, \quad 0 \le k \le n, \quad a_k = 0, \quad k > n\,. \tag{9.8}$$

From the criterion of dissipativity, the criterion of conservativity can be derived easily.

Theorem 6 ([Criterion of conservativity]). *Let*

$$P(t) = a_0 t^{2m} + a_2 t^{2m-2} + \dots + a_{2m-2} t^2 + a_{2m} \tag{9.9}$$

be a polynomial with strictly positive coefficients a_{2l}, $0 \le l \le m$:

$$a_0 > 0,\, a_2 > 0,\, \dots,\, a_{2m-2} > 0,\, a_{2m} > 0\,. \tag{9.10}$$

For the polynomial P to be conservative, it is necessary and sufficient that all the determinants D_k, $k = 1, 2, \ldots, 2m-1, 2m$, be strictly positive:

$$D_1 > 0,\ D_2 > 0,\ D_3 > 0,\ \ldots,\ D_{2m-1} > 0,\ D_{2m} > 0, \tag{9.11}$$

where the determinants D_k are constructed from the coefficients of the polynomial P according to the following rule. Determinant D_k are the determinant Δ_k, (9.7), whose entries a_{2l}, $0 \leq l \leq m$, are the coefficients of the polynomial P, and $a_{2l+1} = (m-l)\, a_{2l}$, $0 \leq l \leq m-1$:

$$D_1 = m\, a_0, \quad D_2 = \begin{vmatrix} m\, a_0 & (m-1)a_2 \\ a_0 & a_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} ma_0 & (m-1)a_2 & (m-2)a_4 \\ a_0 & a_2 & a_4 \\ 0 & m\, a_0 & (m-1)a_2 \end{vmatrix},$$

$$D_4 = \begin{vmatrix} m\, a_0 & (m-1)\, a_2 & (m-2)a_4 & (m-4)a_6 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & m\, a_0 & (m-1)\, a_2 & (m-2)a_4 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \quad \ldots \tag{9.12}$$

$$D_{2m} = \begin{vmatrix} m\, a_0 & (m-1)a_2 & (m-2)a_4 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & ma_0 & (m-1)a_2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a_{2m} \end{vmatrix}.$$

Proof. This theorem is the immediate consequence of Hermite-Bieler theorem and Lemma 7.2. Q.E.D.

Applying the Conservativity criterion to investigate whether the Weyl polynomial $W^{\infty}_{\mathcal{M}}$ is dissipative, we should take, according to (9.2) and (9.9),

$$a_{2l} = \frac{(n+1)!}{2^{m-l}(m-l)!(2l+n-2m)!} v_{2l+n-2m}(V)\,, \tag{9.13}$$
$$0 \leq l \leq m, \quad a_{2l} = 0, \quad l > m\,, \quad \text{where } m = \left[\tfrac{n}{2}\right]$$

10. THE CASE OF LOW DIMENSION: PROOF OF THEOREMS 2.2 AND 2.3.

Proof of Theorem 2.2. We apply the Routh-Hurwitz criterion of dissipativity to the Minkowski polynomial $M^{\mathbb{R}^n}_V$. 'Opening' the Hurwitz determinants Δ_k, (9.7) , and taking into account that $a_k = 0$ for $k > n$, we obtain that for $n \leq 5$,

$$\Delta_1 = a_1 \tag{10.1.1}$$

$$\Delta_2 = a_1 a_2 - a_0 a_3\,, \tag{10.1.2}$$

$$\Delta_3 = a_1 a_2 a_3 + a_0 a_1 a_5 - a_0 a_3^2 - a_1^2 a_4\,, \tag{10.1.3}$$

(10.1.4)
$$\Delta_4 = a_1 a_2 a_3 a_4 + a_0 a_2 a_3 a_5 + 2 a_0 a_1 a_4 a_5 - a_1^2 a_4^2 - a_0^2 a_5^2 - a_0 a_3^2 a_4 - a_1 a_2^2 a_5,$$

$$\Delta_5 = a_5 \Delta_4\,, \tag{10.1.5}$$

where we should take a_k as in (9.8).

According to Routh-Hurwitz criterion, we have to prove that $\Delta_1 > 0$, $\Delta_2 > 0, \ldots, \Delta_n > 0$. The cases $n = 2, 3, 4, 5$ will be considered separately. Since V

is solid, $v_k(V) > 0$, $0 \leq k \leq n$. (Corollary 8.1.b and (8.8) .)
Thus, the determinant $\Delta_1 = \binom{n}{1} v_1(V)$ is always positive.

The cases $n = 2, 3, 4, 5$ will be considered separately. To shorten notation, we right v_k instead $v_k(V)$.

$n = 2$. In this case,

$$a_0 = v_0,\ a_1 = 2v_1,\ a_2 = v_2,$$
$$\Delta_2 = 2v_2v_1\,.$$

The inequality $\Delta_2 > 0$ is evident. Thus, the Minkowski polynomial $M_V^{\mathbb{R}^2}$ is dissipative.

$n = 3$. In this case,

$$a_0 = v_0, \quad a_1 = 3v_1, \quad a_2 = 3v_2, \quad a_3 = v_3\,, \quad a_k = 0,\, k > 3\,.$$

Substituting these expressions for a_k into (10.1), we obtain

$$\Delta_2 = 9v_1v_2 - v_0v_3, \quad \Delta_3 = v_3\Delta_2\,.$$

Thus, the property of $M_V^{\mathbb{R}^3}$ to be dissipative is equivalent to the inequality

$$9v_1v_2 > v_0v_3, \tag{10.2}$$

$n = 4$. In this case,

$$a_0 = v_0, \quad a_1 = 4v_1, \quad a_2 = 6v_2, \quad a_3 = 4v_3 \quad a_4 = v_4, \quad a_k = 0,\, k > 4\,.$$

Substituting these expressions for a_k into (10.1), we obtain

$$\Delta_2 \;=\; 24v_1v_2 - 4v_0v_3, \quad \Delta_3 \;=\; 96v_1v_2v_3 - 16v_0v_3^2 - 16v_1^2v_4\,, \quad \Delta_4 \;=\; v_4\Delta_3\,.$$

Thus, the property of $M_V^{\mathbb{R}^4}$ to be dissipative is equivalent to the pair of inequalities

$$6v_1v_2 > v_0v_3, \tag{10.3.2}$$
$$6v_1v_2v_3 > v_0v_3^2 + v_1^2v_4\,. \tag{10.3.3}$$

$n = 5$. In this case,

$$a_0 = v_0,\ a_1 = 5v_1,\ a_2 = 10v_2,\ a_3 = 10v_3,\ a_4 = 5v_4,\ a_5 = v_5,\ a_k = 0,\ k > 5\,.$$

Substituting these expressions for a_k into (10.1), we obtain

$$\begin{aligned} \Delta_2 = 50v_1v_2 - 10v_0v_3, \quad \Delta_3 = 500v_1v_2v_3 + 5v_0v_1v_5 - 100v_0v_3^2 - 125v_1^2v_4\,, \\ \Delta_4 = 2500v_1v_2v_3v_4 + 100v_0v_2v_3v_5 + 50v_0v_1v_4v_5 \\ - 625v_1^2v_4^2 - v_0^2v_5^2 - 500v_0v_3^2v_4 - 500v_0v_1^2v_5\,, \quad \Delta_5 = v_5\Delta_4\,. \end{aligned}$$

Thus, the property of $M_V^{\mathbb{R}^5}$ to be dissipative is equivalent to the triple of inequalities

$$5v_1v_2 > v_0v_3, \tag{10.4.2}$$
$$100v_1v_2v_3 + v_0v_1v_5 > 20v_0v_3^2 + 25v_1^2v_4\,, \tag{10.4.3}$$
$$\begin{aligned} 2500v_1v_2v_3v_4 + 100v_0v_2v_3v_5 + 50v_0v_2v_4v_5 > \\ > 625v_1^2v_4^2 + 500v_0v_3^2v_4 + 500v_1v_2^2v_5 + v_0^2v_5^2\,. \end{aligned} \tag{10.4.4}$$

As it is claimed in Lemma 10.1 below, the inequalities (10.2), (10.3), (10.4), where $v_k = v_k(V)$ are the cross-sectional measures of the solid compact set V of the appropriate dimension, are the consequences of the Alexandrov-Fenchel inequalities. This completes the proof. Q.E.D.

Remark 10.1. *All this business works up to certain n, but it* does not *work for all n. If n is large enough, then the conditions $v_k^2 \geq v_{k-1}v_{k+1}, \quad 1 \leq k \leq n-1$, posed on positive numbers v_k does not imply the inequalities $\Delta_k \geq 0$ for all $k = 1, \ldots, n$, where Δ_k is constructed from $a_k = \binom{n}{k} v_k$. Already by $n = 30$, $\Delta_5 < 0$ for certain v_k satisfying these conditions. Moreover, as we will see later, for n large enough, there exist examples of such compact convex sets $V \subset \mathbb{R}^n$ for which the Minkowski polynomial M_V is not dissipative, and the Weyl polynomial $W^1_{\partial V}$ is not conservative. In such examples the sets V are although solid, but 'almost degenerated'.*

Lemma 10.1. *Let v_k, $0 \leq k \leq n$, be strictly positive numbers satisfying the inequalities*

$$v_k^2 \geq v_{k-1}v_{k+1}, \quad 1 \leq k \leq n-1\,. \tag{10.5}$$

Then:

1. *If $n = 3$, then the inequality* (10.2) *holds;*
2. *If $n = 4$, then the inequalities* (10.3) *holds;*
3. *If $n = 5$, then the inequalities* (10.4) *holds.*

Proof of Lemma 10.1. Given k, $1 \leq k \leq n-2$, multiplying the inequalities

$$v_k^2 \geq v_{k-1}v_{k+1}, \quad v_{k+1}^2 \geq v_k v_{k+2},$$

an then cancelling on $v_k v_{k+1}$, we obtain the inequality

$$v_k v_{k+1} \geq v_{k-1}v_{k+2}\,, \quad 1 \leq k \leq n-2\,. \tag{10.6}$$

In particular, for $n = 3$, $k = 1$, as well as for $n = 4$, $k = 1$ and $n = 5$, $k = 1$.

$$v_1 v_2 \geq v_0 v_3\,.$$

This inequality implies the inequality (10.2), (10.3.2) and (10.4.2). Multiplying the inequality $v_1v_2 \geq v_0v_3$ with the positive number v_3, we obtain

$$v_1v_2v_3 \geq v_0v_3^2 \tag{10.7a}$$

For $n = 4$, $k = 2$, the inequality (10.6) means

$$v_2v_3 \geq v_1v_4.$$

Multiplying the inequality $v_2v_3 \geq v_1v_4$ with v_1, we get

$$v_1v_2v_3 \geq v_1^2v_4. \tag{10.7b}$$

The inequalities (10.7) imply the inequality (10.3.3) and (10.4.3) .
Multiplying the inequality $v_2v_3 \geq v_1v_4$ (this is (10.6) for $k = 2, n = 5$) with v_1v_4, we obtain

$$v_1v_2v_3v_4 \geq v_1^2v_4^2\,. \tag{10.8a}$$

Multiplying the inequality $v_1v_2 \geq v_0v_3$ (this is (10.6) for $k = 1, n = 5$) with v_3v_4, we obtain

$$v_1v_2v_3v_4 \geq v_0v_3^2v_4\,. \tag{10.8b}$$

Multiplying the inequality $v_3v_4 \geq v_2v_5$ (this is (10.6) for $k=3, n=5$) with the positive number v_1v_2, we obtain

$$v_1v_2v_3v_4 \geq v_1v_2^2v_5\,. \tag{10.8c}$$

Further, multiplying the inequalities $v_1v_2 \geq v_0v_3$ (this is (10.6) for $k=1, n=5$) and $v_3v_4 \geq v_2v_5$ (this is (10.6) for $k=3, n=5$), we obtain that $v_1v_4 \geq v_0v_5$. Rising it to square, we obtain

$$v_1^2v_4^2 \geq v_0^2v_5^2\,.$$

Multiplying this inequality with the inequality $v_2v_3 \geq v_1v_4$ (this is (10.6) for $k=2, n=5$), we obtain that

$$v_1v_2v_3v_4 \geq v_0^2v_5^2\,. \tag{10.8d}$$

Since $2500 > 625+500+500+1$, the inequality (10.4.4) it the consequence of the inequalities (10.8). (The number 2500 is the coefficient before the monomial $v_1v_2v_3v_4$ in the left hand side of (10.4.4), the numbers 625, 500, 500, 1 are the coefficients before the monomials $v_1^2v_4^2$, $v_0v_3^2v_4$, $v_1v_2^2v_5$ and $v_0^2v_5^2$ in the right hand side of (10.4.4) respectively.) Q.E.D.

Proof of Theorem 2.3. We apply the Criterion of conservativity, which was formulated in the previous section, to the Weyl polynomial $W^{\infty}_{\mathcal{M}}$. Opening the determinants Δ_k, (9.12) , and taking into account that $a_{2l}=0$ for $l > \left[\frac{n}{2}\right]$, we obtain that for $n \leq 5$, that is[25] for $m = \left[\frac{n}{2}\right] \leq 2$,

$$D_1 = ma_0 \tag{10.9.1}$$

$$D_2 = a_0a_2\,, \tag{10.9.2}$$

$$D_3 = a_0\big((m-1)a_2^2 - 2ma_0a_4\big)\,, \tag{10.9.3}$$

$$D_4 = a_0a_4(a_2^2 - 4a_0a_4)\,. \tag{10.9.4}$$

where we should take a_{2l} as in (9.13).

According to the Conservativity criterion, we have to prove that $D_1 > 0$, $D_2 > 0, \ldots, D_{2m} > 0$, where $m = \left[\frac{n}{2}\right]$.
Since V is solid, $v_k(V) > 0$, $0 \leq k \leq n+1$. (Corollary 8.1.b and (8.8) .) Thus, the determinants D_1, D_2 are always positive.

Therefore, if $n=2$, or if $n=3$, that is if $m=1$, the Weyl polynomial $W^{\infty}_{\mathcal{M}}$ is conservative. Of course, this fact is evident without referring to the conservativity criterion:

In the case $\boldsymbol{n=2}$, according to (9.8) or (9.2),

$$W^{\infty}_{\mathcal{M}}(t) = 3v_2 + 3v_0t^2\,.$$

In the case $\boldsymbol{n=3}$, according to (9.8) or (9.2),

$$W^{\infty}_{\mathcal{M}}(t) = v_3 + 3v_1t^2\,.$$

Evidently, in both cases, $n=2$ or $n=3$, the polynomial $W^{\infty}_{\mathcal{M}}$ is conservative.

In the cases $n=4,\ n=5$, to what corresponds $m=2$,

$$D_3 = a_0(a_2^2 - 4a_0a_4), \quad D_4 = a_4D_3\,.$$

[25] Recall that $n = \dim \mathcal{M}$, $n+1 = \dim V : \mathcal{M} = \partial V$.

According to (9.13), we have to take in the cases:
In the case $\boldsymbol{n=4}$

$$a_0 = 15v_0, \quad a_2 = 30v_2, \quad a_4 = 5v_4\,.$$

Thus,

$$D_3 = 15v_0(900v_2^2 - 300v_0v_4)\,.$$

The conditions $D_3 > 0$, $D_4 > 0$ take the form

$$3v_2^2 > v_0v_4\,. \tag{10.10}$$

In the case $\boldsymbol{n=5}$

$$a_0 = 90v_1, \quad a_2 = 60v_3, \quad a_4 = 6v_5\,.$$

Thus,

$$D_3 = 90v_1(900v_2^2 - 300v_0v_4)\,.$$

The conditions $D_3 > 0$, $D_4 > 0$ take the form

$$5v_3^2 > 3v_1v_5\,. \tag{10.11}$$

So, in the cases $n = 4$ and $n = 5$ the property of the Weyl polynomial $W^{\infty}_{\mathcal{M}}$ be conservative is equivalent to the inequality (10.10) and (10.11) respectively, where $v_k = v_k(V)$ are the cross-sectional measures of the solid compact set V generating the surface $\mathcal{M}$: $\mathcal{M} = \partial V$. In its turn, the inequalities (10.10) and (10.11) are evident consequences of the inequalities $v_2^2 \geq v_0v_4$ and $v_3^2 \geq v_1v_5$ respectively. The latter inequalities are special cases of the inequalities (9.3). (See Lemma 9.2.) Thus, in the cases $n = 4$ and $n = 5$ the Weyl $W^{\infty}_{\mathcal{M}}$ polynomial of infinite index is conservative. By Lemma 2.1, all Weyl polynomials $W^{p}_{\mathcal{M}}$, $p = 1, 2, 3, \ldots$, are conservative as well. $\square$

11. EXTENDING OF THE AMBIENT SPACE.

Adjoint convex sets. Let V be a compact convex set, $V \subset \mathbb{R}^n$. However we may consider the space $\mathbb{R}^n$ as a subspace of the space $\mathbb{R}^{n+q}$ of a higher dimension $q = 1, 2, 3, \ldots$. The embedding $\mathbb{R}^n$ to $\mathbb{R}^{n+q}$ is standard:

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+q} : \quad (\xi_1, \ldots, \xi_n) \to (\xi_1, \ldots, \xi_n; \underbrace{0, \ldots, 0}_{q}).$$

Thus, the set V, which originally was considered as a subset of $\mathbb{R}^n$, may also be considered as a subset of $\mathbb{R}^{n+q}$. In other words, we identify the set $V \subset \mathbb{R}^n$ with the set $V \times 0_q$, which is the Cartesian product of the set V and the zero point 0^q of the space $\mathbb{R}^q$: $V \times 0^q \subset \mathbb{R}^{(n+q)}$.

Definition 11.1. *Given a compact convex set V, $V \subset \mathbb{R}^n$, and a number q, $q = 0, 1, 2, 3, \ldots$, the q-th adjoint to V set $V^{(q)}$ is defined as:*

$$V^{(q)} \stackrel{\text{def}}{=} V \times 0^q, \quad V^{(q)} \subset \mathbb{R}^{n+q}\,, \tag{11.1}$$

where 0^q is the zero point of the space $\mathbb{R}^q$, and the space $\mathbb{R}^{n+q}$ is considered as the Cartesian product: $\mathbb{R}^{n+q} = \mathbb{R}^n \times \mathbb{R}^q$.
The Minkowski polynomial $M^{\mathbb{R}^{n+q}}_{V \times 0^q}$ of the q-th adjoint set $V^{(q)}$,

$$M^{\mathbb{R}^{n+q}}_{V \times 0^q} = \mathrm{Vol}_{n+q}(V \times 0^q + tB^{n+q}), \tag{11.2}$$

is said to be the q-th adjoint Minkowski polynomial for the set V.

For $q = 0$, the set $V^{(0)}$ coincides with V, and the polynomial $M^{\mathbb{R}^{n+0}}_{V\times 0^0}$ coincides with $M^{\mathbb{R}^n}_V$. For $q = 1$, the set $V^{(1)}$ is what we are called the squeezed cylinder with the base V.

Minkowski polynomials for adjoint sets. Let us answer the natural question:

How the polynomials $M^{\mathbb{R}^n}_V(t)$ and $M^{\mathbb{R}^{n+q}}_{V\times 0^q}(t)$ are related?

The answer this question will be done by an inductive reasoning. Lemma 11.1 below provides the step of the induction.

Lemma 11.1. *Let V be a compact convex set in $\mathbb{R}^n$, and*

$$M^{\mathbb{R}^n}_V(t) = \sum_{0\leq k\leq n} m^{\mathbb{R}^n}_k(V)t^k \tag{11.3}$$

be the Minkowski polynomial with respect to the ambient space $\mathbb{R}^n$. Then the Minkowski polynomial $M^{\mathbb{R}^{n+1}}_{V\times 0}(t)$ is equal to:

$$M^{\mathbb{R}^{n+1}}_{V\times 0^1}(t) = t\sum_{0\leq k\leq n} \frac{\pi^{1/2}\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2}+1)}\, m^{\mathbb{R}^n}_k(V)\, t^k\,. \tag{11.4}$$

The following theorem completes the inductional reasoning:

Theorem 11.1. *Let V be a compact convex set in $\mathbb{R}^n$, and*

$$M^{\mathbb{R}^n}_V(t) = \sum_{0\leq k\leq n} m^{\mathbb{R}^n}_k(V)t^k \tag{11.5}$$

be the Minkowski polynomial of the set V. Then the q-th adjoint Minkowski polynomial $M^{\mathbb{R}^{n+q}}_{V\times 0^q}(t)$ of the set V is equal to:

$$M^{\mathbb{R}^{n+q}}_{V\times 0^q}(t) = \sum_{0\leq k\leq n} m^{\mathbb{R}^n}_k(V)\,\gamma^{(q)}_k\; t^{k+q}\,, \tag{11.6}$$

where

$$\gamma^{(q)}_k = \pi^{q/2}\,\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+q}{2}+1)}\,, \quad k = 0,\,1,\,2,\,\dots\,;\quad q = 0,\,1,\,2,\,\dots. \tag{11.7}$$

A sketch of proof of this theorem can be found in [28], Chapter VI, Section 6.1.9. A detailed proof is presented below.

Remark 11.1. *Theorem 11.1 means that the sequence of the coefficients $\{m^{\mathbb{R}^{n+q}}_k(V\times 0^q)\}_{0\leq k\leq n+q}$ of the polynomial $M^{\mathbb{R}^{n+q}}_{V\times 0^q}$:*

$$M^{\mathbb{R}^{n+q}}_{V\times 0^q}(t) = \sum_{0\leq k\leq n+q} m^{\mathbb{R}^{n+q}}_k(V\times 0^q)\, t^k \tag{11.8}$$

are obtained from the sequence of the coefficients $\{m_k(V)\}_{0\leq k\leq n}$ of the polynomial $M^{\mathbb{R}^n}_V$, (11.5), by means of shift and multiplication:

$$m^{\mathbb{R}^{n+q}}_k(V\times 0^q) = 0, \qquad 0\leq k < q; \tag{11.9a}$$

$$m^{\mathbb{R}^{n+q}}_{k+q}(V\times 0^q) = m^{\mathbb{R}^n}_k(V)\,\gamma^{(q)}_k,\quad 0\leq k\leq n\,. \tag{11.9b}$$

Remark 11.2. *According to* Theorem 11.1*, the transformation which maps the polynomial* $M_V^{\mathbb{R}^n}$ *into the polynomial* $M_{V\times 0^q}^{\mathbb{R}^{n+q}}$ *is essentially of the form*

$$\sum_{0\leq k\leq n} m_k t^k \to \sum_{0\leq k\leq n} \gamma_k m_k t^k, \tag{11.10}$$

where γ_k *is a certain sequence of* multipliers. (*The factor* t^q *before the sum in* (11.6) *is not essential*). *The transformations of the form* (11.10) *were already discussed is Section 5. There such transformations were considered in relation with location of roots of polynomials and entire functions.*

Lemma 11.2. *For any* q*,* $q = 1, 2, 3, \ldots$*, the sequence* $\{\gamma_k^{(q)}\}_{k=0,1,2,\ldots}$ *is not a multiplier sequence in the sense of* Definition 5.3*.*

Proof. In Section 13 we explain that the entire function

$$\mu_q(t) = \sum_{0\leq k<\infty} \frac{\gamma_k^{(q)}}{k!} t^k \tag{11.11}$$

has infinitely many non-real roots. The entire function $\mu_q(t)$, (11.11), appears as the function $\mathcal{M}_{B^n\times 0^q}(t)$ in Section 5. (Up to a constant factor which is not essential for study the roots.) According to the Polya-Schur Theorem, which was formulated in Section 5, the sequence $\{\gamma_k^{(q)}\}_{k=0,1,2,\ldots}$ is not a multiplier sequence. □

Remark 11.3. *In Section 5 we study the function* $\mu_q(t)$ *in much more details that it is needed to prove* Lemma 11.2*. The study of section 5 is aimed to clarify for which* q *the roots of the function in question are located in the left half plane. The question whether there are non-real roots is much more rough. This question may be answered from very general considerations. The function* μ_q *admits the integral representation:*

$$\mu_q(t) = q\omega_q \int_0^1 (1-\xi^2)^{\frac{q}{2}-1}\xi\, e^{\xi t}\, d\xi\,. \tag{11.12}$$

(Expanding the exponential $e^{\xi t}$ *into the Taylor series, we see that the Taylor coefficients of the function in the right hand side of* (11.12) *are the numbers* $\frac{\gamma_k^{(q)}}{k!}$*.) From* (11.12) *it follows that the function* $\mu_q(t)$ *is an entire function of exponential type, and that its indicator diagram is the interval* $[0, 1]$*. Moreover,* $\sup_{-\infty<t<\infty} |\mu_q(it)| < \infty$*. In particular, the function* $\mu_q(it)$ *belongs to the class of entire functions which is denoted by* C *in* [38]*, Lecture 17. From* Theorem of Cartwright-Levinson (Theorem 1 of the Lecture 17 from [38]) *it follows that the function* $\mu_q(t)$ *has infinitely many roots, these roots have positive density, and the 'majority' of these roots is located 'near' the rays* $\arg t = \frac{\pi}{2}$ *and* $\arg t = -\frac{\pi}{2}$*. In particular, the function* $\mu_q(t)$ *has infinitely many non-real roots. (We already used this reasoning proving* Statement 2 of Theorem 6.2*.)* □

Proof of Lemma 11.1. Let $(x, s) \in \mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$, and $s \in \mathbb{R}$. Then by Pythagorean theorem,

$$\mathrm{dist}^2_{\mathbb{R}^{n+1}}((x, s), V\times 0) = \mathrm{dist}^2_{\mathbb{R}^n}(x, V) + s^2.$$

Therefore, the equivalence holds:

$$\Big(\operatorname{dist}_{\mathbb{R}^{n+1}}((x,s),V\times 0)\leq t\Big)\iff\Big(\operatorname{dist}_{\mathbb{R}^n}(x,V)\leq\sqrt{t^2-s^2}\Big)\tag{11.13}$$

Let

$$\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)=\{(x,s)\in\mathbb{R}^{n+1}:\,\operatorname{dist}_{\mathbb{R}^{n+1}}((x,s),V\times 0^1)\leq t\}.\tag{11.14}$$

be the t-neighborhood of the set $V\times 0^1$ with respect to the ambient space $\mathbb{R}^{n+1}$. Thus,

$$\operatorname{Vol}_{n+1}(\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t))=M^{\mathbb{R}^{n+1}}_{V\times 0^1}(t).\tag{11.15}$$

For fixed $s\in\mathbb{R}$, let $\mathfrak{S}(s)$ be the 'horizontal section' of the set $\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)$ on the 'vertical level' s:

$$\mathfrak{S}(s)=\{x\in\mathbb{R}^n:\,(x,s)\in\mathfrak{T}^{\mathbb{R}^{n+1}}_{V}(t)\}.$$

It is clear that

$$\operatorname{Vol}_{n+1}(\mathfrak{T}^{\mathbb{R}^{n+1}}_{V\times 0^1}(t))=\int\operatorname{Vol}_n(\mathfrak{S}(s))ds\,.\tag{11.16}$$

The equivalence (11.13) means that

$$\mathfrak{S}(s)=\mathfrak{T}^{\mathbb{R}^n}_{V}(\sqrt{t^2-s^2})=\{x\in\mathbb{R}^n:\,\operatorname{dist}_{\mathbb{R}^n}(x,V)\leq\sqrt{t^2-s^2}\}.$$

Thus,

$$\operatorname{Vol}_n(\mathfrak{S}(s))=M^{\mathbb{R}^n}_{V}(\sqrt{t^2-s^2}).\tag{11.17}$$

From (11.16) and (11.17) it follows that

$$M^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)=\int\limits_{-t}^{t}M^{\mathbb{R}^n}_{V}(\sqrt{t^2-s^2})ds\,.$$

Changing variable $s\to ts^{1/2}$, we obtain

$$M^{\mathbb{R}^{n+1}}_{V}(t)=t\int\limits_{0}^{1}M^{\mathbb{R}^n}_{V}(t(1-s)^{1/2})s^{-1/2}ds.$$

Substituting the expression (1.31) for $M^{\mathbb{R}^n}_{V}$ into the last formula, we obtain

$$M^{\mathbb{R}^{n+1}}_{V\times 0^1}(t)=t\sum_{0\leq k\leq n}m_k(V)\,t^k\int\limits_{0}^{1}(1-s)^{k/2}s^{-1/2}ds.$$

According to Euler,

$$\int\limits_{0}^{1}(1-s)^{k/2}s^{-1/2}ds=\mathrm{B}\big(\tfrac{k}{2}+1,\tfrac{1}{2}\big)=\frac{\Gamma\big(\frac{1}{2}\big)\,\Gamma\big(\frac{k}{2}+1\big)}{\Gamma\big(\frac{k+1}{2}+1\big)}=\pi^{1/2}\frac{\Gamma\big(\frac{k}{2}+1\big)}{\Gamma\big(\frac{k+1}{2}+1\big)}.$$

Thus, (11.4) holds. ∎

Proof of Theorem 11.1. For $q=0$, the statement of the Theorem is self-evident. Let us show how to pass from q to $q+1$. Since $V\times 0^{q+1}=(V\times 0^q)\times 0^1$, and $\mathbb{R}^{n+q+1}=\mathbb{R}^{n+q}\times\mathbb{R}^1$, we can apply Lemma 11.1 to the convex set $V\times 0^q$

whose Minkowski polynomial is (11.6) by the induction assumption. The induction assumption can be formulated as

$$m_k^{\mathbb{R}^{n+q}}(V \times 0^q) = 0, \qquad 0 \le k < q; \tag{11.18a}$$

$$m_k^{\mathbb{R}^{n+q}}(V \times 0^q) = m_{k-q}^{\mathbb{R}^n}(V)\,\gamma_{k-q}^{(q)}, \quad q \le k \le q+n\,. \tag{11.18b}$$

By Lemma 1.2,

(11.19a)

$$m_k^{\mathbb{R}^{(n+q)+1}}((V \times 0^q) \times 0^1) = 0, \quad k = 0\,;$$

(11.19b)

$$m_k^{\mathbb{R}^{(n+q)+1}}((V \times 0^q) \times 0^1) = m_{k-1}^{\mathbb{R}^{(n+q)}}((V \times 0^q)) \cdot \gamma_{k-1}^{(1)}\,, \quad 1 \le k \le n+q+1\,.$$

In view of the identity

$$\gamma_k^{(q)} \cdot \gamma_{k+q}^{(1)} = \gamma_k^{(q+1)}\,,$$

(11.19) takes the form (11.18) with q replaced by $q+1$. □

Remark 11.4. *From* (1.6) *and* (11.7) *it follows that*

$$\gamma_k^{(q)} = \frac{\omega_{k+q}}{\omega_k}\,. \tag{11.20}$$

Thus, the equalities (11.9b) *can be rewritten as*

$$\frac{m_{k+q}^{\mathbb{R}^{n+q}}(V \times 0^q)}{\omega_{k+q}} = \frac{m_k^{\mathbb{R}^n}(V)}{\omega_k}, \quad q = 0,\,1,\,2,\,\dots\,. \tag{11.21}$$

The equality (11.21) *holds for* $k = 0,\,1,\,\dots\,,n$. *For other* k, *the value* $m_k^{\mathbb{R}^n}(V)$ *is not yet defined. Let us agree that*

$$\omega_k = 1 \ \text{ for } \ k < 0, \quad m_k^{\mathbb{R}^n}(V) = 0 \ \text{ for } \ k < 0 \ \text{ and for } \ k > n\,. \tag{11.22}$$

Under this agreement, the equality (11.21) *holds for* every $k \in \mathbb{Z}$: *for* $k > n$ *or for* $k < -q$ (11.21) *is trivial, for* $-q \le k \le -1$ *it coincides with* (11.9a), *for* $0 \le k \le n$ *– with* (11.9b).

The Minkowski polynomials for the q-th adjoint to the ball B^n. In particular, applying Theorem 11.1 to the case $V = B^n$, $B^n \subset \mathbb{R}^n$, we obtain:

$$M_{B^n \times 0^q}^{\mathbb{R}^{n+q}}(t) = \omega_n \omega_q\, t^q \mathcal{M}_{B^n \times 0^q}(nt)\,, \tag{11.23}$$

where the normalized Minkowski polynomial $\mathcal{M}_{B^n \times 0^q}$ is defined as

$$\mathcal{M}_{B^n \times 0^q}(t) = \sum_{0 \le k \le n} \frac{n!}{(n-k)!\,n^k} \frac{\Gamma(\frac{q}{2}+1)\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+q}{2}+1)} \frac{t^k}{k!}\,. \tag{11.24}$$

The polynomial $\mathcal{M}_{B^n \times 0^q}$ is the Jensen polynomial associated with the entire functions $M^{B^n \times 0^q}$:

$$\mathcal{M}_{B^n \times 0^q}(t) = \mathfrak{J}_n(M_{B^\infty \times 0^q}; t), \tag{11.25}$$

where

$$\mathcal{M}_{B^\infty \times 0^q}(t) = \sum_{0 \le k < \infty} \frac{\Gamma(\frac{q}{2}+1)\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+q}{2}+1)} \frac{t^k}{k!}\,. \tag{11.26}$$

Comparing with (11.12), we obtain Comparing with (11.12), we obtain

$$\mathcal{M}_{B^\infty\times 0^q}(t) = q\int\limits_0^1 (1-\xi^2)^{\frac{q}{2}-1}\xi\, e^{\xi t}\, d\xi\,. \tag{11.27}$$

For every $q = 0, 1, 2, \ldots$, the function $\mathcal{M}_{B^\infty\times 0^q}$ is an entire function of the exponential type one.

Lemma 11.3.

1. *For $q = 0, 1, 2, 4$, the entire function $\mathcal{M}_{B^\infty\times 0^q}$ is in the Hurwitz class $\mathcal{H}$;*
2. *For $q \geq 5$, the entire function $\mathcal{M}_{B^\infty\times 0^q}$ is not in the Hurwitz class: it has infinitely many roots in the open right half plane $\{z :\ \mathrm{Im} z > 0\}$..*

Proof of this Lemma is presented in Section 13. Statement 2 is a consequence of the asymptotic calculation of the function $\mathcal{M}_{B^\infty\times 0^q}$. (See Lemma 13.1.)

For $q = 0$, the function $\mathcal{M}_{B^\infty\times 0^0} = e^t$, thus it is of type I in the Laguerre-Polya class: $\mathcal{M}_{B^\infty\times 0^0} \in \mathcal{L}$-$\mathcal{P}$-I. For $q = 2$ and $q = 4$ the functions $\mathcal{M}_{B^\infty\times 0^q}$ can be calculated explicitly and investigated by elementary methods. The case $q = 1$ is more involved. The case $q = 3$ remains open. $\square$

Proof of Statement 1 of Theorem 2.7. Let $q \geq 5$ be given. According to statement 2 of Lemma 11.3, the function $\mathcal{M}_{B^\infty\times 0^q}$ has infinitely many roots in the open right half-plane. In view of the approximation property of the Jensen polynomials (Lemma 4.1), for $n \geq n(q)$ some roots of the Jensen polynomial $\mathcal{J}_n(M_{B^\infty\times 0^q}; t)$ are located in the open right half-plane. In view of (11.23) and (11.25), some roots of the Minkovski polynomial $M_{B^n\times 0^q}$ of the (non-solid) convex set $B^n \times 0^q$, $B^n \times 0^q \subset \mathbb{R}^{n+q}$, are located in the open right half-plane. Fix $n : n \geq n(q)$. Consider the ellipsoids $E_{n,\,q,\,\varepsilon}$ defined in (2.9), $E_{n,\,q,\,\varepsilon} \subset \mathbb{R}^{n+q}$. For $\varepsilon > 0$, the ellipsoid $E_{n,\,q,\,\varepsilon}$ is a solid convex set with respect to the ambient space $\mathbb{R}^{n+q}$. The family of the convex sets $\{E_{n,\,q,\,\varepsilon}\}_{\varepsilon>0}$ is monotonic, (See Remark 1.6 and footnote 7), and

$$\lim_{\varepsilon\to+0} E_{n,\,q,\,\varepsilon} = B^n \times 0^q\,. \tag{11.28}$$

It is known that the Minkowski polynomials $M_V(t)$ depends on the set V continuously: see Section 8 and footnote 23. Therefore,

$$\lim_{\varepsilon\to 0} M^{\mathbb{R}^{n+q}}_{E_{n,\,q,\,\varepsilon}}(t) = M^{\mathbb{R}^{n+q}}_{B^n\times 0^q}(t) \tag{11.29}$$

locally uniformly in $\mathbb{C}$. Hence, there exists $\varepsilon(q, n)$, $\varepsilon(q, n) > 0$ such that the Minkowski polynomial $M^{\mathbb{R}^{n+q}}_{E_{n,\,q,\,\varepsilon}}$ has roots located in the open right half-plane. $\square$

The Weyl polynomials for the surfaces of the adjoint convex sets. Passing to define the so-called adjoint Weyl polynomials $W^{\,p}_{V\times 0^q}$, we do this following Definition 1.9 as a sample.

Definition 11.2. *Given a convex compact set V, $V \subset \mathbb{R}^n$, and a number q, $q = 0, 1, 2, 3, \ldots,$, the q-th adjoint Weyl polynomial $W^{\,1}_{\partial(V\times 0^q)}$ of the index 1 for the* *convex surface $\partial(V \times 0^q)$ is defined by means of the odd part of the q-th adjoint Minkowskii polynomial $M^{\mathbb{R}^{n+q}}_{V\times 0^q}$:*

$$2tW^1_{\partial(V\times 0^q)}(t) \overset{\text{def}}{=} M^{\mathbb{R}^{n+q}}_{V\times 0^q}(t) - M^{\mathbb{R}^{n+q}}_{V\times 0^q}(-t)\,, \tag{11.30}$$

where $M^{\mathbb{R}^{n+q}}_{V^{(q)}}$ is the q-th adjoint Minkowskii polynomial which was introduced in Definition 11.1. *In more detail*[26],

$$W^1_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} m^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)t^{2l}\,. \tag{11.31}$$

From (11.31) we may define the Weyl coefficients $k_{2l}(\partial(V\times 0^q))$ according to Definition 1.10:

$$k_{2l}(\partial(V\times 0^q)) = m^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)\frac{(2\pi)^l\omega_1}{\omega_{1+2l}}\,. \tag{11.32}$$

Then we define the Weyl polynomials $W^p_{\partial(V\times 0^q)}$ with higher p according to[27] Definition 1.11:

Definition 11.3.

$$W^p_{\partial(V\times 0^q)}(t) \overset{\text{def}}{=} \sum_{l\in\mathbb{Z}} k_{2l}(\partial(V\times 0^q))(2\pi)^{-l}\frac{\omega_{2l+p}}{\omega_p}t^{2l}\,. \tag{11.33}$$

Thus,

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} \frac{m^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)}{\omega_{2l+1}}\,\omega_1\omega_{2l+p}\,t^{2l+p}\,. \tag{11.34}$$

Let us clarify how the Weyl polynomials for the convex surfaces ∂V and $\partial(V\times 0^q)$ are related. Here we also have to distinguish the cases even and odd q.

Lemma 11.4. *Let V, $V\subset\mathbb{R}^n$, be a solid compact convex set, and let $p>0$, $q>0$ be integers. Then*

1. *For even q*

$$\omega_p t^p\cdot W^p_{\partial(V\times 0^q)}(t) = \omega_{p+q}t^{p+q}\cdot W^{p+q}_{\partial V}(t)\,; \tag{11.35}$$

2. *For odd q*

$$\omega_p t^p\cdot W^p_{\partial(V\times 0^q)}(t) = \omega_{p+q-1}t^{p+q-1}W^{p+q-1}_{\partial(V\times 0^1)}(t)\,. \tag{11.36}$$

Proof of Lemma 11.4. We distinguish cases of even and odd q.

1. q is even. The equality (11.21) with $k=2l+1-q$ takes the form

$$\frac{m^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)}{\omega_{2l+1}} = \frac{m^{\mathbb{R}^n}_{2l+1-q}(V)}{\omega_{2l+1-q}}.$$

[26] According to the agreement (11.22), $m^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)=0$ for $2l+1<0$ or $2l+1>n+q$.

[27] Remark that $\dfrac{2^{-l}\,\Gamma(\frac{p}{2}+1)}{\Gamma(\frac{p}{2}+l+1)} = (2\pi)^{-l}\dfrac{\omega_{2l+p}}{\omega_p}$.

From this and (11.34) it follows that

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} \frac{m^{\mathbb{R}^n}_{2l+1-q}(V)}{\omega_{2l+1-q}}\,\omega_1\omega_{2l+p}\,t^{2l+p}\,.$$

Changing the summation variable: $l \to l + \frac{q}{2}$, we obtain

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} \frac{m^{\mathbb{R}^n}_{2l+1}(V)}{\omega_{2l+1}}\,\omega_1\omega_{2l+p+q}\,t^{2l+p+q}\,.$$

The expression in the right hand side of the last equality has the same structure that the expression in the right hand side of (11.34), with $V\times 0^q$ replaced to V, p replaced by $p+q$, q replaced by 0. So, (11.35) is proved.

2. q is odd. The equality (11.21) implies the equality

$$\frac{m^{\mathbb{R}^{n+q}}_{2l+1}(V\times 0^q)}{\omega_{2l+1}} = \frac{m^{\mathbb{R}^{n+1}}_{2l+1-(q-1)}(V\times 0^1)}{\omega_{2l+1-(q-1)}}.$$

From this and (11.34) it follows that

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} \frac{m^{\mathbb{R}^{n+1}}_{2l+1-(q-1)}(V\times 0^1)}{\omega_{2l+1-(q-1)}}\,\omega_1\omega_{2l+p}\,t^{2l+p}\,.$$

Changing the summation variable: $l \to l + \frac{q-1}{2}$, we obtain

$$\omega_p t^p W^p_{\partial(V\times 0^q)}(t) = \sum_{l\in\mathbb{Z}} \frac{m^{\mathbb{R}^{n+1}}_{2l+1}(V\times 0^1)}{\omega_{2l+1}}\,\omega_1\omega_{2l+p+q-1}\,t^{2l+p+q-1}\,.$$

The expression in the right hand side of the last equality has the same structure that the expression in the right hand side of (11.34), with $V\times 0^q$ replaced to $V\times 0^1$, p replaced by $p+q-1$, q replaced by 1. So, (11.36) is proved. □

The meaning of Lemma 11.4 lies in the following. Studying the location of roots of Weyl polynomials related to convex surfaces there is no need to consider the boundary surfaces $\partial(V\times 0^q)$ of q-th adjoint convex sets $V\times 0^q$ for arbitrary large q. It is enough to restrict the consideration to the cases $q=0$ and $q=1$ only, that is to the case of the set V itself and to the case of the squeezed cylinder with the base V.

Proof of Statement 2 of Theorem 2.7. By Statement 2 of Theorem 6.4, the entire function $\mathcal{W}^{p+q-1}_{\partial(B^\infty\times 0)}$ has infinitely many non-real roots which. (We have assumed that $p+q-1 \geq 5$.) If n is large enough, the Jensen polynomial $\mathcal{W}^{p+q-1}_{\partial(B^{n+1}\times 0)}(t)=\mathcal{J}_{2[n/2]}(\mathcal{W}^{p+q-1}_{\partial(B^\infty\times 0)};t)$ also has non-real roots. According to (4.12b) and (4.13b), the Weyl polynomial $W^{p+q-1}_{\partial(B^n\times 0)}(t)$ has roots which do not belong to the imaginary axis. By Statement 2 of Lemma 11.4,

$$W^p_{\partial(B^n\times 0^q)} = \tfrac{\omega_{p+q-1}}{\omega_p}t^{q-1}W^{p+q-1}_{\partial(B^n\times 0)}\,.$$

Thus, the Weyl polynomial $W^p_{\partial(B^n\times 0^q)}$ has roots which do not belong to the imaginary axis. For fixed q, n and a positive ε, consider the ellipsoid $E_{n,q,\varepsilon}$ defined by (2.9). Since $E_{n,q,\varepsilon} \to B^n\times 0^q$ as $\varepsilon\to+0$, also $W^p_{E_{n,q,\varepsilon}} \to W^p_{\partial(B^n\times 0^q)}$ as $\varepsilon\to+0$.

Hence, if ε is small enough: $0 < \varepsilon \leq \varepsilon(n, p, q)$, the polynomial $W_{E_{n,\,q,\,\varepsilon}}$ has roots which do not belong to the imaginary axis. $\square$

12. THE MINKOWSKI POLYNOMIAL OF THE CARTESIAN PRODUCT OF CONVEX SETS.

Let V_1 and V_2 be compact convex sets,

$$V_1 \subset \mathbb{R}^{n_1}, \quad V_2 \subset \mathbb{R}^{n_2}.$$

Then the Cartesian product $V_1 \times V_2$ is a compact convex set embedded into the Cartesian product $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Since $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ can be naturally identified with $\mathbb{R}^{n_1+n_2}$, we can consider $V_1 \times V_2$ as being embedded into $\mathbb{R}^{n_1+n_2}$:

$$V_1 \times V_2 \subset \mathbb{R}^{n_1+n_2}.$$

The natural question arises:
How to express the Minkowski polynomial $M^{\mathbb{R}^{n_1+n_2}}_{V_1 \times V_2}$ for the Cartesian product $V_1 \times V_2$ in terms of the Minkowski polynomials[28] $M^{\mathbb{R}^{n_1}}_{V_1}$ *and* $M^{\mathbb{R}^{n_2}}_{V_2}$ *for the Cartesian factors* V_1 *and* V_2 *?*
To answer this question, we introduce a special multiplication in the set of polynomials, the so-called *M-multiplication*, which is suitable for this goal.

Definition 12.1. *The M-product $t^k \circ t^l$ of two monomials t^k and t^l is defined as*

$$t^k \circ t^l \stackrel{\text{def}}{=} \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}\, t^{k+l}, \quad k \geq 0,\, l \geq 0. \tag{12.1}$$

It is clear that

$$\text{a}).\ t^0 \circ t^k = t^k, \quad \text{b}).\ t^k \circ t^l = t^l \circ t^k, \quad \text{c}).\ (t^k \circ t^l) \circ t^m = t^k \circ (t^l \circ t^m). \tag{12.2}$$

The M-multiplication (12.1) *of monomials can be extended to the multiplication of polynomials by linearity:*

$$\text{For}\quad A(t) = \sum_{0 \leq k \leq n_1} a_k t^k, \quad B(t) = \sum_{0 \leq l \leq n_2} b_l t^l, \tag{12.3a}$$

$$(A \circ B)(t) = \sum_{\substack{0 \leq k \leq n_1,\\ 0 \leq l \leq n_2}} a_k b_l (t^k \circ t^l) = \sum_{0 \leq k \leq n_1, 0 \leq l \leq n_2} a_k b_l \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}\, t^{k+l},$$

and finally, the M-product $A \circ B$ of the polynomials A and B is defined as

$$(A \circ B)(t) = \sum_{0 \leq r \leq n_1+n_2} \Bigg(\sum_{k \geq 0,\, l \geq 0, k+l=r} a_k\, b_l\, \frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)} \Bigg) t^r\,. \tag{12.3b}$$

From (12.2.b) and (12.2.c) it follows that

$$A \circ B = B \circ A, \quad (A \circ B) \circ C = A \circ (B \circ C)$$

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[28]The Minkowski polynomials M_{V_1}, M_{V_2}, $M_{V_1 \times V_2}$ are considered with respect to the ambient spaces $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, $\mathbb{R}^{n_1+n_2}$ respectively.

for every polynomials A, B, C. In particular, the 'triple product' $A\circ B\circ C$ is well defined. This triple product can be explicitly expressed in terms of the coefficients of the factors: if

$$A(t)=\sum_{0\leq k\leq n_1} a_k t^k,\quad B(t)=\sum_{0\leq l\leq n_2} b_l t^l,\quad C(t)=\sum_{0\leq m\leq n_3} c_m t^m,$$

then

$$(A\circ B\circ C)(t)=\sum_{0\leq r\leq n_1+n_2+n_3}\Bigg(\sum_{\substack{k\geq 0,\,l\geq 0,\,m\geq 0\\ k+l+m=r}} a_k\,b_l\,c_m\,\frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)\Gamma\big(\frac{m}{2}+1\big)}{\Gamma\big(\frac{k+l+m}{2}+1\big)}\Bigg)\,t^r\,.$$

It is clear, that for every number λ and for every polynomials A and B,

$$(\lambda A)\circ B=\lambda(A\circ B).$$

Moreover, if

$$\mathbb{I}(t)\equiv 1,\quad \mathbb{T}(t)\equiv t,$$

then

$$\mathbb{I}\circ A=A.$$

Thus, *the polynomial* $\mathbb{I}$ *is the unity with respect to the* M*-Multiplication.*
It is worthy to mention that

$$t^{(\circ\, k)}\stackrel{\text{def}}{=}\underbrace{t\circ t\circ\ \cdots\circ t}_{k}=\frac{(\sqrt{\pi}/2)^k}{\Gamma(\frac{k}{2}+1)}t^k. \tag{12.4}$$

Remark 12.1. *The M-multiplication by the polynomial* $\mathbb{T}$ *is related to the transformation of the form* (11.10)*:*

(12.5a) If $$A(t)=\sum_{0\leq k\leq n} a_k t^k,$$

(12.5b) then $$(\underbrace{\mathbb{T}\circ\ \cdots\circ\mathbb{T}}_{p}\circ A)(t)=2^{-p}t^p\sum_{0\leq k\leq n} a_k\gamma^{(p)}_k t^k,$$

where the 'multipliers' $\gamma^{(p)}_k$ *are defined by* (11.7)*.*

Lemma 12.1. *The M-product* $A\circ B$ *of polynomials* A *and* B *admits the integral*[29] *representation*[30]

$$(A\circ B)(t)=A(0)B(t)+\int_0^t A\big((t^2-\tau^2)^{1/2}\big)\,dB(\tau)\,, \tag{12.6a}$$

as well as

$$(A\circ B)(t)=A(t)B(0)+\int_0^t B\big((t^2-\tau^2)^{1/2}\big)\,dA(\tau)\,.$$

[29]The integrals in the right hand sides of (12.6) are Stieltjes integrals.
[30]At least, for $t>0$.

PROOF. First of all, the expressions in the right hand sides of (12.6) are equal: Integrating by parts and replacing the variable $\tau \to (t^2-\tau^2)^{1/2}$, we obtain

$$A(0)B(t)+\int\limits_0^t A\big((t^2-\tau^2)^{1/2}\big)\,dB(\tau)=A(t)B(0)+\int\limits_0^t B\big((t^2-\tau^2)^{1/2}\big)\,dA(\tau).$$

So, the expressions in the right hand sides of (12.6) which at the first glance are asymmetric with respect to A and B actually are symmetric. Let

$$A(t)=\sum_{0\leq k\leq n_1} a_k t^k,\quad B(t)=\sum_{0\leq l\leq n_2} b_l t^l$$

be the expressions for the polynomials A and B in terms of their coefficients. Let us substitute these polynomials into the right hand side of (12.6a):

$$A(0)B(t)+\int\limits_0^t A\big((t^2-\tau^2)^{1/2}\big)\,dB(\tau)=$$

$$a_0\sum_{0\leq l\leq n_2} b_l t^l+\int\limits_0^t\Big(\sum_{0\leq k\leq n_1} a_k(t^2-\tau^2)^{k/2}\Big)\cdot\Big(\sum_{1\leq l\leq n_2} l\,b_l\tau^{l-1}\Big)\,d\tau=$$

$$\sum_{0\leq l\leq n_2} a_0 b_l t^l+\sum_{\substack{0\leq k\leq n_1\\1\leq l\leq n_2}} a_k b_l\cdot l\int\limits_0^t (t^2-\tau^2)^{k/2}\tau^{l-1}d\tau\,.$$

Changing variable $\tau\to t\tau^{1/2}$, we get

$$l\int\limits_0^t (t^2-\tau^2)^{k/2}\tau^{l-1}d\tau=t^{k+l}(l/2)\int\limits_0^1(1-\tau)^{k/2}\tau^{l/2-1}d\tau=$$

$$t^{k+l}(l/2)B\Big(\frac{k}{2}+1;\frac{l}{2}\Big)\,.$$

Now, according to Euler,

$$(l/2)B\Big(\frac{k}{2}+1;\frac{l}{2}\Big)=\frac{\Gamma\big(\frac{k}{2}+1\big)\frac{l}{2}\Gamma\big(\frac{l}{2}\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}=\frac{\Gamma\big(\frac{k}{2}+1\big)\Gamma\big(\frac{l}{2}+1\big)}{\Gamma\big(\frac{k+l}{2}+1\big)}\,.$$

Thus, the right hand side of (12.6a) can be transformed into the right hand side of (12.3b). Q.E.D.

Theorem 12.1. *Given the compact convex sets V_1 and V_2, $V_1\subset\mathbb{R}^{n_1}, V_2\subset\mathbb{R}^{n_2}$, let $M_{V_1}^{\mathbb{R}^{n_1}}(t)$, $M_{V_2}^{\mathbb{R}^{n_2}}(t)$ be the Minkowski polynomials for the sets V_1 and V_2. Then the Minkowski polynomial $M_{V_1\times V_2}^{\mathbb{R}^{n_1+n_2}}$ of the Cartesian product $V_1\times V_2$ is equal to the M-product of the polynomials $M_{V_1}^{\mathbb{R}^{n_1}}$ and $M_{V_2}^{\mathbb{R}^{n_1 2}}$:*

$$M_{V_1\times V_2}^{\mathbb{R}^{n_1+n_2}}=M_{V_1}^{\mathbb{R}^{n_1}}\circ M_{V_2}^{\mathbb{R}^{n_2}}\,. \tag{12.7}$$

A sketch of proof of this theorem can be found in [28], Chapter VI, Section 6.1.9. A detailed proof is presented below.

Remark 12.2. *Let S be 'the origin' of $\mathbb{R}^1$, that is the one-point set: $S = \{t : t = 0\}$. Then $M_S(t) = 2t$, that is*

$$M_S(t) = 2\,\mathbb{T}(t). \tag{12.8}$$

Let V be a compact convex set embedded into R^n. The Cartesian product $V \times \underbrace{S \times \cdots \times S}_{p}$ can be identified with the convex set $V \times 0^p$, $V \times 0^p \subset \mathbb{R}^{n+p}$. Thus,

$$M_{V \times \underbrace{S \times \cdots \times S}_{p}}(t) = M^{\mathbb{R}^{n+p}}_{V \times 0^p}(t)\,,$$

or

$$2^p(\underbrace{\mathbb{T} \circ \cdots \circ \mathbb{T}}_{p}) \circ M^{\mathbb{R}^n}_V = M^{\mathbb{R}^{n+p}}_{V \times 0^p}. \tag{12.9}$$

In view of (12.5) *and* (12.8)*, the equality* (12.9) *is another form of the equality* (11.6)*.*

Proof of Theorem 12.1. Denote

$$V = V_1 \times V_2\,.$$

According to the identification $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we present a point $x \in \mathbb{R}^{n_1+n_2}$ as a pair $x = (x_1,\, x_2)$, where $x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}$. It is clear that

$$\operatorname{dist}^2_{\mathbb{R}^{n_1+n_2}}(x,\, V) = \operatorname{dist}^2_{\mathbb{R}^{n_1}}(x_1,\, V_1) + \operatorname{dist}^2_{\mathbb{R}^{n_2}}(x_2,\, V_2). \tag{12.10}$$

For $\tau > 0$, $\tau' > 0$, $\tau'' > 0$, let $V(\tau), V_1(\tau')$ and $V_2(\tau'')$ be the τ-neighborhood of V with respect to $\mathbb{R}^{n_1+n_2}$, the τ'-neighborhood of V_1 w. r. t. $\mathbb{R}^{n_1}$ and τ''-neighborhood of V_2 w. r. t. $\mathbb{R}^{n_2}$ respectively:

$$\begin{gathered} V(\tau) = V + \tau B_{n_1+n_2}, \quad V_1(\tau') = V_1 + \tau' B_{n_1}, \quad V_2(\tau'') = V_2 + \tau'' B_{n_2}; \\ V,\, B_{n_1+n_2} \subset \mathbb{R}^{n_1+n_2}; \quad V_1,\, B_{n_1} \subset \mathbb{R}^{n_1}; \quad V_2,\, B_{n_2} \subset \mathbb{R}^{n_2}. \end{gathered}$$

Here B^n be the Euclidean ball of the radius one in $\mathbb{R}^n$. (With $n = n_1 + n_2, n_1,\, n_2$ respectively.)

Given a number t, $t > 0$, consider the t-neighborhood $V(t)$ of $V = V_1 \times V_2$, and let

$$0 = \tau_0 < \tau_1 < \ldots < \tau_{N-1} < \tau_N = t$$

be a partition of the interval $[0,\, t]$. From (12.10) it follows that

$$\begin{gathered} \big(V_1(0) \times V_2(t)\big) \cup \Big(\bigcup_{1 \le k \le N} \big(V_1(\tau_k) \setminus V_1(\tau_{k-1})\big) \times V_2\big((t^2 - \tau_k^2)^{1/2}\big)\Big) \\ \subseteq V(t) \subseteq \\ \big(V_1(0) \times V_2(t)\big) \cup \Big(\bigcup_{1 \le k \le N} \big(V_1(\tau_k) \setminus V_1(\tau_{k-1})\big) \times V_2\left((t^2 - \tau_{k-1}^2)^{1/2}\right)\Big)\,. \end{gathered} \tag{12.11}$$

Since $V_1(\tau_k) \supseteq V_1(\tau_{k-1})$,

$$\operatorname{Vol}_{n_1}\big(V_1(\tau_k) \setminus V_1(\tau_{k-1}\big) = \operatorname{Vol}_{n_1}\big(V_1(\tau_k)\big) - \operatorname{Vol}_{n_1}\big(V_1(\tau_{k-1})\big),$$

thus

$$\operatorname{Vol}_{n_1+n_2}\Big(\big(V_1(\tau_k)\setminus V_1(\tau_{k-1})\big)\times V_2\left((t^2-\tau_l^2)^{1/2}\right)\Big)=$$
$$\Big(\operatorname{Vol}_{n_1}\big(V_1(\tau_k)\big)-\operatorname{Vol}_{n_1}\big(V_1(\tau_{k-1})\big)\Big)\cdot\operatorname{Vol}_{n_2}\Big(V_2\big((t^2-\tau_l^2)^{1/2}\big)\Big),$$
$$l=k-1 \text{ or } l=k.$$

Moreover

$$\operatorname{Vol}_{n_1+n_2}\Big(V_1(0)\times V_2(t)\Big)=\operatorname{Vol}_{n_1}\big(V_1(0)\big)\cdot\operatorname{Vol}_{n_2}\big(V_2(t)\big)\,.$$

In the notation of Minkowski polynomials, the last equalities take the form

$$\operatorname{Vol}_{n_1+n_2}\Big(\big(V_1(\tau_k)\setminus V_1(\tau_{k-1})\big)\times V_2\left((t^2-\tau_l^2)^{1/2}\right)\Big)=$$
$$\Big(M_{V_1}(\tau_k)-M_{V_1}(\tau_{k-1})\Big)\cdot M_{V_2}\big((t^2-\tau_l^2)^{1/2}\big),\quad l=k-1 \text{ or } l=k, \tag{12.12a}$$

$$\operatorname{Vol}_{n_1+n_2}\Big(V_1(0)\times V_2(t)\Big)=M_{V_1}(0)\cdot M_{V_2}(t)\,, \tag{12.12b}$$

and also

$$\operatorname{Vol}_{n_1+n_2}\big(V(t)\big)=M_V(t). \tag{12.12c}$$

Since the sets $V_1(\tau_k)\setminus V_1(\tau_{k-1})$ for different k do not intersect, and none of these sets intersects with the set $V_1(0)$, it follows from (12.11) and (12.12) that

$$M_{V_1}(0)\cdot M_{V_2}(t)+\sum_{1\le k\le N}\Big(M_{V_1}(\tau_k)-M_{V_1}(\tau_{k-1})\Big)\cdot M_{V_2}\big((t^2-\tau_k^2)^{1/2}\big)$$
$$\le M_V(t)\le \tag{12.13}$$
$$M_{V_1}(0)\cdot M_{V_2}(t)+\sum_{1\le k\le N}\Big(M_{V_1}(\tau_k)-M_{V_1}(\tau_{k-1})\Big)\cdot M_{V_2}\big((t^2-\tau_{k-1}^2)^{1/2}\big)\,.$$

Passing to the limit as $\max(\tau_k-\tau_{k-1})\to 0$ in the last inequality, we express the Minkowski polynomial $M_V(t)$ as the Stieltjes integral

$$M_V(t)=M_{V_1}(0)\cdot M_{V_2}(t)+\int\limits_0^t M_{V_2}\big((t^2-\tau^2)^{1/2}\big)\,dM_{V_1}(\tau)\,. \tag{12.14}$$

According to Lemma 12.1, the expression in the right hand side of (12.14) is equal to $\big(M_{V_1}\circ M_{V_2}\big)(t)$. Q.E.D.

13. PROPERTIES OF ENTIRE FUNCTIONS GENERATING THE MINKOWSKI AND WEYL POLYNOMIALS FOR THE DEGENERATED CONVEX SETS $B^{n+1}\times 0^q$.

In this section we investigate location of roots of the entire functions generating the Minkowski and Weyl polynomials related to the 'degenerated' convex sets $B^{n+1}\times 0^q$. These are:

- The entire functions $\mathcal{M}_{B^n\times 0^q}$ which appears in (11.26), in particular for $q=1$ in (4.7b) .

- The entire function $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}$, $1\leq p<\infty$, which appears in (4.11c).
- The entire function $\mathcal{W}^{\infty}_{\partial(B^\infty\times 0)}$ which appears in (4.11d).

The functions $\mathcal{M}_{B^n\times 0^q}$, $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}$, $1\leq p<\infty$ can not be calculated explicitly (except a very special values of the parameters p or q), but they can be calculated asymptotically.

The above mentioned functions admit integrable representations:

$$\mathcal{M}_{B^n\times 0^q}(t)=q\int\limits_0^1(1-\xi^2)^{\frac{q}{2}-1}\xi\, e^{\xi t}\,d\xi\,; \tag{13.1}$$

$$\mathcal{W}^{p}_{\partial(B^\infty\times 0)}(t)=p\int\limits_0^1(1-\xi^2)^{\frac{p}{2}-1}\xi\cos t\xi\,d\xi\,; \tag{13.2}$$

These integral representation can be used for the asymptotic calculation of the functions $\mathcal{M}_{B^n\times 0^q}$, $\mathcal{W}^{p}_{\partial(B^\infty\times 0)}$.

Another way to to calculate the functions (13.1), (13.2) asymptotically is to use the structure of their Taylor series. The Taylor coefficients of each of these functions are ratios of factorials: these functions belong to the so-called *Fox-Wright function*, [17].

The Fox-Wright function is defined as

$$ {}_p\Psi_q\left\{\begin{matrix}\alpha_1\ \alpha_2\ \ \alpha_p\\ \beta_1\ \beta_2\ \ \beta_p\end{matrix};\begin{matrix}\rho_1\ \rho_2\ \ \rho_q\\ \sigma_1\ \sigma_2\ \ \sigma_q\end{matrix};z\right\}\stackrel{\text{def}}{=}\sum_{0\leq k<\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j\,k+\beta_j)}{\prod_{j=1}^{q}\Gamma(\rho_j\,k+\sigma_j)}\cdot\frac{x^k}{k!}\,. \tag{13.3}$$

Comparing (13.3) with the Taylor expansions (11.26), (4.11c), (4.11d), we see that

$$\mathcal{M}_{B^n\times 0^q}(t)=\Gamma\left(\frac{q}{2}+1\right)\cdot{}_1\Psi_1\left\{\begin{matrix}\frac{1}{2}\\ 1\end{matrix};\begin{matrix}\frac{1}{2}\\ 1+\frac{q}{2}\end{matrix};t\right\},\quad 1\leq q<\infty\,, \tag{13.4}$$

$$\mathcal{W}^{p}_{\partial(B^\infty\times 0^1)}(t)=\Gamma(\tfrac{1}{2})\cdot\Gamma(\tfrac{p}{2}+1)\cdot{}_1\Psi_2\left\{\begin{matrix}1\\ 1\end{matrix};\begin{matrix}1\\ \frac{1}{2}\end{matrix}\ \begin{matrix}1\\ \frac{p}{2}+1\end{matrix};-\frac{t^2}{4}\right\},\quad 1\leq p<\infty\,, \tag{13.5}$$

Asymptotic behavior of the functions ${}_p\Psi_q(z)$ has been studied by E. Barnes, [7], G.N. Watson, [64], G. Fox, [23]), E.M. Wright, [71], [72].

Analysis of the function $\mathcal{M}_{B^n\times 0^q}(t)$: We would like to investigate for which q this function belongs to the Hurwitz class $\mathcal{H}$. According to (13.4), we may readdress the question to the proportional function ${}_1\Psi_1\left\{\begin{matrix}\frac{1}{2}\\ 1\end{matrix};\begin{matrix}\frac{1}{2}\\ 1+\frac{q}{2}\end{matrix};z\right\}$. From the Taylor expansion it is clear that this function is an entire function of exponential type. Since the Taylor coefficients of the function are positive, its defect [31] is non-negative. So, *the function $\mathcal{M}_{B^n\times 0^q}(t)$ is in the Hurwitz class $\mathcal{H}$ is and only if all roots of the function ${}_1\Psi_1\left\{\begin{matrix}\frac{1}{2}\\ 1\end{matrix};\begin{matrix}\frac{1}{2}\\ 1+\frac{q}{2}\end{matrix};z\right\}$ are situated in the open left half plane.* To investigate the location of roots of the last function, we use the following asymptotic approximation, which can be derived from the results stated in [71],

[31] We recall that the defect of an entire function H of exponential type is defined by (5.1).

[72]:
For any $\varepsilon:\ 0<\varepsilon<\dfrac{\pi}{2}$,

$$(13.6)\qquad {}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}=$$

$$=\begin{cases}2^{\frac{q}{2}}z^{-\frac{q}{2}}e^{z}\left(1+r_1(z)\right), & |\arg z|\leq\frac{\pi}{2}-\varepsilon\,;\\ \frac{2}{\Gamma(\frac{q}{2})}z^{-2}+r_2(z), & |\arg z-\pi|\leq\frac{\pi}{2}-\varepsilon\,;\\ 2^{\frac{q}{2}}z^{-\frac{q}{2}}e^{z}+\frac{2}{\Gamma(\frac{q}{2})}z^{-2}+r_3(z)\,, & |\arg z\mp\frac{\pi}{2}|\leq\varepsilon\,.\end{cases}$$

The reminders admit the estimates:

$$(13.7)\qquad |r_1(z)|\leq C_1(\varepsilon)|z|^{-1}, |\arg z|\leq\frac{\pi}{2}-\varepsilon\,;$$
$$|r_2(z)|\leq C_2(\varepsilon)|z|^{-2}, |\arg z-\pi|\leq\frac{\pi}{2}-\varepsilon\,;$$
$$|r_3(z)|\leq C_3\left(|z|^{-2}+|e^{z}||z|^{-(\frac{q}{2}+1)}\right)\,,\quad |\arg z\mp\frac{\pi}{2}|\leq\varepsilon\,,$$

where the values $C_1(\varepsilon)<+\infty,\ C_2(\varepsilon)<+\infty,\ C_3(\varepsilon)<+\infty$ do not depend on z.

From (13.6), (13.7) it follows that for any $\varepsilon>0$ the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ has not more that finitely many roots outside the angular domain $\{z:\ |\arg z\mp\frac{\pi}{2}|\leq\varepsilon$. Inside this domain the analyzed function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ has infinitely many roots, and these roots are asymptotically close to the roots of the approximating function $f_q(z)$:

$$f_q(z)=2^{\frac{q}{2}}z^{-\frac{q}{2}}\left(e^{z}+\frac{2^{1-\frac{q}{2}}}{\Gamma(\frac{q}{2})}z^{\frac{q}{2}-2}\right).$$

Investigating the location of roots of the approximating function $f_q(z)$, one should distinguish several cases:
$\boldsymbol{q=4}$. In this case, the equation $f_q(z)=0$ is the equation $e^{z}+\frac{1}{2}=0$, so, the roots of the approximating function can be found explicitly: these root form an arithmetical progression located on the straight line $\{z=x+iy:\ x=-\ln 2,\ -\infty<y<\infty\}$. From this and (13.6)-(13.7) it follows that the roots of the the analyzed function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ function are asymptotically close to the above appeared straight line. Thus, *for $q=4$ all roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ but finitely many are disposed in the open left half plane.* Actually, for $q=4$ *all* roots of this function are disposed in the open left half plane. To establish this, one need the further analysis. This will be done a little bit later.
$\boldsymbol{q\neq 4}$. In this case, the equation $f_q(z)=0$ is the equation

$$e^{z}+c_q\,z^{\frac{q}{2}-2}=0\,,\quad c_q=\frac{2^{1-\frac{q}{2}}}{\Gamma(\frac{q}{2})}\,,$$

where the exponent $\frac{q}{2}-2$ is different from zero. The last equation has infinitely many roots which have no finite accumulation points and which are asymptotically close to the 'logarithmic parabola'

$$(13.8)\qquad x=(\tfrac{q}{2}-1)\ln\left(|y|+1\right)+ln|c|,\ -\infty<y<\infty\,,\quad (z=x+iy).$$

From this and from (13.6)-(13.7) it follows that the roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ are asymptotically close to the logarithmic parabola (13.8). Now we should distinguish the cases $q<4$ and $q>4$.

$\boldsymbol{q<4}$. In this case, the logarithmic parabola, (13.8) except may by its compact subset, is located inside the left half plane. Since the roots of the analyzed function are asymptotically close to this parabola, all roots but finitely many are located in the left half plane.

$\boldsymbol{q>4}$. In this case, the logarithmic parabola, (13.8) except may by its compact subset, is located inside the right half plane. So, all roots of the function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$, except finitely many, are located in the right half plane.

Let us formulate this result as

Lemma 13.1. *If $q>4$, then the entire function $\mathcal{M}_{B^n\times 0^q}$ has infinitely many roots within the right half plane. In particular, this function does not belong to the Hurwitz class $\mathcal{H}$.*

Claim 2 of Lemma 11.3 is a consequence of Lemma 13.1.

Lemma 13.2. *For $q:\ 0\le q\le 2$, the function $\mathcal{M}_{B^n\times 0^q}$ belongs to the Hurwitz class $\mathcal{H}$.*

Proof. For $q=0$, the assertion is evident: the function in question is equal to e^z. To investigate the case $q>0$, we use the integral representation

$$ {}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}=\frac{q}{\Gamma(\frac{q}{2}+1)}\,I_q(z)\,, \tag{13.9}$$

where

$$ I_q(z)=\int\limits_0^1(1-\xi^2)^{\frac{q}{2}-1}\xi\,e^{\xi t}\,d\xi\,. \tag{13.10}$$

The defect of the entire function ${}_1\Psi_1\left\{\begin{smallmatrix}\frac{1}{2}\\1\end{smallmatrix};\begin{smallmatrix}\frac{1}{2}\\1+\frac{q}{2}\end{smallmatrix};z\right\}$ is non-negative. So it is enough to prove that this function has no roots in the closed right half plane. The function $I_q(z)$ is of the form

$$ I_q(z)=\int\limits_0^1\varphi_q(\xi)e^{\xi z}d\xi\,, \tag{13.11}$$

where

$$ \varphi_q(\xi)=(1-\xi^2)^{\frac{q}{2}-1}\xi,\quad 0\le\xi\le 1\,. \tag{13.12}$$

The crucial circumstance is:

For $q:\ 0\le q\le 2$, the function $\varphi_q(\xi)$ is positive and strictly increasing on the interval $(0,1)$.

Lemma 13.3. [Polya] *If $\varphi(\xi)$ is a non-negative increasing function on the interval $[0,1]$, then the entire function*

$$I(z) = \int\limits_0^1 \varphi(\xi) e^{\xi z} d\xi \tag{13.13}$$

has no zeros in the closed right half plane.

This lemma is a continual analog of one Theorem of one theorem of S. Kakeya. Proof of this Lemma and the reference to the paper of S. Kakeya could be found in [48], § 1. We give another proof. We have learned the idea of this proof from [46]. (See Lemma 4 there.)

Proof of Lemma 13.3. Let $z = x + iy$. Since $f(x) > 0$ for $x \geq 0$, $f(x)$ has no zeros for $0 \leq x < \infty$. Let us show that $f(z)$ has no zeros for $0 \leq x < \infty$, $y \neq 0$. It is enough to consider the case $y > 0$ only. We prove that $\operatorname{Im}(e^{-z}f(z)) < 0$ for $z = x + iy,\ x \geq 0,\ y > 0$, thus $f(z) \neq 0$ for $z = x + iy,\ x \geq 0,\ y > 0$. To prove this, we use the integral representation

$$e^{-z} f(z) = \int\limits_0^\infty \psi(\xi) e^{-i\xi y} d\xi\,, \tag{13.14}$$

where

$$\psi(\xi) = \begin{cases} \varphi(1-\xi)e^{-x\xi}, & 0 \leq \xi \leq 1\,, \\ 0, & 1 < \xi < \infty\,. \end{cases} \tag{13.15}$$

In particular,

$$-(\operatorname{Im} e^{-z} f(z)) = \int\limits_0^\infty \psi(\xi) \sin \xi y\, d\xi\,, \quad z = x + iy, \quad x \geq 0,\, y > 0\,, \tag{13.16}$$

where the function $\psi(\xi)$ is non-negative and decreasing on $[0,\infty)$, strictly decreasing on some non-empty open interval, and $\psi(\infty) = 0$. Further,

$$\begin{gathered} \int\limits_0^\infty \psi(\xi) \sin \xi y\, d\xi = \sum_{k=0}^\infty \int\limits_{\frac{k\pi}{y}}^{\frac{(k+1)\pi}{y}} \psi(\xi) \sin \xi y\, d\xi = \\ \int\limits_0^{\frac{\pi}{y}} \left(\sum_{k=0}^\infty (-1)^k \psi(\xi + \tfrac{k\pi}{y}) \right) \sin \xi y\, d\xi > 0\ : \end{gathered} \tag{13.17}$$

The series under the last integral is a Leibnitz type series. Thus the sum of this series is non-negative on the interval of integration, and is strictly positive on some subinterval. ∎

Lemma 13.4. *For $q = 4$, the function $\mathcal{M}_{B^n \times 0^q}$ belongs to the Hurwitz class $\mathcal{H}$.*

Proof. For $q = 4$, the integral in (13.10) can be calculated explicitly:

$$I_4(z) = \frac{(2z^2 - 6z + 6)e^z + (z^2 - 6)}{z^4}\,. \tag{13.18}$$

Our goal is to prove that the function $\frac{(2z^2-6z+6)e^z+(z^2-6)}{z^4}$ has no roots in the closed right half plane. Instead to investigate this function, we will investigate the function

$$f(z) = (2z^2 - 6z + 6) + (z^2 - 6)e^{-z}\,. \tag{13.19}$$

We prove that the function $f(z)$ has no roots in the closed right half plane other than the root at the point $z = 0$ of multiplicity four. The function f is of the form

$$f(z) = g(z) + h(z)\,,\ \text{where}\ \ g(z) = (2z^2 - 6z + 6),\ h(z) = (z^2 - 6)e^{-z}\,. \tag{13.20}$$

In the right half plane the function h is subordinate to the function g in the following sense. For $R > 0$, let us consider the contour Γ_R which consists of the interval I_R of the imaginary axis and of the semicircle C_R located in the right half plane:

$$\Gamma_R = I_R \cup C_R\,,\ \text{where}\ I_R = [-iR, iR],\ C_R = \{z:\ |z| = R,\ \mathrm{Re}\, z \geq 0\,. \tag{13.21}$$

It is clear that $|g(z)| \geq 1.75\,|z|^2,\ \ |h(z)| \leq 1.25\,|z|^2$ if $z \in C_R$ and R is large enough. In particular, $|g(z)| > |h(z)|$ if $z \in C_R$ and R is large enough. On the imaginary axis,

$$|g(iy)|^2 = 36 + 12y^2 + 4y^4,\ \ |h(iy)|^2 = 36 + 12y^2 + y^4\,,\ -\infty < y < \infty, \tag{13.22}$$

In particular, $|g(z)| \geq |h(z)|$ for $z \in I_R$, and the inequality is strict for $z \neq 0$. Thus,

$$|g(z)| \geq |h(z)| \quad \text{for} \quad z \in \Gamma_R \quad \text{and } R \text{ is large enough.} \tag{13.23}$$

For $0 < \varepsilon < 1$, consider the function

$$f_\varepsilon(z) = g(z) + (1 - \varepsilon)h(z)\,. \tag{13.24}$$

The function g, which is a polynomial, has two simple roots: $z_{1,2} = \frac{3\pm i\sqrt{3}}{2}$. They are located in the open right half plane. In view of (13.24) and Rouche's theorem, for $\varepsilon > 0$ the function $f_\varepsilon(z)$ has precisely two roots $z_1(\varepsilon)$, $z_2(\varepsilon)$ in the open right half plane. For ε positive an very small, the roots $z_1(\varepsilon)$, $z_2(\varepsilon)$ are located very close to the boundary point $z = 0$. This can be shown by the asymptotic calculation. Since $f_\varepsilon(z) = 6\varepsilon + \frac{1-\varepsilon}{4}z^4 + o(|z|^4)$ as $z \to 0$, the equation $f_\varepsilon(z) = 0$ has the roots $z_1(\varepsilon)$, $z_2(\varepsilon)$ which behave as

$$z_{1,2}(\varepsilon) = \varepsilon^{\frac{1}{4}} 24^{\frac{1}{4}} e^{\pm\frac{\pi}{4}}(1 + o(\varepsilon)) \ \ \text{as}\ \varepsilon \to\ +0\,.$$

Since for $\varepsilon > 0$ the function f_ε has only two roots in the open right half plane, there are no roots other than $z_1(\varepsilon)$, $z_2(\varepsilon)$ there. Since $f(z) = \lim_{\varepsilon\to+0} f_\varepsilon(z)$, the function $f(z)$ has no roots in the open right half plane. (We apply Hurwitz's theorem.) From (13.22) it follows that the function f does not vanish on the imaginary axis except the point $z = 0$. At this point the function f has the root of multiplicity four. Thus, for $q = 4$ the function $I_q(z)$ is in the Hurwitz class. □

Claim 1 of Lemma 11.3 is a consequence of Lemma 13.2 and Lemma 13.4.

Remark 13.1. *From* (13.9) *and* (13.18)*, the explicit expression follows:*

$$ {}_1\Psi_1 \left\{ {\textstyle\frac{\frac{1}{2}}{1}} ; {\textstyle\frac{\frac{1}{2}}{1+\frac{q}{2}}} ; z \right\} = 2\frac{(2z^2 - 6z + 6)e^z + (z^2 - 6)}{z^4} \quad \text{for} \quad q = 4\,. \tag{13.25}$$

This expression agrees with the asymptotic (13.6).

Analysis of the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$: We may calculate the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ asymptotically expressing it in terms of the appropriate Fox-Wright function, (13.5), and then refer to the asymptotic expansion of this Fox-Wright function. However, we derive the asymptotic of the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ from the asymptotic of the function $\mathcal{M}_{B^n\times 0^p}(t)$. From (13.1) and (13.1) it follows that

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t) = \frac{1}{2}\big(\mathcal{M}_{B^n\times 0^p}(it) + \mathcal{M}_{B^n\times 0^p}(-it)\big)\,. \tag{13.26}$$

Comparing (13.26) with (13.6), we see that

$$\mathcal{W}^p_{\partial(B^\infty\times 0)}(t) = \tag{13.27}$$
$$= \begin{cases} 2^{\frac{p}{2}}\, t^{-\frac{p}{2}}\, \cos\,(t-\frac{\pi p}{4}) + \frac{2}{\Gamma(\frac{p}{2})}t^{-2} + r_1(t), & |\arg t| \leq \varepsilon\,, \\ 2^{\frac{q}{2}}\,(te^{-i\pi})^{-\frac{p}{2}}\cos\,(t+\frac{\pi p}{4}) + \frac{2}{\Gamma(\frac{p}{2})}t^{-2} + r_2(t), & |\arg t - \pi| \leq \varepsilon\,, \\ 2^{\frac{p}{2}}\,(t\,e^{\mp\frac{i\pi}{2}})^{-\frac{p}{2}}\,e^{\mp it}(1+r_3(t))\,, & |\arg t \pm \frac{\pi}{2}| \leq \frac{\pi}{2} - \varepsilon\,, \end{cases}$$

where the reminders $r_1(t)$, $r_2(t)$, $r_2(t)$ admit the estimates

$$|r_1(t)| \leq C_1(\varepsilon)\left(|t|^{-(1+\frac{p}{2})}e^{|\operatorname{Im} t|} + |t|^{-3}\right)\,, \qquad |\arg t| \leq \varepsilon\,, \tag{13.28a}$$

$$|r_2(t)| \leq C_2(\varepsilon)\left(|t|^{-(1+\frac{p}{2})}e^{|\operatorname{Im} t|} + |t|^{-3}\right)\,, \quad |\arg t - \pi| \leq \varepsilon\,, \tag{13.28b}$$

$$|r_3(t)| \leq C_3(\varepsilon)\,|t|^{-1}|, \qquad |\arg t \mp \frac{\pi}{2}| \leq \frac{\pi}{2} - \varepsilon\,, \tag{13.28c}$$

and $C_1(\varepsilon) < \infty$, $C_2(\varepsilon) < \infty$, $C_3(\varepsilon) < \infty$ for every $\varepsilon :\ 0 < \varepsilon < \frac{\pi}{2}$. Moreover, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ is even function of t, and takes real values at real t.

From (13.27), (13.28) it follows that for every $\varepsilon > 0$ the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ may have not more that finitely many roots within the angles $\{t :\ |\arg\,(t \mp \frac{\pi}{2}| \leq \frac{\pi}{2} - \varepsilon\}$. Within the angle $\{t :\ |\arg t| \leq \varepsilon\}$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ has infinitely many roots, and these roots are asymptotically close to the roots of the approximating function

$$f_p(t) = 2^{\frac{p}{2}}t^{-\frac{p}{2}}\cos\,(t-\tfrac{\pi p}{4}) + \frac{2}{\Gamma(\frac{p}{2})}t^{-2}\,, \quad |\arg t| \leq \varepsilon\,.$$

(Since the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ is even, there is no need to study its behavior within the angle $\{t :\ |\arg t - \pi| \leq \varepsilon\}$.) The behavior of roots of the approximating equation $f_p(t) = 0$, that is the equation

$$\cos\,(t-\tfrac{\pi p}{4}) + \frac{2^{1-\frac{p}{2}}}{\Gamma(\frac{p}{2})}\,t^{\frac{p}{2}-2} = 0\,, \quad |\arg t| \leq \varepsilon\,, \tag{13.29}$$

depends on p.

> If $0 < p < 4$, then all but finitely many roots of the equation (13.29) are real and simple, and these roots are asymptotically close to the roots of the equation $\cos\,(t-\frac{\pi p}{4})$.
> If $p = 4$, then all but finitely many roots of the equation (13.29) are real and simple, and these roots are asymptotically close to the roots of the equation $\cos\,(t-\pi) + \frac{1}{2} = 0 = 0$.

If $p > 4$, then all but finitely many roots of the equation (13.29) are non-real and simple, they are located symmetrically with respect to the real axis, and are asymptotically close to the 'logarithmic parabola'

$$|y| = (\tfrac{p}{2} - 1)\ln(|x|+1) + \ln c_p\,, \quad c_p = \frac{2^{1-\frac{p}{2}}}{\Gamma(\frac{p}{2})}\,, \quad 0 \leq x < \infty\,.$$

Thus we prove the following

Lemma 13.5. *For each $p:\ 0 \leq p < \infty$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ has infinitely many roots. All but finitely many these roots are simple. They lie symmetric with respect to the point $z = 0$.*

1. *If $0 \leq p \leq 4$, then all but finitely many these roots are real;*
2. *If $4 < p$, then all but finitely many these roots are non-real. In particular, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ does not belong to the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$.*

Lemma 13.6. *For $0 < p \leq 2$, as well as for $p = 4$, the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}$ belongs to the Laguerre-Polya class.*

Proof. The equality (13.26) is a starting point of our reasoning. If the function $\mathcal{M}_{B^n\times 0^p}(t)$ is in the Hurwitz class $\mathcal{H}$, then the function

$$\Omega(t) = \mathcal{M}_{B^n\times 0^p}(it) \tag{13.30}$$

is in the class $\mathcal{P}$ in the sense of of [37].

Definition 13.1. [B.Levin, [37],Chapter VII, Section 4.] *An entire function $\Omega(t)$ of exponential type belongs to the class $\mathcal{P}$ if:*

1. *$\Omega(t)$ has no roots in the closed lower half-plane $\{t : \operatorname{Im} t \leq 0\}$.*
2. *The defect d_Ω of the function ω is non-negative, where*

$$2d_\Omega = \overline{\lim_{r\to+\infty}}\,\frac{\ln|\Omega(-ir)|}{r} - \overline{\lim_{r\to+\infty}}\,\frac{\ln|\Omega(ir)|}{r}\,.$$

In the book [37] of B.Ya.Levin, the following version of the Hermite-Bieler Theorem is proved:

Theorem 7. [[37], Chapter VII, Section 4, Theorem 7] *If an entire function $\Omega(t)$ is in class $\mathcal{P}$, then its real and imaginary parts ${}^{\mathcal{R}}\Omega(t)$ and ${}^{\mathcal{I}}\Omega(t)$:*

$${}^{\mathcal{R}}\Omega(t) = \frac{\Omega(t) + \overline{\Omega(\overline{t})}}{2}, \quad {}^{\mathcal{I}}\Omega(t) = \frac{\Omega(t) - \overline{\Omega(\overline{t})}}{2i},$$

possess the properties:

1. *The roots of each of the functions ${}^{\mathcal{R}}\Omega(t)$ and ${}^{\mathcal{I}}\Omega(t)$ are real and simple;*
2. *The root sets of the functions ${}^{\mathcal{R}}\Omega(t)$ and ${}^{\mathcal{I}}\Omega(t)$ interlace.*

Let us apply this theorem to the function $\Omega(t)$ defined by (13.30):

$$\Omega(t) = \mathcal{M}_{B^n\times 0^p}(it)\,,$$

taking into account that the function $\mathcal{M}_{B^n\times 0^p}$ is real:
$\mathcal{M}_{B^n\times 0^p}(t) \equiv \overline{\mathcal{M}_{B^n\times 0^p}(\overline{t})}$, or what is the same, $\mathcal{M}_{B^n\times 0^p}(-it) \equiv \overline{\mathcal{M}_{B^n\times 0^p}(i\overline{t})}$.
Hence,

$$^{\mathcal{R}}\Omega(t) = \frac{1}{2}\big(\mathcal{M}_{B^n\times 0^p}(it) + \mathcal{M}_{B^n\times 0^p}(-it)\big)\,,$$

or taking (13.26) into account,

$$^{\mathcal{R}}\Omega(t) = \mathcal{W}^p_{\partial(B^\infty\times 0)}(t)\,.$$

Thus, the following result holds:

Lemma 13.7. *If the function $\mathcal{M}_{B^n\times 0^p}(t)$ belongs to the Hurwitz class $\mathcal{H}$, then the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ belongs to the Laguerre-Polya class $\mathcal{L}$-$\mathcal{P}$.*

Combining Lemma 13.7 with Lemmas 13.2 and 13.4, we obtain Lemma 13.6. □

Remark 13.2. *For the function $\Omega(t)$ of the form (13.30), its real part $^{\mathcal{R}}\Omega(t)$ $^{\mathcal{I}}\Omega(t)$ has infinitely many non-real roots if $p > 4$. Nevertheless, all roots of the imaginary part $^{\mathcal{I}}\Omega(t)$ are real for every $p \geq 0$.*

Indeed, according to (13.1) and (13.30),

$$^{\mathcal{I}}\Omega(t) = p\int\limits_0^1 (1-\xi^2)^{\frac{p}{2}-1}\xi\,\sin\xi t\,d\xi\,. \tag{13.31}$$

According to A.Hurwitz (see [64], section 15.27), for every $\nu \geq -1$, all roots of the entire function

$$\Big(\frac{t}{2}\Big)^{-\nu}J_\nu(t) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,\Gamma(\nu+l+1)}\Big(\frac{t^2}{4}\Big)^l$$

are real. ($J_\nu(t)$ is the Bessel function of the index ν.) For $\nu > \frac{1}{2}$, the function $(\frac{t}{2})^{-\nu}J_\nu(t)$ admits the integral representation

$$\frac{1}{2}\Big(\frac{t}{2}\Big)^{-\nu}J_\nu(t) = \frac{1}{\Gamma(\nu+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int\limits_0^1(1-\xi^2)^{\nu-\frac{1}{2}}\cos t\xi\,d\xi\,. \tag{13.32}$$

Thus, for $\nu > -\frac{1}{2}$ all roots of the entire function $\int\limits_0^1(1-\xi^2)^{\nu-\frac{1}{2}}\cos t\xi\,d\xi$ are real. If all roots of a real entire function of exponential type are real, then all roots of its derivative are real as well. Thus, for $\nu > -\frac{1}{2}$ all roots of the function $\int\limits_0^1(1-\xi^2)^{\nu-\frac{1}{2}}\xi\sin t\xi\,d\xi$ are real. However, for $\nu = \frac{p-1}{2}$, the last function coincides with the function $\frac{1}{p}\,{}^{\mathcal{I}}\Omega(t)$. □

For $p = 3$, we do not know whether the function $\mathcal{W}^p_{\partial(B^\infty\times 0)}(t)$ belongs to the Laguerre-Polya class or not. Our conjecture is that YES. Let us formulate our conjectures more precisely. Let us formulate our conjectures in terms of the Fox-Wright functions.

CONJECTURE 1. *For $0 \leq \lambda \leq 2$, all roots of the Fox-Wright function*

$$ {}_1\Psi_1 \left\{ \begin{smallmatrix} \frac{1}{2} \\ 1 \end{smallmatrix}; \begin{smallmatrix} \frac{1}{2} \\ 1+\lambda \end{smallmatrix}; t \right\} = \sum_{0 \leq k < \infty} \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+1+\lambda)} \frac{t^k}{k!} \tag{13.33} $$

lie in the open left half plane.

We proved that the answer is affirmative for $0 \leq \lambda \leq 1$, and for $\lambda = 2$.

CONJECTURE 2. *For $0 \leq \lambda \leq 2$, all roots of the Fox-Wright function*

$$ {}_1\Psi_2 \left\{ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}; \begin{smallmatrix} 1 \\ \frac{1}{2} \end{smallmatrix} \begin{smallmatrix} 1 \\ 1+\lambda \end{smallmatrix}; t \right\} = \sum_{0 \leq l < \infty} \frac{\Gamma(l+1)}{\Gamma(l+\frac{1}{2})\Gamma(l+1+\lambda)} \frac{t^l}{l!} \tag{13.34} $$

are negative and simple.

We proved that the answer is affirmative for $0 \leq \lambda \leq 1$, and for $\lambda = 2$.

From Hermite-Bieler theorem it follows that if Conjecture 1 holds for some λ, then for this λ Conjecture 2 holds as well.

The conjectures 1 and 2 are related to some deep questions related to 'meromorphic multiplier sequences'. (See [17], Problem 1.1.)

14. CONCLUDING REMARKS.

1. In the present paper we use geometric consideration to a small extent. The only general geometric tool which we used was the Alexandrov-Fenchel inequalities. We did not use the monotonicity properties of the coefficients of the Minkowski polynomials. If V_0, V_1, V_2 are convex sets, such that

$$ V_1 \subseteq V_0 \subseteq V_2 $$

and $M_{V_0}(t)$, $M_{V_1}(t)$, $M_{V_2}(t)$ are their Minkowskii polynomials,

$$ M_{V_j}(t) = \sum_{0 \leq k \leq n} m_k(V_j) t^k, \quad j = 0,\, 1,\, 2, $$

then for the coefficients of these polynomials the inequalities

$$ m_k(V_1) \leq m_k(V_0) \leq m_k(V_2)\,, \quad 0 \leq k \leq n. $$

hold. In this connection, the use of the *Kharitonov criterion of stability* may be helpful. (Concerning the Haritonov criterion see Chapters 5 and 7 of the book [9] and the literature quoted there.) The Kharitonov criterion deals with the 'interval stability' of polynomials. In its simplest form, this criterion allow to determine whether the polynomial

$$ f(t) = \sum_{0 \leq k \leq n} a_k t^k \tag{14.1} $$

with the real coefficients a_k is stable from the information that these coefficients belongs to some intervals:

$$ a_k^- \leq a_k \leq a_k^+\,, \quad 0 \leq k \leq n\,. \tag{14.2} $$

Applying this criterion, one need to construct certain polynomials from the given numbers a_k^-, a_k^+, $0 \leq k \leq n$. (There are finitely many such polynomials.) If all these polynomials are stable, then the arbitrary polynomial $f(t)$, (14.1), whose coefficients satisfy the inequalities (14.2), is stable.

2. In the example of a convex set V whose Minkowski polynomial is not dissipative, the set V are very 'flattened' in some direction. (See Theorem 2.7.)

What one can say about Minkowski polynomials of those convex set V which are 'isotropic'?

The notion of *isotropy* may be defined in the following way.

Definition 14.1. *The solid convex set V, $V \subset \mathbb{R}^n$, is said to be isotropic (with respect to the point 0), if the integral*

$$\int\limits_V |\langle x, e\rangle|^2 dv_n(x)$$

takes the same value (i.e. is constant with respect to e) for every vector $e \in \mathbb{R}^n$ such that $\langle e, e\rangle = 1$. Here $\langle\, .\, ,\, .\,\rangle$ is the standard scalar product in $\mathbb{R}^n$, and $dv_n(x)$ is the standard n-dimensional element on volume.

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