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ASYMPTOTICS FOR NONLINEAR DAMPED WAVE EQUATIONS WITH LARGE INITIAL DATA

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Abstract. We study the one dimensional nonlinear damped wave equation

$$\left\{ \begin{array}{c} u_{tt}+u_t-u_{xx}=\lambda \left|u\right|^{\sigma} u,\ x\in \mathbf{R},\ t>0, \\ u\left(0,x\right)=u_0\left(x\right),\ x\in \mathbf{R}, \\ u_t\left(0,x\right)=u_1\left(x\right),\ x\in \mathbf{R}, \end{array}\right. \tag{0.1}$$

where $\sigma>0$, $\lambda\in\mathbf{R}$. Our aim is to prove the large time asymptotic formulas for solutions of the Cauchy problem (0.1) without any restriction on the size of the initial data.

1. Introduction

We study the one dimensional nonlinear damped wave equation

$$\left\{ \begin{array}{c} u_{tt}+u_t-u_{xx}=\lambda \left|u\right|^{\sigma} u,\ x\in \mathbf{R},\ t>0, \\ u\left(0,x\right)=u_0\left(x\right),\ u_t\left(0,x\right)=u_1\left(x\right),\ x\in \mathbf{R}, \end{array}\right. \tag{1.1}$$

where $\sigma>0$, $\lambda\in\mathbf{R}$. Recently much attention was drawn to nonlinear wave equations with dissipative terms. The blow-up results were proved in [27] for the case of nonlinearity $-\left|v\right|^{1+\sigma}$, with $\sigma<2$, when the initial data are such that $\int_{\mathbf{R}} v_0\left(x\right)dx>0$, and $\int_{\mathbf{R}} v_1\left(x\right)dx>0$. Blow-up results for the critical and subcritical cases $\sigma\leq 2$ were obtained in [21]. The global existence of solutions to the Cauchy problem (1.1) with nonlinearities $\pm\left|v\right|^{1+\sigma}$ or $\pm\left|v\right|^{\sigma} v$ for supercritical power $\sigma>2$ was proved in paper [27], where the large time decay estimates were obtained if the initial data are sufficiently small and have a compact support. The large time asymptotic behavior of solutions to (1.1) for the supercritical case $\sigma>2$

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was obtained in paper [13] in the framework of the usual $\mathbf{L}^2$ - theory. When the initial data are in the usual Sobolev space $\partial^\alpha u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, $|\alpha| \leq 1$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, problem (1.1) was considered in papers [23], [25]. Applying the energy type estimates of [22] and [19] it was proved in paper [17] that solutions of the nonlinear damped wave equation (1.1) in the supercritical case $\sigma > 3$ with arbitrary initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{L}^1$, $u_1 \in \mathbf{L}^2 \cap \mathbf{L}^1$ (i.e. without smallness assumption on the initial data) have the same large time asymptotics as that for the linear heat equation $\partial_t - \partial_x^2$, that is

$$\|u(t) - MG_0(t)\|_{\mathbf{L}^p} = o\left(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\right)$$

for $t \to \infty$, where $2 \leq p \leq \infty$, and $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel, M is a constant. Recently the critical case $\sigma = 2$ was considered in paper [12], where it was proved that the large time decay estimate of solutions of (1.1) have an additional logarithmic correction, i.e.

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C(1+t)^{-\frac{1}{2}} (1+\log(1+t))^{-\frac{1}{2}}.$$

The large time asymptotics of solutions to the Cauchy problem (1.1) with small initial data in the subcritical case $\sigma \in (0,2)$, when σ is close to 2, was obtained in paper [14]. Note that similar behavior of solutions was discovered for the nonlinear heat equation

$$v_t - v_{xx} - v^{1+\sigma} = 0$$

first in the critical case $\sigma = 2$, comparing with the linear heat equation, (see [6], [9]). For blow-up results we refer [5], [7], [20]. Large time behavior of solutions to the nonlinear heat equations in the subcritical cases $\sigma \in (0,2)$ was obtained in papers [2], [3], [8], [18], [29].

Sharp time decay estimates in $\mathbf{L}^p$ norm of solutions to (1.1) in the subcritical case $\sigma \in (0,2)$ were obtained recently in paper [24], under the condition that the initial data decay exponentially at infinity without any restriction on the size, where $2 \leq p \leq \frac{2}{n-2}$ for the space dimension $n \geq 3$, $2 \leq p < \infty$ for $n = 2$ and $2 \leq p \leq \infty$ for $n = 1$. The method used in [24] can not be applied to the supercritical case. As far as we know the large time asymptotic formulas for solutions of (1.1) for $\sigma < 3$ were not considered previously for the case of arbitrary (not small) initial data.

The aim of the present paper is to prove large time asymptotic formulas for the solutions of the Cauchy problem (1.1) without any restriction on the size of the initial data. We restrict our attention in this paper to the one dimensional case since our method here is based on the following a priori estimate of solutions $\int_0^\infty \|u_{tt}\|_{\mathbf{L}^1} dt < \infty$ stated below in Lemma 3.1 under the condition $\sigma > 2\left(\sqrt{3}-1\right)$, so that this lower bound of order of the nonlinearity becomes greater than the critical value $\sigma = \frac{2}{n}$ for higher space dimensions.

By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denoted by $\mathbf{L}^p$, $1 \leq p \leq \infty$, the weighted Lebesgue space $\mathbf{L}^{1,a}$ is defined by

$$\mathbf{L}^{p,a} = \left\{\phi \in \mathbf{L}^p; \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle x\rangle^a \phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $\{x\} = \frac{|x|}{\langle x\rangle}$, $a \geq 0$. Weighted Sobolev spaces we define as follows

$$\mathbf{W}_p^{k,a} = \left\{ \phi \in \mathbf{L}^p; \|\phi\|_{\mathbf{W}_p^{k,a}} = \sum_{j=0}^{k} \left\|\partial^j \phi\right\|_{\mathbf{L}^{p,a}} < \infty \right\},$$

where $k \geq 0$, $a \geq 0$, $1 \leq p \leq \infty$. For simplicity we denote $\mathbf{W}_p^k = \mathbf{W}_p^{k,0}$, $\mathbf{H}^{k,a} = \mathbf{W}_2^{k,a}$. Also we define the norm of the usual Sobolev space $\mathbf{H}^k = \mathbf{H}^{k,0}$ as follows $\|\phi\|_{\mathbf{H}^k}^2 = \sum_{j=0}^{k} \left\|\partial^j \phi\right\|_{\mathbf{L}^2}^2$. Below $\mathcal{F}_{x\to\xi}\phi$ or $\hat{\phi}(\xi)$ is the Fourier transform of $\phi(x)$ defined by

$$\hat{\phi}(\xi) = \mathcal{F}_{x\to\xi}\phi = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{-ix\xi}\phi(x)\,dx$$

and

$$\check{\phi}(x) = \overline{\mathcal{F}}_{\xi\to x}\phi = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{ix\xi}\phi(\xi)\,d\xi$$

is the inverse Fourier transform of $\phi(\xi)$.

We now consider the case $\lambda < 0$, then using the method of paper [16] we can remove the smallness condition on the initial data $u_0(x)$, $u_1(x)$ in the supercritical case $\sigma > 2$.

Theorem 1.1. *Let $\lambda < 0$, $\sigma > 2$. Suppose that the initial data $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. Then there exists a unique solution $u \in \mathbf{C}([0,\infty); \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a})$ to the Cauchy problem (1.1). Moreover the asymptotics is true*

$$u(t,x) = A(4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} + O\left(t^{-\frac{1}{2}-\gamma}\right) \tag{1.2}$$

for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $0 < \gamma < \frac{1}{2}\min(a, \sigma-2)$ and

$$A = \int_{\mathbf{R}} (u_0(x) + u_1(x))\,dx + \lambda \int_0^{\infty} \int_{\mathbf{R}} |u|^{\sigma} u(\tau,x)\,dx d\tau$$

Consider the subcritical case of $\sigma < 2$. Denote the heat kernel

$$G_0(x) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}$$

and $\theta = \int_{\mathbf{R}} (u_0(x) + u_1(x))\,dx$.

Theorem 1.2. *Let $\lambda < 0$. We assume that the initial data $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a} \cap \mathbf{C}^3$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a} \cap \mathbf{C}^2$, $a \in (0,1)$, are such that $\theta > 0$. Also we suppose that the value $\sigma < 2$ is close to 2. Then the Cauchy problem (1.1) has a unique global solution $u \in \mathbf{C}([0,\infty); \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a} \cap \mathbf{C}^3)$, satisfying the following time decay estimates*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{\sigma}}$$

for large $t > 0$. Furthermore there exist a constant b and a function $V \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$ such that the asymptotic formula

$$u(t,x) = bt^{-\frac{1}{\sigma}} V\left(xt^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right),$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\gamma = \frac{1}{2}\min\left(a, 1-\frac{\sigma}{2}\right)$, and $V \in \mathbf{L}^{1,a} \cap \mathbf{L}^\infty$ is a solution of the integral equation

$$V(\xi) = G_0(\xi) - \frac{1}{\Omega}\int_0^1 \frac{dz}{z(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}} G_0\left(\left(\xi - yz^{\frac{1}{2}}\right)(1-z)^{-\frac{1}{2}}\right) F(y)\,dy,$$

with

$$\Omega = \frac{\sigma}{1-\frac{\sigma}{2}} \int_{\mathbf{R}} V^{1+\sigma}(y)\, dy,$$

and

$$F(y) = V^{1+\sigma}(y) - V(y) \int_{\mathbf{R}} V^{1+\sigma}(\xi)\, d\xi.$$

We organize the rest of the paper as follows. In Section 2 we obtain some preliminary estimates of the Green operator solving the linearized Cauchy problem corresponding to (1.1). Section 3 is devoted to the proof of Theorem 1.1. Finally in Section 4 we prove Theorem 1.2.

2. Preliminaries

Consider the linear Cauchy problem

$$\begin{cases} u_{tt} + u_t - u_{xx} = f(t,x), \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \\ \partial_t u(0,x) = u_1(x), \ x \in \mathbf{R}. \end{cases} \tag{2.1}$$

The solution of (2.1) can be written by the Duhamel formula

$$u(t) = (\partial_t + 1)\mathcal{G}(t) u_0 + \mathcal{G}(t) u_1 + \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau, \tag{2.2}$$

where the Green operator $\mathcal{G}(t) = \mathcal{F}^{-1} L(t,\xi) \mathcal{F}$ with a symbol

$$L(t,\xi) = e^{-\frac{t}{2}} \frac{\sin\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)}{\sqrt{|\xi|^2 - \frac{1}{4}}}.$$

Note that the symbol $L(t,\xi)$ is a smooth and bounded function $L(t,\xi) \in \mathbf{C}^\infty(\mathbf{R}\times\mathbf{R})$. We first collect some preliminary estimates for the Green operator $\mathcal{G}(t)$ in the weighted Lebesgue norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, for $a \in (0,1)$, $1 \le p \le \infty$. Also we show that the operator $\mathcal{G}(t)$ behaves asymptotically as a Green operator $\mathcal{G}_0(t)$ for the heat equation

$$\mathcal{G}_0(t)\psi = t^{-\frac{1}{2}} \int_{\mathbf{R}} G_0\left((x-y)t^{-\frac{1}{2}}\right) \psi(y)\, dy$$

with a heat kernel $G_0(x) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}$. Denote by $\vartheta = \int_{\mathbf{R}} \phi(x)\, dx$ the mean value of the function ϕ.

Lemma 2.1. *The estimates are fulfilled*

$$\left\|\partial_t^k \mathcal{G}(t)\phi\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-k} \|\phi\|_{\mathbf{W}_p^k}$$

and

$$\left\|\partial_t^2 \mathcal{G}(t)\phi + e^{-t}\phi\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-1} \left\|\partial_x^2\phi\right\|_{\mathbf{L}^p}$$

for all $t > 0$, where $k \ge 0$, $1 \le p \le \infty$. Also the estimates are valid

$$\left\|\mathcal{G}(t)\phi - \vartheta t^{-\frac{1}{2}} G_0\left((\cdot)t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-\frac{1}{2}-\frac{a}{2}} \|\phi\|_{\mathbf{L}^{1,a}}$$

and

$$\left\|\mathcal{G}(t)\phi - \vartheta t^{-\frac{1}{2}} G_0\left((\cdot)t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^{1,a}} \le C\|\phi\|_{\mathbf{L}^{1,a}}$$

for all $t > 0$, where $a \in [0,1]$, provided that the right-hand sides are finite.

Proof. We have

$$\mathcal{G}(t)\psi = \frac{1}{2}e^{-\frac{t}{2}}\int_{|y|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)\psi(x-y)\,dy,$$

where $I_0(x)$ is the modified Bessel function of order 0 (see [28]). Note that the function $I_0(x)$ has the following asymptotics

$$I_0(x) \sim \frac{e^x}{\sqrt{2\pi x}}\sum_{m=0}^{\infty}\frac{((2m-1)!!)^2}{(2x)^m 2^{2m} m!}$$

for $x\to+\infty$ (see [4]). Thus we have the estimates

$$\left|\frac{d^k}{dt^k}\left(e^{-\frac{t}{2}}I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)\right)\right| \le Ct^{-\frac{1}{2}-k}e^{-C\frac{y^2}{t}}$$

for all $|y|\le\frac{t}{2}$, $t>0$. And in the domain $\frac{t}{2}\le|y|\le t$, $t>0$ we apply the estimates

$$\sup_{x\ge0}\left|e^{-x}\frac{d^k}{dx^k}I_0(x)\right|\le C.$$

We rewrite the Green operator in the form

$$\begin{aligned}\mathcal{G}(t)\psi &= \frac{1}{2}\int_{|y|\le\frac{t}{2}} e^{-\frac{t}{2}}I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)\psi(x-y)\,dy\\ &+\frac{1}{2}\int_{\frac{1}{2}<|z|\le1} te^{-\frac{t}{2}}I_0\left(\frac{t}{2}\sqrt{1-z^2}\right)\psi(x-zt)\,dz,\end{aligned}$$

hence

$$\begin{aligned}&\left\|\partial_t^k\mathcal{G}(t)\psi\right\|_{\mathbf{L}^p}\le Ct^{-\frac{1}{2}-k}\left\|\int_{|y|\le\frac{t}{2}}e^{-C\frac{y^2}{t}}\psi(x-y)\,dy\right\|_{\mathbf{L}^p}\\ &+Ce^{-Ct}\|\psi\|_{\mathbf{W}_p^k}\le Ct^{-k}\|\psi\|_{\mathbf{W}_p^k}\end{aligned}$$

for all $t>0$, $k\ge0$. Thus the first estimate is true.

To prove the second estimate we represent the symbol

$$\begin{aligned}&\frac{1}{\xi^2}\left(\partial_t^2L(t,\xi)+e^{-t}\right)\\ =&\ \frac{1}{\xi^2}\left(\frac{e^{-\frac{t}{2}}\left(1-2\xi^2\right)\sin\left(t\sqrt{\xi^2-\frac{1}{4}}\right)}{2\sqrt{\xi^2-\frac{1}{4}}}-e^{-\frac{t}{2}}\cos\left(t\sqrt{\xi^2-\frac{1}{4}}\right)+e^{-t}\right)\\ =&\ -e^{-\frac{t}{2}}\frac{\sin t\xi}{\xi}+\widehat{R}(t,\xi),\end{aligned}$$

where

$$\begin{aligned}\widehat{R}(t,\xi)&=\frac{e^{-\frac{t}{2}}}{\xi^2}\left(\frac{\sinh\left(\frac{t}{2}\sqrt{1-4\xi^2}\right)}{\sqrt{1-4\xi^2}}-\sinh\left(\frac{t}{2}\right)\right.\\ &\left.-\cosh\left(\frac{t}{2}\sqrt{1-4\xi^2}\right)+\cosh\left(\frac{t}{2}\right)\right)-\frac{e^{-\frac{t}{2}}\sin\left(t\sqrt{\xi^2-\frac{1}{4}}\right)}{\sqrt{\xi^2-\frac{1}{4}}}+e^{-\frac{t}{2}}\frac{\sin t\xi}{\xi}.\end{aligned}$$

Note that $\widehat{R}(t) \in \mathbf{C}^{\infty}(\mathbf{R})$. As $\xi \to 0$ we have

$$\begin{aligned}
\widehat{R}(t,\xi) &= \frac{\left(1-2\xi^2-\sqrt{1-4\xi^2}\right)\exp\left(\frac{t}{2}\sqrt{1-4\xi^2}-\frac{t}{2}\right)}{\xi^2\sqrt{1-4\xi^2}} \\
&\quad -\frac{e^{-\frac{t}{2}}}{2\xi^2}\left(\left(1+\left(1-4\xi^2\right)^{-\frac{1}{2}}\right)\exp\left(-\frac{t}{2}\sqrt{1-4\xi^2}\right)-2e^{-\frac{t}{2}}\right) \\
&\quad +e^{-\frac{t}{2}}\frac{\exp\left(-\frac{t}{2}\sqrt{1-4\xi^2}\right)}{\sqrt{1-4\xi^2}} = O\left(\xi^2 e^{-Ct\xi^2}\right)+O\left(e^{-\frac{t}{2}}\right).
\end{aligned}$$

Moreover $\widehat{R}(t,\xi)$ decays at infinity along with all derivatives with respect to ξ, so that the estimate is true

$$\left|\partial_\xi^m \widehat{R}(t,\xi)\right| \le C\langle t\rangle^{\frac{m}{2}-1} e^{-Ct\xi^2}+Ce^{-\frac{t}{2}}\langle\xi\rangle^{-2}$$

for all $t>0$, $\xi\in\mathbf{R}$, $m=0,1,2$. Therefore there exists an inverse Fourier transform $R(t,x)=\mathcal{F}_{\xi\to x}\widehat{R}(t,\xi)$, which satisfy the estimate

$$\sup_{x\in\mathbf{R}}\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^2 |R(t,x)| \le C\langle t\rangle^{-\frac{3}{2}}.$$

Hence applying the Young inequality we obtain

$$\begin{aligned}
&\left\|\partial_t^2\mathcal{G}(t)\phi+e^{-t}\phi\right\|_{\mathbf{L}^p} \le C\left\|e^{-\frac{t}{2}}\int_{|x-y|<t}\phi_{yy}(y)\,dy\right\|_{\mathbf{L}^p} \\
&+C\left\|\int_{\mathbf{R}}R(t,x-y)\,\phi_{yy}(y)\,dy\right\|_{\mathbf{L}^p} \le Ct^{-1}\left\|\phi_{xx}\right\|_{\mathbf{L}^p}
\end{aligned}$$

for all $t>0$, where $1\le p\le\infty$.

Now we prove the last two estimates of the lemma. We have

$$\begin{aligned}
&\frac{1}{2}e^{-\frac{t}{2}}I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{y^2}{4t}} \\
&= \frac{e^{\frac{1}{2}\sqrt{t^2-y^2}-\frac{t}{2}}}{\sqrt{4\pi}\sqrt[4]{t^2-y^2}}-\frac{1}{\sqrt{4\pi t}}e^{-\frac{y^2}{4t}} = O\left(t^{-\frac{3}{2}}e^{-\frac{y^2}{4t}}\right)
\end{aligned}$$

for all $|y|\le\frac{t}{2}$, $t>0$. Therefore

$$\begin{aligned}
&\left(\mathcal{G}(t)-\mathcal{G}_0(t)\right)\psi = \int_{|y|\le\frac{t}{2}} O\left(t^{-\frac{3}{2}}e^{-\frac{y^2}{4t}}\right)\psi(x-y)\,dy \\
&+\frac{1}{2}e^{-\frac{t}{2}}\int_{\frac{t}{2}<|y|\le t} I_0\left(\frac{1}{2}\sqrt{t^2-y^2}\right)\psi(x-y)\,dy \\
&-\frac{1}{\sqrt{4\pi t}}\int_{|y|>\frac{t}{2}} e^{-\frac{y^2}{4t}}\psi(x-y)\,dy.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\left\|\left(\mathcal{G}(t)-\mathcal{G}_0(t)\right)\psi\right\|_{\mathbf{L}^p} \\
\le\ & Ct^{-\frac{3}{2}}\|\psi\|_{\mathbf{L}^p}\int_{-\infty}^{\infty}e^{-\frac{y^2}{4t}}\,dy+Ce^{-Ct}\|\psi\|_{\mathbf{L}^p}\le Ct^{-1}\|\psi\|_{\mathbf{L}^p}
\end{aligned}$$

and

$$\|(\mathcal{G}(t)-\mathcal{G}_0(t))\psi\|_{\mathbf{L}^{1,a}} \le Ct^{-\frac{3}{2}}\|\psi\|_{\mathbf{L}^1}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}}|y|^a\,dy$$

$$+Ct^{-\frac{3}{2}}\|\psi\|_{\mathbf{L}^{1,a}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4t}}dy + Ce^{-Ct}\|\psi\|_{\mathbf{L}^{1,a}}$$

$$\le \quad Ct^{\frac{a}{2}-1}\|\psi\|_{\mathbf{L}^1} + Ct^{-1}\|\psi\|_{\mathbf{L}^{1,a}} \le C\|\psi\|_{\mathbf{L}^{1,a}}.$$

Taking into account the corresponding estimates of the heat kernel (see [11]) the last two estimates of the lemma follow. Lemma 2.1 is proved.

By using a standard contraction mapping principle we have the following result.

Proposition 2.1. *Let $\sigma > 0$, $\lambda \in \mathbf{R}$. Let $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. Then there exists a positive time T and a unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}\right)$ to the Cauchy problem (1.1).*

We now prove a global existence of solutions to the Cauchy problem (1.1) in the supercritical case of $\sigma > 2$. The main term of the large time asymptotics of solutions has a quasi linear character.

Proposition 2.2. *Let $\sigma > 2$, $\lambda \in \mathbf{R}$. Let $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. Suppose that the local solutions u constructed in Proposition 2.1 satisfy a priori estimate*

$$\sup_{t\in[0,T]} \langle t\rangle^{\frac{1}{2}}\|u(t)\|_{\mathbf{L}^\infty} \le C,$$

where $C > 0$ does not depend on the existence time T. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}\right)$ of the Cauchy problem (1.1). Moreover the asymptotics (1.2) is valid for large time $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $0 < \gamma < \min\left(\frac{a}{2}, \frac{\sigma}{2}-1\right)$ and a constant A is defined in Theorem 1.1.

Proof of Proposition 2.2. We write problem (1.1) as an integral equation

$$u = (\partial_t + 1)\mathcal{G}(t)u_0 + \mathcal{G}(t)u_1 + \lambda\int_0^t \mathcal{G}(t-\tau)|u(\tau)|^\sigma u(\tau)\,d\tau.$$

Let us prove a priori estimate

$$\sup_{t\in[0,T]}\left(\langle t\rangle^{-\frac{a}{2}}\|u(t)\|_{\mathbf{L}^{1,a}} + \langle t\rangle^{\frac{1}{2}}\|u(t)\|_{\mathbf{L}^\infty} + \|u(t)\|_{\mathbf{W}_1^2} + \langle t\rangle^{\frac{1}{4}}\|u(t)\|_{\mathbf{H}^2}\right) \le C, \tag{2.3}$$

where $C > 0$ does not depend on the existence time T. Applying Lemma 2.1 we obtain

$$\|u\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}}\left(\|u_0\|_{\mathbf{L}^{1,a}} + \|u_1\|_{\mathbf{L}^{1,a}}\right)$$

$$+C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{L}^{1,a}}d\tau + C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{L}^1}\langle t-\tau\rangle^{\frac{a}{2}}d\tau$$

$$\le \quad C\langle t\rangle^{\frac{a}{2}} + C\int_0^t\langle\tau\rangle^{-\frac{\sigma}{2}}\|u(\tau)\|_{\mathbf{L}^{1,a}}d\tau + C\int_0^t\langle\tau\rangle^{-\frac{\sigma}{2}}\langle t-\tau\rangle^{\frac{a}{2}}\|u(\tau)\|_{\mathbf{L}^1}d\tau,$$

$$\|u\|_{\mathbf{W}_1^2} \le C\left(\|u_0\|_{\mathbf{W}_1^2} + \|u_1\|_{\mathbf{W}_1^1}\right) + C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{W}_1^1}d\tau$$

$$\le \quad C + C\int_0^t\langle\tau\rangle^{-\frac{\sigma}{2}}\|u(\tau)\|_{\mathbf{W}_1^1}d\tau,$$

and

$$\begin{aligned}
\|u\|_{\mathbf{H}^2} &\le C\langle t\rangle^{-\frac14}\left(\|u_0\|_{\mathbf{W}_1^2}+\|u_0\|_{\mathbf{H}^2}+\|u_1\|_{\mathbf{W}_1^1}+\|u_1\|_{\mathbf{H}^1}\right)\\
&\quad +C\int_0^{\frac t2}\langle t-\tau\rangle^{-\frac14}\left\||u|^\sigma u(\tau)\right\|_{\mathbf{W}_1^1}d\tau+C\int_{\frac t2}^t\left\||u|^\sigma u(\tau)\right\|_{\mathbf{H}^1}d\tau\\
&\le C+C\langle t\rangle^{-\frac14}\int_0^{\frac t2}\langle\tau\rangle^{-\frac\sigma2}\|u(\tau)\|_{\mathbf{W}_1^1}d\tau+C\int_{\frac t2}^t\langle\tau\rangle^{-\frac\sigma2}\|u(\tau)\|_{\mathbf{H}^1}d\tau.
\end{aligned}$$

Therefore the Gronwall inequality gives the desired estimate (2.3). Now the global existence of solutions to the Cauchy problem (1.1) follows by Proposition 2.1 via a standard continuation argument.

We now prove the asymptotics (1.2). We use the integral representation (2.2) with $f=\lambda|u|^\sigma u$. By virtue of the third estimate of Lemma 2.1 we have the following asymptotic representation for the Green operator

$$\begin{aligned}
&(\partial_t+1)\mathcal{G}(t)u_0+\mathcal{G}(t)u_1\\
&=\theta t^{-\frac12}G_0\left(xt^{-\frac12}\right)+O\left(t^{-\frac{1+a}{2}}\left(\|u_0\|_{\mathbf{L}^{1,a}}+\|u_1\|_{\mathbf{L}^{1,a}}\right)\right)
\end{aligned}\tag{2.4}$$

for large time $t\to\infty$ uniformly with respect to $x\in\mathbf{R}$, where $G_0(x)=(4\pi)^{-\frac12}e^{-\frac{x^2}{4}}$, $a\in(0,1]$ and $\theta=\int_{\mathbf{R}}(u_0(x)+u_1(x))\,dx$. Denote also

$$\vartheta(\tau)=\lambda\int_{\mathbf{R}}|u|^\sigma u(\tau)\,dy,$$

and consider the difference

$$\begin{aligned}
&\lambda\int_0^t\mathcal{G}(t-\tau)|u|^\sigma u(\tau)\,d\tau-t^{-\frac12}G_0\left(xt^{-\frac12}\right)\int_0^\infty\vartheta(\tau)\,d\tau\\
&=\lambda\int_{\frac t2}^t\mathcal{G}(t-\tau)|u|^\sigma u(\tau)\,d\tau\\
&\quad+\lambda\int_0^{\frac t2}\left(\mathcal{G}(t-\tau)|u|^\sigma u(\tau)-\vartheta(\tau)(t-\tau)^{-\frac12}G_0\left(x(t-\tau)^{-\frac12}\right)\right)d\tau\\
&\quad+\int_0^{\frac t2}\left((t-\tau)^{-\frac12}G_0\left(x(t-\tau)^{-\frac12}\right)-t^{-\frac12}G_0\left(xt^{-\frac12}\right)\right)\vartheta(\tau)\,d\tau\\
&\quad+t^{-\frac12}G_0\left(xt^{-\frac12}\right)\int_{\frac t2}^\infty\vartheta(\tau)\,d\tau.
\end{aligned}\tag{2.5}$$

In the domain $0<\tau<\frac t2$ we apply the third estimate of Lemma 2.1 to get

$$\begin{aligned}
&\left\|\lambda\mathcal{G}(t-\tau)|u|^\sigma u(\tau)-\vartheta(\tau)(t-\tau)^{-\frac12}G_0\left(x(t-\tau)^{-\frac12}\right)\right\|_{\mathbf{L}^\infty}\\
&\le C(t-\tau)^{-\frac12-\frac a2}\left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^{1,a}}
\end{aligned}$$

and in the domain $\frac t2\le\tau<t$ we use the estimate

$$\left\|\lambda\mathcal{G}(t-\tau)|u|^\sigma u(\tau)\right\|_{\mathbf{L}^\infty}\le C\left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^\infty}.$$

Therefore by virtue of estimates (2.3) we have

$$\begin{aligned}&\left\|\lambda\int_{\frac{t}{2}}^{t}\mathcal{G}\left(t-\tau\right)\left|u\right|^{\sigma}u\left(\tau\right)d\tau\right\|_{\mathbf{L}^{\infty}}\\ \leq\ & C\left\|u\right\|_{\mathbf{X}}^{\sigma+1}\int_{\frac{t}{2}}^{t}\left\langle\tau\right\rangle^{-\frac{1}{2}(\sigma+1)}d\tau\leq C\left\langle t\right\rangle^{-\frac{1}{2}-\gamma},\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_{0}^{\frac{t}{2}}\left(\lambda\mathcal{G}\left(t-\tau\right)\left|u\right|^{\sigma}u\left(\tau\right)-\vartheta\left(\tau\right)\left(t-\tau\right)^{-\frac{1}{2}}G_{0}\left(x\left(t-\tau\right)^{-\frac{1}{2}}\right)\right)d\tau\right\|_{\mathbf{L}^{\infty}}\\ \leq\ & C\left\|u\right\|_{\mathbf{X}}^{\sigma+1}\int_{0}^{\frac{t}{2}}\left(t-\tau\right)^{-\frac{1}{2}-\frac{a}{2}}\left\langle\tau\right\rangle^{-\frac{\sigma}{2}}d\tau\leq C\left\langle t\right\rangle^{-\frac{1}{2}-\gamma},\end{aligned}$$

where $0<\gamma<\min\left(\frac{a}{2},\frac{\sigma}{2}-1\right)$; the norm $\left\|\cdot\right\|_{\mathbf{X}}$ is defined by

$$\left\|\phi\right\|_{\mathbf{X}}=\sup_{t>0}\left(\left(1+t\right)^{\frac{1}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{\infty}}+\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1}}+\left(1+t\right)^{-\frac{a}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,a}}\right).$$

Now we estimate the third summand in (2.5)

$$\begin{aligned}&\left\|\int_{0}^{\frac{t}{2}}\left(G_{0}\left(t-\tau,x\right)-t^{-\frac{1}{2}}G_{0}\left(xt^{-\frac{1}{2}}\right)\right)\vartheta\left(\tau\right)d\tau\right\|_{\mathbf{L}^{\infty}}\\ \leq\ & C\int_{0}^{\frac{t}{2}}\left\|\left(t-\tau\right)^{-\frac{1}{2}}G_{0}\left(x\left(t-\tau\right)^{-\frac{1}{2}}\right)-t^{-\frac{1}{2}}G_{0}\left(xt^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^{\infty}}\left\langle\tau\right\rangle^{-\frac{\sigma}{2}}d\tau\\ \leq\ & C\int_{0}^{\frac{t}{2}}\left\langle t-\tau\right\rangle^{-\frac{1}{2}-\gamma}\left\langle\tau\right\rangle^{\gamma-\frac{\sigma}{2}}d\tau\leq C\left\langle t\right\rangle^{-\frac{1}{2}-\gamma}.\end{aligned}$$

For the last summand in (2.5) we have

$$\left\|t^{-\frac{1}{2}}G_{0}\left(xt^{-\frac{1}{2}}\right)\int_{\frac{t}{2}}^{\infty}\vartheta\left(\tau\right)d\tau\right\|_{\mathbf{L}^{\infty}}\leq Ct^{-\frac{1}{2}}\int_{\frac{t}{2}}^{\infty}\left\langle\tau\right\rangle^{-\frac{\sigma}{2}}d\tau\leq C\left\langle t\right\rangle^{-\frac{1}{2}-\gamma}.$$

Thus in view of (2.4) we see from the integral equation (2.2) that there exists a constant

$$A=\theta+\int_{0}^{\infty}\vartheta\left(\tau\right)d\tau \tag{2.6}$$

such that asymptotics (1.2) is valid. Proposition 2.2 is proved.

In the case of arbitrary sign of the coefficient λ we have to assume a smallness condition for the initial data u_0 and u_1 to be able to prove the a priori estimate $\left\langle t\right\rangle^{\frac{1}{2}}\left\|u\left(t\right)\right\|_{\mathbf{L}^{\infty}}\leq C$, which ensures a global existence result in view of Proposition 2.2.

Proposition 2.3. *Let* $\sigma>2$, $\lambda\in\mathbf{R}$. *Let* $u_0\in\mathbf{W}_1^2\cap\mathbf{H}^2\cap\mathbf{L}^{1,a}$, $u_1\in\mathbf{W}_1^1\cap\mathbf{H}^1\cap\mathbf{L}^{1,a}$, $a\in(0,1]$. *Suppose that the norm* $\left\|u_0\right\|_{\mathbf{L}^1}+\left\|u_0\right\|_{\mathbf{L}^{\infty}}+\left\|u_1\right\|_{\mathbf{L}^1}+\left\|u_1\right\|_{\mathbf{L}^{\infty}}=\varepsilon$, *where* $\varepsilon>0$ *is sufficiently small. Then the local solution constructed in Proposition 2.1 satisfies the a priori estimate*

$$\sup_{t\in[0,T]}\left\langle t\right\rangle^{\frac{1}{2}}\left\|u\left(t\right)\right\|_{\mathbf{L}^{\infty}}+\sup_{t\in[0,T]}\left\|u\left(t\right)\right\|_{\mathbf{L}^{1}}<40\varepsilon.$$

Proof of Proposition 2.3. Consider a maximal time $T_0 > 0$ such that

$$\sup_{t\in[0,T_0]} \langle t\rangle^{\frac{1}{2}} \|u(t)\|_{\mathbf{L}^\infty} + \sup_{t\in[0,T_0]} \|u(t)\|_{\mathbf{L}^1} = 40\varepsilon.$$

Applying Lemma 2.1 we obtain

$$\begin{aligned} \|u\|_{\mathbf{L}^\infty} &\le 10\varepsilon \langle t\rangle^{-\frac{1}{2}} + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{1}{2}} \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^1} d\tau \\ &+C\int_{\frac{t}{2}}^t \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^\infty} d\tau \\ \le\ & 10\varepsilon\langle t\rangle^{-\frac{1}{2}} + C\varepsilon^{\sigma+1}\int_0^{\frac{t}{2}} \langle\tau\rangle^{-\frac{\sigma}{2}} \langle t-\tau\rangle^{-\frac{1}{2}} d\tau \\ &+C\varepsilon^{\sigma+1}\int_{\frac{t}{2}}^t \langle\tau\rangle^{-\frac{1}{2}(\sigma+1)} d\tau < 20\varepsilon\langle t\rangle^{-\frac{1}{2}} \end{aligned}$$

since $\varepsilon > 0$ is sufficiently small and $\sigma > 2$. Similarly we estimate the norm

$$\begin{aligned} \|u(t)\|_{\mathbf{L}^1} &\le \|u_0\|_{\mathbf{L}^1} + \|u_1\|_{\mathbf{L}^1} + C\int_0^t \left\||u|^\sigma u(\tau)\right\|_{\mathbf{L}^1} d\tau \\ \le\ & 10\varepsilon + C\varepsilon^{\sigma+1}\int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}} d\tau < 20\varepsilon \end{aligned}$$

for all $t \in [0, T_0]$. Therefore we obtain

$$\sup_{t\in[0,T_0]} \langle t\rangle^{\frac{1}{2}} \|u(t)\|_{\mathbf{L}^\infty} + \sup_{t\in[0,T_0]} \|u(t)\|_{\mathbf{L}^1} < 40\varepsilon.$$

The contradiction yields the result of Proposition 2.3.

3. Proof of Theorem 1.1

Define the norms

$$\|\phi\|_{p,q} \equiv \left\| \|\phi(t,x)\|_{\mathbf{L}^q(\mathbf{R}_x)} \right\|_{\mathbf{L}^p(0,\infty)}.$$

We first prove a global existence result for large data.

Proposition 3.1. *Let $\lambda < 0, \sigma > 0$. Suppose that the initial data $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}\right)$ to the Cauchy problem (1.1). Moreover the a priori estimates of a solution are valid*

$$\|u\|_{\infty,2} + \|u\|_{\infty,\sigma+2} + \|u_t\|_{\infty,2} + \|u_x\|_{\infty,2} \le C, \tag{3.1}$$

and

$$\|u\|_{\sigma+2,\sigma+2} + \|u_x\|_{2,2} + \|u_t\|_{2,2} \le C. \tag{3.2}$$

Proof. Let u be a solution constructed in Proposition 2.1. We now multiply equation (1.1) by $2(2u_t + u)$. Then integrating the result with respect to $x \in \mathbf{R}$ we get

$$\begin{aligned} & 2\int_{\mathbf{R}} \left((u_t + u)(u_{tt} + u_t) + u_t u_{tt}\right) dx \\ = \ & -2\int_{\mathbf{R}} (u_t)^2 dx + 2\int_{\mathbf{R}} (2u_t + u) u_{xx} dx - 2|\lambda| \int_{\mathbf{R}} (2u_t + u)|u|^\sigma u dx, \end{aligned}$$

therefore

$$\frac{d}{dt}\left(\|u_t+u\|_{\mathbf{L}^2}^2+\|u_t\|_{\mathbf{L}^2}^2+2\|u_x\|_{\mathbf{L}^2}^2+\frac{4|\lambda|}{\sigma+2}\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}\right)$$
$$= \quad -2\|u_t\|_{\mathbf{L}^2}^2-2\|u_x\|_{\mathbf{L}^2}^2-2|\lambda|\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}$$

from which the a priori estimate $\|u(t)\|_{\mathbf{L}^\infty}^2 \le C\|u_x(t)\|_{\mathbf{L}^2}\|u(t)\|_{\mathbf{L}^2} \le C$ follows. In the same way as in the proof of Proposition 2.2, applying Lemma 2.1 we obtain

$$\|u\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{2}}\left(\|u_0\|_{\mathbf{L}^{1,a}}+\|u_1\|_{\mathbf{L}^{1,a}}\right)$$
$$+C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{L}^{1,a}}d\tau + C\int_0^t \||u|^\sigma u(\tau)\|_{\mathbf{L}^1}\langle t-\tau\rangle^{\frac{a}{2}}d\tau$$
$$\le \quad C\langle t\rangle^{\frac{a}{2}}+C\int_0^t\|u(\tau)\|_{\mathbf{L}^{1,a}}d\tau + C\int_0^t\langle t-\tau\rangle^{\frac{a}{2}}\|u(\tau)\|_{\mathbf{L}^1}d\tau,$$

$$\|u\|_{\mathbf{W}_1^2} \le C\left(\|u_0\|_{\mathbf{W}_1^2}+\|u_1\|_{\mathbf{W}_1^1}\right)+C\int_0^t\||u|^\sigma u(\tau)\|_{\mathbf{W}_1^1}d\tau$$
$$\le \quad C+C\int_0^t\|u(\tau)\|_{\mathbf{W}_1^1}d\tau,$$

and

$$\|u\|_{\mathbf{H}^2}\le C\left(\|u_0\|_{\mathbf{H}^2}+\|u_1\|_{\mathbf{H}^1}\right)+C\int_0^t\||u|^\sigma u(\tau)\|_{\mathbf{H}^1}d\tau$$
$$\le \quad C+C\int_0^t\|u(\tau)\|_{\mathbf{H}^1}d\tau$$

for all $t\in[0,T]$. Then the Gronwall lemma yields the estimate

$$e^{-Ct}\left(\|u(t)\|_{\mathbf{L}^{1,a}}+\|u(t)\|_{\mathbf{W}_1^2}+\|u(t)\|_{\mathbf{H}^2}\right)\le C$$

for all $t\in[0,T]$, where $C>0$ does not depend on T. Therefore we can prolong the local solution to the global one. Moreover we have the desired estimates (3.1) and (3.2). Proposition 3.1 is proved.

We now prepare several lemmas.

Lemma 3.1. *Let $\sigma>2\left(\sqrt{3}-1\right)$, $\lambda<0$. Suppose that the initial data $u_0\in\mathbf{W}_1^2\cap\mathbf{H}^2\cap\mathbf{L}^{1,a}$, $u_1\in\mathbf{W}_1^1\cap\mathbf{H}^1\cap\mathbf{L}^{1,a}$, $a\in(0,1]$. Let u be a global solution constructed in Proposition 3.1. Then the estimate is true*

$$\|u_{tt}\|_{1,1}\le C.$$

Proof. Since $\|u\|_{\infty,2}+\|u\|_{\sigma+2,\sigma+2}\le C$, by the Hölder inequality we get for $2\le p\le\sigma+2$

$$\|u(t)\|_{\mathbf{L}^p}\le\|u(t)\|_{\mathbf{L}^2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)}\|u(t)\|_{\mathbf{L}^{2+\sigma}}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)}$$

hence

$$\|u\|_{s,p}\le\|u\|_{\infty,2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)}\|u\|_{\sigma+2,\sigma+2}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)}\le C \tag{3.3}$$

for $s=\frac{\sigma p}{p-2}$, $2\le p\le\sigma+2$. Since $\|u\|_{\infty,2}+\|u_x\|_{2,2}\le C$, by the Cauchy-Schwarz inequality we have $\|u\|_{\mathbf{L}^\infty}^2\le 2\|u\|_{\mathbf{L}^2}\|u_x\|_{\mathbf{L}^2}$, hence

$$\|u\|_{4,\infty}\le C. \tag{3.4}$$

Combining estimates (3.3) and (3.4) we obtain

$$\|u\|_{s,p} \le C \tag{3.5}$$

for $s = \frac{4p}{p+2-\sigma}$, $2+\sigma \le p \le \infty$.

Now we estimate the second derivative u_{xx}. By the integral representation (2.2) with $f = \lambda |u|^{\sigma} u$ we find

$$\begin{aligned} u_{xx} &= (\partial_t + 1)\,\partial_x^2 \mathcal{G}(t)\,u_0 + \partial_x^2 \mathcal{G}(t)\,u_1 \\ &\quad + \lambda \int_0^t \partial_x^2 \mathcal{G}(t-\tau)\,|u(\tau)|^{\sigma} u(\tau)\,d\tau. \end{aligned}$$

In view of estimates (3.3), (3.4) and $\|u_x\|_{2,2} \le C$, using Lemma 2.1 and the Young inequality we obtain

$$\begin{aligned} &\left\| \left\| \int_0^t \partial_x^2 \mathcal{G}(t-\tau)\,|u(\tau)|^{\sigma} u(\tau)\,d\tau \right\|_{\mathbf{L}_x^2} \right\|_{\mathbf{L}_t^s(0,\infty)} \\ \le\ & C \left\| \int_0^t \langle t-\tau\rangle^{-1} \left(\|u(\tau)\|_{\mathbf{L}^{2(\sigma+1)}}^{\sigma+1} + \|u(\tau)\|_{\mathbf{L}^2}^{\frac{\sigma}{2}} \|u_x(\tau)\|_{\mathbf{L}^2}^{1+\frac{\sigma}{2}} \right) d\tau \right\|_{\mathbf{L}_t^s(0,\infty)} \\ \le\ & C \left\| \langle t\rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,T)} \left\| \|u(t)\|_{\mathbf{L}^{2(\sigma+1)}}^{\sigma+1} \right\|_{\mathbf{L}_t^s(0,\infty)} \\ & + C \left\| \langle t\rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,\infty)} \|u\|_{\infty,2}^{\frac{\sigma}{2}} \|u_x\|_{2,2}^{1+\frac{\sigma}{2}} \\ \le\ & C \|u\|_{(\sigma+1)s_2, 2(\sigma+1)} + C \|u\|_{\infty,2}^{\frac{\sigma}{2}} \|u_x\|_{2,2}^{1+\frac{\sigma}{2}}, \end{aligned}$$

where $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} - 1 < \frac{1}{s_2}$ and $\frac{1}{s} = \frac{1}{s_1} + \frac{2+\sigma}{4} - 1 < \frac{1}{s_2}$, since $s_1 > 1$, $s_2 = \frac{8}{\sigma+4}$. Thus we get the estimate

$$\|u_{xx}\|_{s,2} \le C \tag{3.6}$$

for $s > \frac{8}{\sigma+4}$, if $\sigma > 1$.

Next we estimate $\partial_t^2 u$. We have by the integral representation (2.2) with $f = \lambda |u|^{\sigma} u$

$$\begin{aligned} &\partial_t^2 u(t) = (\partial_t + 1)\,\partial_t^2 \mathcal{G}(t)\,u_0 + \partial_t^2 \mathcal{G}(t)\,u_1 \\ &+ \lambda \partial_t \mathcal{G}(t)\,|u_0|^{\sigma} u_0 + \lambda \int_0^t \partial_t \mathcal{G}(t-\tau)\,\partial_\tau |u(\tau)|^{\sigma} u(\tau)\,d\tau. \end{aligned}$$

By virtue of estimates of Lemma 2.1 we obtain

$$\|u_{tt}(t)\|_{\mathbf{L}^1} \le C\langle t\rangle^{-1} + C \int_0^t \langle t-\tau\rangle^{-1} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{\sigma} \|u_\tau(\tau)\|_{\mathbf{L}^2}\,d\tau.$$

In view of estimates (3.3), (3.4) and $\|u_x\|_{2,2} \le C$, using Lemma 2.1 and the Young inequality we obtain

$$\left\|\|u_{tt}(t)\|_{\mathbf{L}^1}\right\|_{\mathbf{L}^s_t(0,\infty)} \le C + C\left\|\left\|\int_0^t \partial_t \mathcal{G}(t-\tau)|u(\tau)|^\sigma u_\tau(\tau)\,d\tau\right\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^s_t(0,\infty)}$$
$$\le C\left\|\int_0^t \langle t-\tau\rangle^{-1}\|u(\tau)\|^\sigma_{\mathbf{L}^{2\sigma}}\|u_\tau(\tau)\|_{\mathbf{L}^2}\,d\tau\right\|_{\mathbf{L}^s_t(0,\infty)}$$
$$\le C\left\|\langle t\rangle^{-1}\right\|_{\mathbf{L}^{s_1}_t(0,\infty)}\left\|\|u(t)\|^\sigma_{\mathbf{L}^{2\sigma}}\right\|_{\mathbf{L}^{s_2}_t(0,\infty)}\left\|\|u_t(t)\|_{\mathbf{L}^2}\right\|_{\mathbf{L}^2_t(0,\infty)}$$
$$\le C\|u\|_{\sigma s_2,2\sigma}\|u_t\|_{2,2},$$

where $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} - \frac{1}{2} < \frac{1}{s_2} + \frac{1}{2}$, since $s_1 > 1$, $s_2 = \frac{8}{\sigma+2}$ for $\sigma \ge 2$, and $s_2 = \frac{2\sigma}{2\sigma-2}$ for $1 < \sigma \le 2$. Thus we get the estimate

$$\|u_{tt}\|_{s,1} \le C \tag{3.7}$$

for $s > \frac{8}{\sigma+6}$ for $\sigma \ge 2$, and $s > \frac{2\sigma}{3\sigma-2}$ for $1 < \sigma \le 2$.

To prove the estimate of the lemma we write

$$\partial_t^2 u(t) = (\partial_t + 1)\partial_t^2 \mathcal{G}(t) u_0 + \partial_t^2 \mathcal{G}(t) u_1$$
$$+\lambda|u(t)|^\sigma u(t) + \lambda\int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau)|u(\tau)|^\sigma u(\tau)\,d\tau$$
$$+\lambda\int_{\frac{t}{2}}^t \left(\partial_t^2\mathcal{G}(t-\tau) + e^{-(t-\tau)}\right)|u(\tau)|^\sigma u(\tau)\,d\tau$$
$$-\lambda\int_{\frac{t}{2}}^t e^{-(t-\tau)}|u(\tau)|^\sigma u(\tau)\,d\tau.$$

Integration by parts in view of equation (1.1) yields

$$-\lambda\int_{\frac{t}{2}}^t e^{-(t-\tau)}|u(\tau)|^\sigma u(\tau)\,d\tau$$
$$= -\lambda|u(t)|^\sigma u(t) + \lambda(\sigma+1)|u(t)|^\sigma u_t(t)$$
$$+\lambda e^{-\frac{t}{2}}\left|u\left(\frac{t}{2}\right)\right|^\sigma\left(u\left(\frac{t}{2}\right) - (\sigma+1)u\left(\frac{t}{2}\right)\right)$$
$$-\lambda\int_{\frac{t}{2}}^t e^{-(t-\tau)}\partial_\tau^2\left(|u(\tau)|^\sigma u(\tau)\right)d\tau$$
$$= -\lambda|u(t)|^\sigma u(t) + \lambda(\sigma+1)|u(t)|^\sigma(u_{xx}(t) - u_{tt}(t))$$
$$+\lambda^2(\sigma+1)|u(t)|^{2\sigma}u(t)$$
$$+\lambda e^{-\frac{t}{2}}\left|u\left(\frac{t}{2}\right)\right|^\sigma\left(u\left(\frac{t}{2}\right) - (\sigma+1)u\left(\frac{t}{2}\right)\right)$$
$$-\lambda\int_{\frac{t}{2}}^t e^{-(t-\tau)}\partial_\tau^2\left(|u(\tau)|^\sigma u(\tau)\right)d\tau.$$

Therefore

$$\partial_t^2 u(t) = (\partial_t + 1)\,\partial_t^2 \mathcal{G}(t)\,u_0 + \partial_t^2 \mathcal{G}(t)\,u_1$$
$$+\lambda e^{-\frac{t}{2}} \left| u\left(\frac{t}{2}\right)\right|^{\sigma} \left(u\left(\frac{t}{2}\right) - (\sigma+1)\,u\left(\frac{t}{2}\right)\right)$$
$$+\lambda(\sigma+1)\,|u(t)|^{\sigma}\,(u_{xx}(t) - u_{tt}(t)) + \lambda^2(\sigma+1)\,|u(t)|^{2\sigma}\,u(t)$$
$$+\lambda \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau)\,|u(\tau)|^{\sigma}\,u(\tau)\,d\tau$$
$$-\lambda \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} \partial_\tau^2 \left(|u(\tau)|^{\sigma}\,u(\tau)\right) d\tau$$
$$+\lambda \int_{\frac{t}{2}}^{t} \left(\partial_t^2 \mathcal{G}(t-\tau) + e^{-(t-\tau)}\right) |u(\tau)|^{\sigma}\,u(\tau)\,d\tau.$$

Hence we have

$$\|u_{tt}(t)\|_{\mathbf{L}^1} \le \left\|(\partial_t+1)\,\partial_t^2\mathcal{G}(t)\,u_0 + \partial_t^2\mathcal{G}(t)\,u_1\right\|_{\mathbf{L}^1}$$
$$+Ce^{-\frac{t}{2}} \left\| \left| u\left(\frac{t}{2}\right)\right|^{\sigma} \left(u\left(\frac{t}{2}\right) - (\sigma+1)\,u\left(\frac{t}{2}\right)\right)\right\|_{\mathbf{L}^1}$$
$$+C\left\||u(t)|^{\sigma}\,u_{xx}(t)\right\|_{\mathbf{L}^1} + C\left\||u(t)|^{\sigma}\,u_{tt}(t)\right\|_{\mathbf{L}^1} + C\,\|u(t)\|_{\mathbf{L}^{1+2\sigma}}^{1+2\sigma}$$
$$+C\left\| \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} \partial_\tau^2\left(|u(\tau)|^{\sigma}\,u(\tau)\right) d\tau\right\|_{\mathbf{L}^1}$$
$$+C\left\| \int_0^{\frac{t}{2}} \partial_t^2\mathcal{G}(t-\tau)\,|u(\tau)|^{\sigma}\,u(\tau)\,d\tau\right\|_{\mathbf{L}^1}$$
$$+C\left\| \int_{\frac{t}{2}}^{t} \left(\partial_t^2\mathcal{G}(t-\tau) + e^{-(t-\tau)}\right)|u(\tau)|^{\sigma}\,u(\tau)\,d\tau\right\|_{\mathbf{L}^1} \tag{3.8}$$

By Lemma 2.1 we have

$$\left\|(\partial_t+1)\,\partial_t^2\mathcal{G}(t)\,u_0 + \partial_t^2\mathcal{G}(t)\,u_1\right\|_{\mathbf{L}^1}$$
$$\le Ce^{-\frac{t}{2}} \left\| \left| u\left(\frac{t}{2}\right)\right|^{\sigma} \left(u\left(\frac{t}{2}\right) - (\sigma+1)\,u\left(\frac{t}{2}\right)\right)\right\|_{\mathbf{L}^1} \le C\,\langle t\rangle^{-2}.$$

In view of estimates (3.5) and (3.6) we get

$$\left\| \left\||u(t)|^{\sigma}\,u_{xx}(t)\right\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,T)} \le \|u_{xx}\|_{s_1,2}\,\|u\|_{\sigma s_2,2\sigma}^{\sigma}$$

where $s_1 > \frac{8}{\sigma+4}$ and $s_2 = \frac{8}{\sigma+2}$ for $\sigma \ge 2$, and $s_2 = \frac{\sigma}{\sigma-1}$ for $1 < \sigma \le 2$; so that $\frac{1}{s_1} + \frac{1}{s_2} > 1$, when $\sigma > 2\left(\sqrt{3}-1\right)$. In the same manner by virtue of (3.5) and (3.7) we find

$$\left\| \left\||u(t)|^{\sigma}\,u_{tt}(t)\right\|_{\mathbf{L}^1_x}\right\|_{\mathbf{L}^1_t(0,\infty)} \le \|u_{tt}\|_{s,1}\,\|u\|_{4,\infty}^{\sigma}$$

where $s > \frac{8}{\sigma+6}$ for $\sigma \ge 2$, and $s > \frac{2\sigma}{3\sigma-2}$ for $1 < \sigma \le 2$; so that $\frac{1}{s} + \frac{\sigma}{4} > 1$, when $\sigma > \sqrt{5}-1$. By (3.5) we get

$$\left\| \|u(t)\|_{\mathbf{L}^{1+2\sigma}}^{1+2\sigma}\right\|_{\mathbf{L}^1_t(0,\infty)} \le C\,\|u\|_{1+2\sigma,1+2\sigma}^{1+2\sigma} \le C,$$

for $\sigma > 1$. Similarly we have

$$\begin{aligned}
&\left\| \left\| \int_{\frac{t}{2}}^{t} e^{-(t-\tau)} \partial_\tau^2 \left(|u(\tau)|^{\sigma} u(\tau) \right) d\tau \right\|_{\mathbf{L}_x^1} \right\|_{\mathbf{L}_t^1(0,\infty)} \\
\leq\ & C \left\| \left\| |u(t)|^{\sigma-1} u_t^2(t) \right\|_{\mathbf{L}_x^1} \right\|_{\mathbf{L}_t^1(0,\infty)} + C \left\| \left\| |u(t)|^{\sigma} u_{tt}(t) \right\|_{\mathbf{L}_x^1} \right\|_{\mathbf{L}_t^1(0,\infty)} \\
\leq\ & C \|u\|_{\infty,\infty}^{\sigma-1} \|u_t\|_{2,2}^{2} + C \|u_{tt}\|_{s,1} \|u\|_{4,\infty}^{\sigma}
\end{aligned}$$

for $\sigma > \sqrt{5} - 1$.

Using Lemma 2.1, we obtain

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau) |u(\tau)|^{\sigma} u(\tau) d\tau \right\|_{\mathbf{L}^1} \\
\leq\ & C \langle t \rangle^{-2} \int_0^{\frac{t}{2}} \left(\left\| |u(\tau)|^{\sigma} u(\tau) \right\|_{\mathbf{L}^1} + \left\| |u(\tau)|^{\sigma} u_x(\tau) \right\|_{\mathbf{L}^1} \right) d\tau.
\end{aligned}$$

In view of estimates (3.3)–(3.6) we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \left(\left\| |u(\tau)|^{\sigma} u(\tau) \right\|_{\mathbf{L}^1} + \left\| |u(\tau)|^{\sigma} u_x(\tau) \right\|_{\mathbf{L}^1} \right) d\tau \\
\leq\ & \int_0^{\frac{t}{2}} \left(\|u(\tau)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1} + \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{\sigma} \|u_x(\tau)\|_{\mathbf{L}^2} \right) d\tau \leq C t^{\frac{1}{\sigma}},
\end{aligned}$$

therefore

$$\begin{aligned}
&\left\| \int_0^{\frac{t}{2}} \partial_t^2 \mathcal{G}(t-\tau) |u(\tau)|^{\sigma} u(\tau) d\tau \right\|_{1,1} \\
\leq\ & \left\| \langle t \rangle^{-2} \int_0^{\frac{t}{2}} \left(\left\| |u(\tau)|^{\sigma} u(\tau) \right\|_{\mathbf{L}^1} + \left\| |u(\tau)|^{\sigma} u_x(\tau) \right\|_{\mathbf{L}^1} \right) d\tau \right\|_{\mathbf{L}_t^1(0,\infty)} \\
\leq\ & C \int_0^{\infty} \langle t \rangle^{-2} t^{\frac{1}{\sigma}} dt \leq C,
\end{aligned}$$

when $\sigma > 1$.

For the last summand in (3.8) by Lemma 2.1, we get

$$\begin{aligned}
&\left\| \int_{\frac{t}{2}}^{t} \left(\partial_t^2 \mathcal{G}(t-\tau) + e^{-(t-\tau)} \right) |u(\tau)|^{\sigma} u(\tau) d\tau \right\|_{\mathbf{L}^1} \\
\leq\ & C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-1-\gamma} \left(\left\| |u(\tau)|^{\sigma-1} u_x^2(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} + \left\| |u(\tau)|^{\sigma} u_{xx}(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} \right) \\
& \times \left(\left\| |u(\tau)|^{\sigma} u(\tau) \right\|_{\mathbf{L}^1}^{\gamma} + \left\| |u(\tau)|^{\sigma} u_x(\tau) \right\|_{\mathbf{L}^1}^{\gamma} \right) d\tau \\
\leq\ & C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-1-\gamma} \left(\left\| |u(\tau)|^{\sigma-1} u_x^2(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} + \left\| |u(\tau)|^{\sigma} u_{xx}(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} \right) d\tau
\end{aligned}$$

with some small $\gamma > 0$. When $\sigma > 2\left(\sqrt{3}-1\right)$, and if $\gamma > 0$ is sufficiently small, by the Hölder inequality we obtain

$$\begin{aligned} &\left\| \|u(\tau)\|_{\mathbf{L}^\infty}^{(\sigma-1)(1-\gamma)} \|u_x(t)\|_{\mathbf{L}^2}^{2-2\gamma} \right\|_{\mathbf{L}_t^1(0,\infty)} \\ \leq\ & \|u\|_{(\sigma-1)(1-\gamma)s_1,\infty}^{(\sigma-1)(1-\gamma)} \|u_x\|_{(2-2\gamma)s_2,2}^{2-2\gamma} \leq C \end{aligned}$$

with $(\sigma-1)(1-\gamma)s_1 \geq 4$, $(2-2\gamma)s_2 \geq 2$; so that $\frac{1}{s_1}+\frac{1}{s_2}=1$ and

$$\begin{aligned} &\left\| \|u_{xx}(\tau)\|_{\mathbf{L}^2}^{1-\gamma} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{(1-\gamma)\sigma} \right\|_{\mathbf{L}_t^1(0,\infty)} \\ \leq\ & \|u_{xx}\|_{(1-\gamma)s_1,2}^{1-\gamma} \|u\|_{(1-\gamma)\sigma s_2,2\sigma}^{(1-\gamma)\sigma} \leq C \end{aligned}$$

with $(1-\gamma)s_1 > \frac{8}{\sigma+4}$, $(1-\gamma)s_2 = \frac{8\sigma}{\sigma+2}$ for $\sigma \geq 2$, and $(1-\gamma)s_2 = \frac{\sigma}{\sigma-1}$ for $1 < \sigma \leq 2$; so that $\frac{1}{s_1}+\frac{1}{s_2}=1$. Therefore

$$\begin{aligned} &\left\| \int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{-1-\gamma} \left(\left\| |u(\tau)|^{\sigma-1} u_x^2(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} + \left\| |u(\tau)|^{\sigma} u_{xx}(\tau) \right\|_{\mathbf{L}^1}^{1-\gamma} \right) d\tau \right\|_{\mathbf{L}_t^1(0,\infty)} \\ \leq\ & C \left\| \|u(\tau)\|_{\mathbf{L}^\infty}^{(\sigma-1)(1-\gamma)} \|u_x(t)\|_{\mathbf{L}^2}^{2-2\gamma} \right\|_{\mathbf{L}_t^1(0,\infty)} \\ & + C \left\| \|u_{xx}(\tau)\|_{\mathbf{L}^2}^{1-\gamma} \|u(\tau)\|_{\mathbf{L}^{2\sigma}}^{(1-\gamma)\sigma} \right\|_{\mathbf{L}_t^1(0,\infty)} \leq C\ . \end{aligned}$$

Thus from (3.8) we have the result of the lemma. Lemma 3.1 is proved.

Now we estimate the decay rate of the $\mathbf{L}^p$ - norms of the solutions.

Lemma 3.2. *Let $\lambda < 0$. Suppose that $u_0 \in \mathbf{W}_1^2 \cap \mathbf{H}^2 \cap \mathbf{L}^{1,a}$, $u_1 \in \mathbf{W}_1^1 \cap \mathbf{H}^1 \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. Let the global solution constructed in Proposition 3.1 satisfy*

$$\|u_{tt}\|_{1,1} \leq C.$$

Then the estimates

$$\|u(t)\|_{\mathbf{L}^1} \leq C,\ \ \|u(t)\|_{\mathbf{L}^2} \leq C\langle t\rangle^{-\frac{1}{4}},$$

and

$$\|u_x(t)\|_{\mathbf{L}^2} \leq C\langle t\rangle^{-\frac{3}{4}}$$

are valid for all $t > 0$.

Proof. We estimate the $\mathbf{L}^1$ - norm. Denote $S(x) = 1$ for all $x > 0$ and $S(x) = -1$ for all $x < 0$; $S(0) = 0$. We multiply equation (1.1) by $S(u(t,x))$ and integrate with respect to x over $\mathbf{R}$ to get

$$\begin{aligned} &\int_{\mathbf{R}} S(u(t,x))\, u_t(t,x)\, dx = \int_{\mathbf{R}} S(u(t,x))\, u_{xx} dx \\ &+ \int_{\mathbf{R}} S(u(t,x)) \left(\lambda |u(t,x)|^{\sigma} u(t,x) - u_{tt}(t,x)\right) dx. \end{aligned}$$

We have

$$\int_{\mathbf{R}} S(u(t,x))\, u_t(t,x)\, dx = \frac{d}{dt}\|u(t)\|_{\mathbf{L}^1},$$

$$\int_{\mathbf{R}} S(u(t,x))\, u_{xx}(t,x)\, dx \leq 0$$

and

$$\int_{\mathbf{R}} S(u(t,x))\,\lambda\,|u(t,x)|^{\sigma}\,u(t,x)\,dx = \lambda\,\|u(t)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1} \le 0.$$

Therefore we find

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^1} \le \|u_{tt}(t)\|_{\mathbf{L}^1}. \tag{3.9}$$

Integration of inequality (3.9) in view of estimate (3.11) yields the first estimate of the lemma.

In particular, we find

$$\sup_{\xi\in\mathbf{R}} |\widehat{u}(t,\xi)| \le (2\pi)^{-\frac{1}{2}}\|u(t)\|_{\mathbf{L}^1} \le C. \tag{3.10}$$

We now multiply equation (1.1) by $2(2u_t+u)$. Then integrating the result with respect to $x\in\mathbf{R}$ we get

$$\begin{aligned} & 2\int_{\mathbf{R}}\left((u_t+u)(u_{tt}+u_t)+u_t u_{tt}\right)dx \\ = & -2\int_{\mathbf{R}}(u_t)^2\,dx + 2\int_{\mathbf{R}}(2u_t+u)\,u_{xx}dx - 2|\lambda|\int_{\mathbf{R}}(2u_t+u)\,|u|^{\sigma}\,udx, \end{aligned}$$

therefore

$$\begin{aligned} & \frac{d}{dt}\left(\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + 2\|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{\sigma+2}\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}\right) \\ = & -2\|u_t\|_{\mathbf{L}^2}^2 - 2\|u_x\|_{\mathbf{L}^2}^2 - 2|\lambda|\,\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}. \end{aligned} \tag{3.11}$$

By the Plansherel theorem using the Fourier splitting method due to [26], we have

$$\begin{aligned} \|u_x\|_{\mathbf{L}^2}^2 &= \int_{|\xi|\le\chi}|\xi\widehat{u}(t,\xi)|^2\,d\xi + \int_{|\xi|\ge\chi}|\xi\widehat{u}(t,\xi)|^2\,d\xi \\ &\ge \chi^2\|u\|_{\mathbf{L}^2}^2 - C\chi^3, \end{aligned}$$

where $\chi>0$. Thus from (3.11) we have the inequality

$$\begin{aligned} & \frac{d}{dt}\left(\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + 2\|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{2+\sigma}\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}\right) \\ \le & -\chi^2\|u\|_{\mathbf{L}^2}^2 - \|u_x\|_{\mathbf{L}^2}^2 - 2\|u_t\|_{\mathbf{L}^2}^2 - 2|\lambda|\,\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} + C\chi^3 \\ \le & -\chi^2\left(\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + \|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{2+\sigma}\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}\right) + C\chi^3. \end{aligned} \tag{3.12}$$

We choose $\chi^2 = 2(t_0+t)^{-1}$, $t_0\ge 2$ and change

$$\|u_t+u\|_{\mathbf{L}^2}^2 + \|u_t\|_{\mathbf{L}^2}^2 + \|u_x\|_{\mathbf{L}^2}^2 + \frac{4|\lambda|}{2+\sigma}\|u\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} = (t_0+t)^{-2}\,W(t).$$

Then we get from (3.12)

$$\frac{d}{dt}W(t) \le C(t_0+t)^{\frac{1}{2}}. \tag{3.13}$$

Integration of (3.13) with respect to time yields

$$W(t) \le C(t_0+t)^{\frac{3}{2}}.$$

Therefore we obtain the second estimate of the lemma.

We now differentiate equation (1.1) with respect to x and multiply the result by $2(2u_{xt}+u_x)$. Then integrating with respect to $x\in\mathbf{R}$ we get

$$\frac{d}{dt}\int_{\mathbf{R}}\left((u_{xt}+u_x)^2+(u_{xt})^2\right)dx=2\int_{\mathbf{R}}(2u_{xt}+u_x)\,u_{xxx}dx$$
$$-2\int_{\mathbf{R}}(u_{xt})^2\,dx-2|\lambda|(\sigma+1)\int_{\mathbf{R}}(2u_{xt}+u_x)\,|u|^{\sigma}u_xdx,$$

hence

$$\begin{aligned}&\frac{d}{dt}\left(\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\right)\\=\;&-2\|u_{xx}\|_{\mathbf{L}^2}^2-2\|u_{xt}\|_{\mathbf{L}^2}^2-2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\\&+2|\lambda|\sigma(\sigma+1)\int_{\mathbf{R}}u_x^2|u|^{\sigma-2}uu_tdx.\end{aligned}\tag{3.14}$$

Then using equation (1.1) we get

$$\begin{aligned}&\int_{\mathbf{R}}u_x^2|u|^{\sigma-2}uu_tdx=\int_{\mathbf{R}}u_x^2|u|^{\sigma-2}uu_{xx}dx+\int_{\mathbf{R}}u_x^2|u|^{\sigma-2}u\left(\lambda|u|^{\sigma}u-u_{tt}\right)dx\\&=-\frac{\sigma-1}{3}\left\||u|^{\sigma-2}u_x^4\right\|_{\mathbf{L}^1}+\lambda\int_{\mathbf{R}}u_x^2|u|^{2\sigma}dx-\int_{\mathbf{R}}u_x^2|u|^{\sigma-2}uu_{tt}dx\\&\le\|u_x\|_{\mathbf{L}^2}^2\|u\|_{\mathbf{L}^\infty}^{\sigma-1}\|u_{tt}\|_{\mathbf{L}^1}\le\frac{\mu(t)}{2|\lambda|\sigma(\sigma+1)}\|u_x\|_{\mathbf{L}^2}^2,\end{aligned}$$

where we denote

$$\mu(t)=2|\lambda|\sigma(\sigma+1)\|u\|_{\mathbf{L}^\infty}^{\sigma-1}\|u_{tt}\|_{\mathbf{L}^1}.$$

By Lemma 3.1 we see that $\int_0^\infty\mu(t)\,dt\le C$. As above by the Plansherel theorem, we have

$$\begin{aligned}\|u_{xx}\|_{\mathbf{L}^2}^2&=\int_{|\xi|\le\chi}\left|\xi^2\widehat{u}(t,\xi)\right|^2d\xi+\int_{|\xi|\ge\chi}\left|\xi^2\widehat{u}(t,\xi)\right|^2d\xi\\&\ge\chi^2\|u_x(t)\|_{\mathbf{L}^2}^2-C\chi^5,\end{aligned}$$

where $\chi>0$. Thus from (3.14) we have the inequality

$$\begin{aligned}&\frac{d}{dt}\left(\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\right)\\\le\;&\left(\mu-\chi^2\right)\|u_x\|_{\mathbf{L}^2}^2-2\|u_{xt}\|_{\mathbf{L}^2}^2-\|u_{xx}\|_{\mathbf{L}^2}^2\\&-2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}+C\chi^5\\\le\;&\left(\mu-\chi^2\right)\left(\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\right)\\&+C\chi^5.\end{aligned}\tag{3.15}$$

We choose $\chi^2=3(t_0+t)^{-1}$, $t_0\ge6$ and change

$$\begin{aligned}&\|u_{xt}+u_x\|_{\mathbf{L}^2}^2+\|u_{xt}\|_{\mathbf{L}^2}^2+2\|u_{xx}\|_{\mathbf{L}^2}^2+2|\lambda|(\sigma+1)\left\||u|^{\sigma}u_x^2\right\|_{\mathbf{L}^1}\\=\;&(t_0+t)^{-3}e^{\int_0^t\mu(\tau)d\tau}W_1(t).\end{aligned}$$

Then we get from (3.15)

$$\frac{d}{dt}W_1(t)\le C(t_0+t)^{\frac12}.\tag{3.16}$$

Integration of (3.16) with respect to time yields

$$W_1(t) \le C(t_0+t)^{\frac{3}{2}}.$$

Therefore we obtain last estimate of the lemma. Furthermore we have the optimal time decay estimate

$$\|u(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{3.17}$$

for all $t>0$, $1\le p\le\infty$ since

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\|u(t)\|_{\mathbf{L}^2}^{\frac{1}{2}}\|u_x(t)\|_{\mathbf{L}^2}^{\frac{1}{2}} \le C\langle t\rangle^{-\frac{1}{2}}$$

and

$$\|u(t)\|_{\mathbf{L}^1} \le C.$$

Lemma 3.2 is proved.

Proof of Theorem 1.1. In view of (3.1) and (3.2) we can apply Lemma 3.1 to obtain $\|u_{tt}\|_{1,1} \le C$. Then from Lemma 3.2 we have the estimate (3.17) for all $t>0$. Thus we have the desired result by Proposition 2.2. Theorem 1.1 is proved.

4. Proof of Theorem 1.2

In order to get the sharp asymptotics of solutions in the subcritical case we need to compare the solutions of the following two problems

$$\begin{cases} u_t - u_{xx} - \lambda|u|^\sigma u = f,\ x\in\mathbf{R},\ t>0, \\ u(0,x) = u_0(x),\ x\in\mathbf{R}, \end{cases} \tag{4.1}$$

and

$$\begin{cases} v_t - v_{xx} - \epsilon\lambda v^{1+\sigma} = |f|,\ x\in\mathbf{R},\ t>0, \\ v(0,x) = \mu|u_0(x)|,\ x\in\mathbf{R}, \end{cases} \tag{4.2}$$

where $0<\sigma<2$, $\lambda<0$.

Lemma 4.1. *Let u and v be classical solutions of (4.1) and (4.2) such that*

$$u,v\in\mathbf{C}\left((0,\infty);\mathbf{C}^2\right)\cap\mathbf{C}^1((0,\infty);\mathbf{C})$$

and $0\le\epsilon\le1$, $\mu\ge1$. Then $|u(t,x)|\le v(t,x)$ for all $t\ge0$, $x\in\mathbf{R}$.

Proof. Define $r=v-u$. Then from (4.1) and (4.2) we obtain

$$\begin{cases} r_t - r_{xx} = |\lambda|\left(|u|^\sigma u - \epsilon v^{\sigma+1}\right) + |f| - f,\ x\in\mathbf{R},\ t>0, \\ r(0,x) = \mu|u_0(x)| - u_0(x),\ x\in\mathbf{R}. \end{cases} \tag{4.3}$$

We need to prove that $r\ge0$ for all $t\ge0$, $x\in\mathbf{R}$. Define $R(t)\equiv\inf_{x\in\mathbf{R}} r(t,x)$. By the contrary, suppose that there exists a time $T>0$ such that $R(T)<0$. By the continuity we can find an interval $[T_1,T]$ such that $R(t)\le0$ for all $t\in[T_1,T]$ and $R(T_1)=0$. By Theorem 2.1 from paper [1] there exists a point $\zeta(t)\in\mathbf{R}$ such that $R(t)=v(t,\zeta(t))-u(t,\zeta(t))=r(t,\zeta(t))$, moreover $R'(t)=\frac{d}{dt}r(t,\zeta(t))=r_t(t,\zeta(t))$ almost everywhere on $t\in[T_1,T]$. We have

$$\begin{aligned} &|u|^\sigma u(t,\zeta(t)) - \epsilon v^{\sigma+1}(t,\zeta(t)) \\ =\ &(v(t,\zeta(t)) - R(t))^{\sigma+1} - \epsilon v^{\sigma+1}(t,\zeta(t)) \ge 0 \end{aligned}$$

for all $t\in[T_1,T]$. At the point of minimum $\zeta(t)$ we have

$$r_{xx}(t,\zeta(t)) \ge 0.$$

Therefore by equation (4.3) we get

$$R'(t) \geq 0$$

almost everywhere on $t \in [T_1, T]$. Integration with respect to time yields $R(t) \geq 0$. This gives a contradiction, hence $u(t,x) \leq |v(t,x)|$ for all $x \in \mathbf{R}$ and $t > T_1$. In the same manner we prove that $v + u \geq 0$ for all $x \in \mathbf{R}$ and $t > T_1$. Lemma 4.1 is proved.

Let u be the global solution constructed in Proposition 3.1 with an additional assumption on the data such that $u_0 \in \mathbf{C}^3$. We can apply Lemma 3.1 and Lemma 3.2 to get a rough time decay estimate (the optimal estimate will be obtained below)

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{1}{2}} \tag{4.4}$$

for all $t > 0$. Now we estimate the $\mathbf{L}^{1,a}$ - norm of the solution. By Lemma 2.1 we have

$$\begin{aligned} &\|u(t)\|_{\mathbf{L}^{1,a}} \leq \|(\partial_t + 1)\mathcal{G}(t)u_0\|_{\mathbf{L}^{1,a}} + \|\mathcal{G}(t)u_1\|_{\mathbf{L}^{1,a}} \\ &+C\int_0^t \left(\left\|u^3(\tau)\right\|_{\mathbf{L}^{1,a}} + \langle t-\tau\rangle^{\frac{a}{2}} \left\|u^3(\tau)\right\|_{\mathbf{L}^1}\right) d\tau \\ \leq\ & C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \langle\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{1,a}} d\tau + C\int_0^t \langle t-\tau\rangle^{\frac{a}{2}}\langle\tau\rangle^{-1} d\tau \\ \leq\ & C\langle t\rangle^{\frac{a}{2}} + C\int_0^t \langle\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^{1,a}} d\tau. \end{aligned}$$

Hence by the Gronwall lemma we obtain

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{2}} \tag{4.5}$$

for all $t > 0$. Next we estimate $\partial_t u$. We have by the integral representation (2.2) with $f = \lambda u^3$

$$\begin{aligned} u_t(t) &= (\partial_t + 1)\partial_t\mathcal{G}(t)u_0 + \partial_t\mathcal{G}(t)u_1 \\ &+\lambda\int_0^t \partial_t\mathcal{G}(t-\tau)u^3(\tau)d\tau, \end{aligned} \tag{4.6}$$

hence

$$\|u_t(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{3}{2}} + C\int_0^t \langle t-\tau\rangle^{-1}\langle\tau\rangle^{-\frac{3}{2}} d\tau \leq C\langle t\rangle^{-1}.$$

Differentiating (4.6) with respect to $t > 0$ we get

$$\begin{aligned} &u_{tt}(t) = (\partial_t + 1)\partial_t^2\mathcal{G}(t)u_0 + \partial_t^2\mathcal{G}(t)u_1 \\ &+2\lambda\partial_t\mathcal{G}\left(\frac{t}{2}\right)u^3\left(\frac{t}{2}\right) + 3\lambda\int_{\frac{t}{2}}^t \partial_t\mathcal{G}(t-\tau)u^2(\tau)u_\tau(\tau)d\tau \\ &+\lambda\int_0^{\frac{t}{2}} \partial_t^2\mathcal{G}(t-\tau)u^3(\tau)d\tau. \end{aligned}$$

Hence

$$\|u_{tt}(t)\|_{\mathbf{L}^\infty} \le \left\|(\partial_t+1)\,\partial_t^2\mathcal{G}(t)\,u_0+\partial_t^2\mathcal{G}(t)\,u_1\right\|_{\mathbf{L}^\infty}$$
$$+C\left\|\partial_t\mathcal{G}\left(\frac{t}{2}\right)u^3\left(\frac{t}{2}\right)\right\|_{\mathbf{L}^\infty}+C\left\|\int_{\frac{t}{2}}^{t}\partial_t\mathcal{G}(t-\tau)\,u^2(\tau)\,u_\tau(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}$$
$$+C\left\|\int_0^{\frac{t}{2}}\partial_t^2\mathcal{G}(t-\tau)\,u^3(\tau)\,d\tau\right\|_{\mathbf{L}^\infty}.$$

By virtue of estimates of Lemma 2.1 we obtain

$$\begin{aligned}\|u_{tt}(t)\|_{\mathbf{L}^\infty} &\le C\langle t\rangle^{-\frac{5}{2}}+C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^\infty}^2\|u_\tau(\tau)\|_{\mathbf{L}^\infty}\,d\tau\\ &\quad+C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2}\|u(\tau)\|_{\mathbf{L}^\infty}^3\,d\tau\\ &\le C\langle t\rangle^{-\frac{5}{2}}+C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}\langle\tau\rangle^{-2}\,d\tau+C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2}\langle\tau\rangle^{-\frac{3}{2}}\,d\tau\\ &\le C\langle t\rangle^{-2}\log(2+t)\end{aligned}\tag{4.7}$$

for all $t>0$. In the same manner we estimate the $\mathbf{L}^{1,a}$ - norm

$$\|u_{tt}(t)\|_{\mathbf{L}^{1,a}}\le C\langle t\rangle^{\frac{a}{2}-2}+C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}\|u(\tau)\|_{\mathbf{L}^\infty}\|u_\tau(\tau)\|_{\mathbf{L}^\infty}\|u(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau$$
$$+C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{\frac{a}{2}-1}\|u(\tau)\|_{\mathbf{L}^\infty}\|u_\tau(\tau)\|_{\mathbf{L}^\infty}\|u(\tau)\|_{\mathbf{L}^1}\,d\tau$$
$$+C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2}\|u(\tau)\|_{\mathbf{L}^\infty}^2\|u(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau$$
$$+C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{\frac{a}{2}-2}\|u(\tau)\|_{\mathbf{L}^\infty}^2\|u(\tau)\|_{\mathbf{L}^1}\,d\tau$$
$$\le C\langle t\rangle^{-2+\frac{a}{2}}+C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-1}\langle\tau\rangle^{\frac{a}{2}-\frac{3}{2}}\,d\tau+C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{\frac{a}{2}-1}\langle\tau\rangle^{-\frac{3}{2}}\,d\tau$$
$$+C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-2}\langle\tau\rangle^{\frac{a}{2}-1}\,d\tau+C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{\frac{a}{2}-2}\langle\tau\rangle^{-1}\,d\tau$$
$$\le C\langle t\rangle^{\frac{a}{2}-2}\log(2+t)\tag{4.8}$$

for all $t>0$, since $\sigma<2$ is sufficiently close to 2. Denote $f(t,x)=u_{tt}$ and consider two auxiliary Cauchy problems

$$\begin{cases}U_t-U_{xx}+|\lambda|\,U^{1+\sigma}=\varepsilon^2|f|\,,\ x\in\mathbf{R},\ t>0,\\ U(0,x)=\varepsilon\,|u_0(x)|\,,\ x\in\mathbf{R},\end{cases}\tag{4.9}$$

and

$$\begin{cases}V_t-V_{xx}+\varepsilon^{2\sigma}|\lambda|\,V^{1+\sigma}=|f|\,,\ x\in\mathbf{R},\ t>0,\\ V(0,x)=\frac{1}{\varepsilon}\,|u_0(x)|\,,\ x\in\mathbf{R}.\end{cases}\tag{4.10}$$

with sufficiently small $\varepsilon>0$. We have $f(t)\in\mathbf{C}^1$ and so by the smoothing property of the heat kernel we find that $U\in\mathbf{C}\left((0,\infty);\mathbf{C}^2\right)\cap\mathbf{C}^1\left((0,\infty);\mathbf{C}\right)$. Note that problem (4.10) can be reduced to problem (4.9) by virtue of the change $V=\varepsilon^{-2}U$.

And problem (4.9) has a sufficiently small initial data $\varepsilon\left|u_0(x)\right|$ and a small force $\varepsilon^2\left|f\right|$. Moreover, the mean value $\theta=\varepsilon\int_{\mathbf{R}}\left|u_0(x)\right|dx=O\left(\varepsilon\right).$ So that the term $\frac{1}{\theta}\varepsilon^2\left|f\right|=O\left(\varepsilon\right)$ is also small. Therefore we can apply the results of papers [11], [9] to calculate the large time asymptotic behavior of solutions $U\left(t,x\right)$ and $V\left(t,x\right)$ to problems (4.9). By Lemma 4.1 $\left|u\left(t,x\right)\right|\leq C\varepsilon^{-2}\left|U\left(t,x\right)\right|$ and we get an optimal time decay estimate for the solution

$$\left\|u\left(t\right)\right\|_{\mathbf{L}^\infty}\leq C\varepsilon^{-1}\left\langle t\right\rangle^{-\frac{1}{2}}\left(1+C\varepsilon\left(2-\sigma\right)^{-\frac{1}{\sigma}}t^{\frac{1}{\sigma}-\frac{1}{2}}\right)^{-1} \tag{4.11}$$

for all $t>0$. Here u is the global solution constructed in Proposition 3.1 with an additional assumption on the data such that $u_0\in\mathbf{C}^3$.

Now we prove the asymptotics of solutions. Taking $u=w_1$ and $\left(1+\partial_x\right)^{-1}u_t=w_2$, with $\left(1+\partial_x\right)^{-1}=\overline{\mathcal{F}}_{\xi\to x}\left(1+i\xi\right)^{-1}\mathcal{F}_{x\to\xi}=e^{-x}\int_{-\infty}^{x}dx'e^{x'}$, we rewrite equation (1.1) in the form of a system of nonlinear evolutionary equations

$$w_t+\mathcal{N}\left(w\right)+\mathcal{L}w=0 \tag{4.12}$$

for the vector $w\left(t,x\right)=\begin{pmatrix} w_1\left(t,x\right)\\ w_2\left(t,x\right)\end{pmatrix}$. The initial data are

$$w\left(0,x\right)=\widetilde{w}\left(x\right)\equiv\begin{pmatrix} u_0\left(x\right)\\ \left(1+\partial_x\right)^{-1}u_1\left(x\right)\end{pmatrix}.$$

The linear part of system (4.12) is a pseudodifferential operator defined by the Fourier transformation as follows

$$\mathcal{L}w=\overline{\mathcal{F}}_{\xi\to x}L\left(\xi\right)\mathcal{F}_{x\to\xi}w,$$

with a matrix - symbol

$$L\left(\xi\right)=\left\{L_{jk}\left(\xi\right)\right\}\big|_{j,k=1,2}=\begin{pmatrix} 0 & -\left(1+i\xi\right)\\ \frac{\xi^2}{1+i\xi} & 1\end{pmatrix}.$$

The nonlinearity is defined by

$$\mathcal{N}\left(w\right)=\begin{pmatrix} 0\\ -\lambda\left(1+\partial_x\right)^{-1}\left(\left|w_1\right|^{\sigma}w_1\right)\end{pmatrix}.$$

Denote by

$$\Lambda_1\left(\xi\right)=\frac{1}{2}+\frac{1}{2}\sqrt{1-4\xi^2},\Lambda_2\left(\xi\right)=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\xi^2}$$

the eigenvalues of the matrix $L\left(\xi\right)$. Note that the matrix

$$Q\left(\xi\right)=\begin{pmatrix} Q_{11}\left(\xi\right) & Q_{12}\left(\xi\right)\\ Q_{21}\left(\xi\right) & Q_{22}\left(\xi\right)\end{pmatrix}=\begin{pmatrix} 1+i\xi & 1+i\xi\\ -\Lambda_1\left(\xi\right) & -\Lambda_2\left(\xi\right)\end{pmatrix}$$

and

$$Q^{-1}\left(\xi\right)=\frac{1}{\left(1+i\xi\right)\left(\Lambda_1\left(\xi\right)-\Lambda_2\left(\xi\right)\right)}\begin{pmatrix} -\Lambda_2\left(\xi\right) & -\left(1+i\xi\right)\\ \Lambda_1\left(\xi\right) & 1+i\xi\end{pmatrix}$$

diagonalize the matrix $L\left(\xi\right)$, i.e.

$$Q^{-1}\left(\xi\right)L\left(\xi\right)Q\left(\xi\right)=\begin{pmatrix} \Lambda_1\left(\xi\right) & 0\\ 0 & \Lambda_2\left(\xi\right)\end{pmatrix}.$$

Consider the system of ordinary differential equations with constant coefficients depending on the parameter $\xi\in\mathbf{R}$

$$\frac{d}{dt}\widehat{w}\left(t,\xi\right)+L\left(\xi\right)\widehat{w}\left(t,\xi\right)=0. \tag{4.13}$$

Multiplying system (4.13) by $Q^{-1}(\xi)$ from the left and changing $\widehat{w}(t,\xi) = Q(\xi) W(t,\xi)$ we diagonalize system (4.13)

$$\frac{d}{dt}\begin{pmatrix} W_1(t,\xi) \\ W_2(t,\xi) \end{pmatrix} = -\begin{pmatrix} \Lambda_1(\xi) & 0 \\ 0 & \Lambda_2(\xi) \end{pmatrix}\begin{pmatrix} W_1(t,\xi) \\ W_2(t,\xi) \end{pmatrix},$$

whence integrating with respect to time $t \geq 0$ we find

$$\begin{pmatrix} W_1(t,\xi) \\ W_2(t,\xi) \end{pmatrix} = \begin{pmatrix} e^{-t\Lambda_1(\xi)} & 0 \\ 0 & e^{-t\Lambda_2(\xi)} \end{pmatrix}\begin{pmatrix} W_1(0,\xi) \\ W_2(0,\xi) \end{pmatrix}$$

Returning to the solution $\widehat{w}(t,\xi)$ we get

$$\widehat{w}(t,\xi) = e^{-tL(\xi)}\begin{pmatrix} \widehat{u}_0(\xi) \\ \widehat{u}_1(\xi) \end{pmatrix},$$

where the fundamental Cauchy matrix has the form

$$\begin{aligned} e^{-tL(\xi)} &= Q(\xi)\begin{pmatrix} e^{-t\Lambda_1(\xi)} & 0 \\ 0 & e^{-t\Lambda_2(\xi)} \end{pmatrix} Q^{-1}(\xi) \\ &= \frac{1}{\sqrt{1-4\xi^2}}\begin{pmatrix} -\Lambda_2(\xi) & -(1+i\xi) \\ \frac{\xi^2}{1+i\xi} & \Lambda_1(\xi) \end{pmatrix} e^{-t\Lambda_1(\xi)} \\ &\quad + \frac{1}{\sqrt{1-4\xi^2}}\begin{pmatrix} \Lambda_1(\xi) & 1+i\xi \\ -\frac{\xi^2}{1+i\xi} & -\Lambda_2(\xi) \end{pmatrix} e^{-t\Lambda_2(\xi)}. \end{aligned}$$

Following the idea of paper [10], we make a change of the dependent variable $w(t,x) = e^{-\varphi(t)} v(t,x)$, then we get from (4.12)

$$v_t + \mathcal{L}v + e^{-\sigma\varphi}\mathcal{N}(v) - \varphi' v = 0. \tag{4.14}$$

We define the real-valued function $\varphi(t)$ by the following zero total mass condition

$$A_0 \int_{\mathbf{R}} \left(e^{-\sigma\varphi}\mathcal{N}(v) - \varphi' v\right) dx = 0, \tag{4.15}$$

where the matrix

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Condition (4.15) is reduced to one equation

$$\begin{aligned} & -e^{-\sigma\varphi(t)}\lambda \int_{\mathbf{R}} (1+\partial_x)^{-1} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \varphi'(t) \int_{\mathbf{R}} (A_0 v(t,x))_1\, dx \\ = & -e^{-\sigma\varphi(t)}\lambda \int_{\mathbf{R}} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \varphi'(t) \int_{\mathbf{R}} (A_0 v(t,x))_1\, dx = 0. \end{aligned}$$

We also assume that $\varphi(0) = 0$. Since $A_0 L(0) = 0$ via equation (4.14) we get

$$\frac{d}{dt}\int_{\mathbf{R}} (A_0 v(t,x))_1\, dx = 0$$

that is

$$\begin{aligned} & \int_{\mathbf{R}} (A_0 v(t,x))_1\, dx = \int_{\mathbf{R}} (A_0 \widetilde{w}(x))_1\, dx \\ = & \int_{\mathbf{R}} \left(u_0(x) + (1+\partial_x)^{-1} u_1(x)\right) dx = \int_{\mathbf{R}} (u_0(x) + u_1(x))\, dx \equiv \theta \end{aligned}$$

for all $t > 0$. Therefore we obtain the equation

$$\begin{aligned}\varphi'(t) &= e^{-\sigma\varphi(t)}\frac{|\lambda|}{\theta}\int_{\mathbf{R}}|v_1(t,x)|^{\sigma} v_1(t,x)\,dx \\ &= \frac{|\lambda|}{\theta}\int_{\mathbf{R}}|u(t,x)|^{\sigma} v_1(t,x)\,dx. \end{aligned} \tag{4.16}$$

Thus from (4.14) we see that the vector v satisfy the following integral equation

$$\begin{aligned} v &= \mathcal{J}(t)\,\widetilde{v} - \int_0^t \mathcal{J}(t-\tau)\left(e^{-\sigma\varphi}\mathcal{N}(v) - \varphi' v\right)d\tau \\ &= \mathcal{J}(t)\,\widetilde{w} - |\lambda|\int_0^t d\tau\, \mathcal{J}(t-\tau)\left(\begin{pmatrix}0\\1\end{pmatrix}(1+\partial_x)^{-1}|u(\tau,x)|^{\sigma} v_1(\tau,x)\right. \\ &\left. -v(\tau)\frac{1}{\theta}\int_{\mathbf{R}}|u(\tau,x)|^{\sigma} v_1(\tau,x)\,dx\right)d\tau. \end{aligned} \tag{4.17}$$

We now collect some preliminary estimates for the Green operator

$$\mathcal{J}(t)\,\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL(\xi)}\hat{\psi}(\xi)\right),$$

in the weighted Lebesgue norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, where $\|\phi\|_{\mathbf{L}^{1,a}} = \|\langle\cdot\rangle^a\phi\|_{\mathbf{L}^1}$, $a \in (0,1)$, $1 \le p \le \infty$. Also we show that $\mathcal{J}(t)$ asymptotically behaves as a Green function $\mathcal{J}_0(t)$ for the heat equation

$$\mathcal{J}_0(t)\,\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-t\xi^2}\hat{\psi}(\xi)\right) = \int_{\mathbf{R}} G_0(t,x-y)\,\psi(y)\,dy$$

with a kernel

$$G_0(t,x) = (2\pi)^{-\frac{1}{2}}\,\overline{\mathcal{F}}_{\xi\to x}\left(e^{-t\xi^2}\right) = (4\pi t)^{-\frac{1}{2}}\,e^{-\frac{x^2}{4t}}.$$

Lemma 4.2. *Suppose that the vector-function* $\phi \in \left(\mathbf{L}^\infty \cap \mathbf{L}^{1,a}\right)^2$, *where* $a \in (0,1)$. *Then the estimates*

$$\|\mathcal{J}(t)\,\phi\|_{\mathbf{L}^p} \le C\,\|\phi\|_{\mathbf{L}^p},$$

and

$$\left\||\cdot|^{\omega}\left(\mathcal{J}(t)\,\phi - (4\pi t)^{-\frac{1}{2}}\,e^{-\frac{x^2}{4t}}A_0\vartheta\right)\right\|_{\mathbf{L}^p} \le Ct^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{a-\omega}{2}}\,\|\phi\|_{\mathbf{L}^{1,a}}$$

are valid for all $t > 0$, $1 \le p \le \infty$, $0 \le \omega \le a$, *where a matrix*

$$A_0 = \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$$

and a vector $\vartheta = \int_{\mathbf{R}}\phi(x)\,dx$.

For the proof of Lemma 4.2, see paper [14]. Applying the identity

$$(1+\partial_x)^{-1}\psi(x) = e^{-x}\int_{-\infty}^{x} e^{y}\psi(y)\,dy$$

we see that

$$\left\|(1+\partial_x)^{-1}\psi\right\|_{\mathbf{L}^\infty} \le C\,\|\psi\|_{\mathbf{L}^\infty},\quad \left\|(1+\partial_x)^{-1}\psi\right\|_{\mathbf{L}^{1,a}} \le C\,\|\psi\|_{\mathbf{L}^{1,a}}.$$

Now let us prove the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le C\,\langle t\rangle^{\frac{a}{2}} \tag{4.18}$$

for all $t > 0$. By estimate (4.11) we have

$$\|u(t)\|_{\mathbf{L}^\infty}^{\sigma} \leq C\langle t\rangle^{-\frac{\sigma}{2}}\left(1 + C\varepsilon^{\sigma}(2-\sigma)^{-1}t^{1-\frac{1}{2}\sigma}\right)^{-1},$$

We use the integral equation (4.17). By virtue of estimate (4.11), Lemma 4.2 we get

$$\begin{aligned} &\left\|\int_0^t d\tau\mathcal{J}(t-\tau)\left(\begin{pmatrix}0\\1\end{pmatrix}(1+\partial_x)^{-1}|u(\tau,x)|^{\sigma}v_1(\tau,x)\right.\right.\\ &\left.\left.-v(\tau)\frac{1}{\theta}\int_{\mathbf{R}}|u(\tau,x)|^{\sigma}v_1(\tau,x)\,dx\right)d\tau\right\|_{\mathbf{L}^{1,a}}\\ \leq\;& C\varepsilon^{-\sigma}(2-\sigma)\int_0^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau \end{aligned}$$

for all $t > 0$. Therefore in view of (4.17) we obtain

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{2}} + C\varepsilon^{-\sigma}(2-\sigma)\int_0^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^{1,a}}\,d\tau$$

for all $t > 0$. Application of the Gronwall lemma yields the estimate (4.18) for all $t > 0$. In the same manner by virtue of estimates (4.11) and Lemma 4.2 we get

$$\begin{aligned} &\|v(t)\|_{\mathbf{L}^p} \leq \|\mathcal{J}(t)\widetilde{w}\|_{\mathbf{L}^p}\\ &+\left\|\int_0^t d\tau\mathcal{J}(t-\tau)\left(\begin{pmatrix}0\\1\end{pmatrix}(1+\partial_x)^{-1}|u(\tau,x)|^{\sigma}v_1(\tau,x)\right.\right.\\ &\left.\left.-v(\tau)\frac{1}{\theta}\int_{\mathbf{R}}|u(\tau,x)|^{\sigma}v_1(\tau,x)\,dx\right)d\tau\right\|_{\mathbf{L}^p}\\ \leq\;& C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} + C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{a}{2}}\langle\tau\rangle^{\frac{a}{2}-1}\,d\tau\\ &+C\varepsilon^{-\sigma}(2-\sigma)\int_{\frac{t}{2}}^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^p}\,d\tau\\ \leq\;& C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} + \epsilon\int_{\frac{t}{2}}^t\langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^p}\,d\tau \end{aligned}$$

for all $t > 0$, since $\sigma < 2$ is close to 2. Application of the Gronwall lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^p} \leq C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$. The above estimates now imply that

$$\|v_1(t) - (\mathcal{J}(t)\widetilde{w})_1\|_{\mathbf{L}^p} \leq C(2-\sigma)\langle t\rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

for all $t > 0$.

Multiplying equation (4.16) by the factor $e^{\sigma\varphi(t)}$, then integrating with respect to time $t > 0$ and making a change of the dependent variable $e^{\sigma\varphi(t)} = h(t)$, we get

$$h(t) = 1 + \frac{|\lambda|\sigma}{\theta}\int_0^t\int_{\mathbf{R}}|v_1(\tau,x)|^{\sigma}v_1(\tau,x)\,dxd\tau. \tag{4.19}$$

Denote $\zeta = \frac{\theta^{1+\sigma}}{(4\pi)^{\frac{\sigma}{2}}\left(1-\frac{\sigma}{2}\right)(1+\sigma)^{\frac{1}{2}}}$. Since $0 < \sigma < 2$ is close to 2, we may suppose that $\zeta \geq 1$.

Lemma 4.3. *We assume that* $\widetilde{w} \in \left(\mathbf{L}^{1,a} \cap \mathbf{L}^{\infty}\right)^2$ *with* $a \in (0,1)$ *and*

$$\theta \equiv \left(A_0 \int_{\mathbf{R}} \widetilde{w}(x)\, dx\right)_1 > 0.$$

Let the function $(v(t,x))_1 = v_1(t,x)$ *satisfy the estimates*

$$\|v_1(t)\|_{\mathbf{L}^{1+\sigma}} \le C \langle t \rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

and

$$\|v_1(t) - (\mathcal{J}(t)\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}} \le C(2-\sigma)\langle t \rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

for all $t > 0$. *Then the following inequality is valid*

$$\left| \int_0^t d\tau \int_{\mathbf{R}} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \zeta t^{1-\frac{\sigma}{2}} \right| \le C t^{1-\frac{\sigma}{2}} \tag{4.20}$$

for all $t > 0$.

Proof. We have

$$\|\phi\|_{\mathbf{L}^p} \le C \|\phi\|_{\mathbf{L}^{\infty}}^{1-\frac{1}{p(1+a)}} \|\langle \cdot \rangle^a \phi\|_{\mathbf{L}^1}^{\frac{1}{p(1+a)}} \tag{4.21}$$

for all $1 \le p \le \infty$. Then via Lemma 4.2 we get

$$\|(\mathcal{J}(t)\widetilde{w})_1 - \theta G_0(t,x)\|_{\mathbf{L}^{1+\sigma}} \le C\langle t \rangle^{-\frac{\sigma}{2(1+\sigma)} - \frac{a}{2}},$$

and via the assumption

$$\|v_1(t) - (\mathcal{J}(t)\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}} \le C(2-\sigma)\langle t \rangle^{-\frac{\sigma}{2(1+\sigma)}}$$

applying the Hölder inequality we have

$$\begin{aligned} &\left\| |v_1(t)|^{\sigma} v_1(t) - \theta^{1+\sigma} G_0^{1+\sigma} \right\|_{\mathbf{L}^1} \le \left\| |v_1(t)|^{\sigma} v_1(t) - (\mathcal{J}(t)\widetilde{w})_1^{1+\sigma} \right\|_{\mathbf{L}^1} \\ &+ \left\| (\mathcal{J}(t)\widetilde{w})_1^{1+\sigma} - \theta^{1+\sigma} G_0^{1+\sigma}(t,x) \right\|_{\mathbf{L}^1} \\ \le\ & C\left(\|v_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma} + \|(\mathcal{J}(t)\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma} \right) \|v_1(t) - (\mathcal{J}(t)\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}} \\ &+ C\left(\|(\mathcal{J}(t)\widetilde{w})_1\|_{\mathbf{L}^{1+\sigma}}^{\sigma} + \theta^{\sigma} \|G_0\|_{\mathbf{L}^{1+\sigma}}^{\sigma} \right) \|(\mathcal{J}(t)\widetilde{w})_1 - \theta G_0(t,x)\|_{\mathbf{L}^{1+\sigma}} \\ \le\ & C(2-\sigma)\langle t \rangle^{-\frac{\sigma}{2}} + C\langle t \rangle^{-\frac{\sigma}{2} - \frac{a}{2}} \end{aligned}$$

for all $t > 0$. By a direct computation we obtain for the heat kernel $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$

$$\int_{\mathbf{R}} (G_0(t,x))^{1+\sigma} dx = (1+\sigma)^{-\frac{1}{2}} (4\pi t)^{-\frac{\sigma}{2}}.$$

Therefore we get

$$\begin{aligned} &\left| \int_{\mathbf{R}} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \frac{\theta^{1+\sigma} t^{-\frac{\sigma}{2}}}{(4\pi)^{\frac{\sigma}{2}} (1+\sigma)^{\frac{1}{2}}} \right| \\ =\ & \left| \int_{\mathbf{R}} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \theta^{1+\sigma} \int_{\mathbf{R}} (G_0(t,x))^{1+\sigma} dx \right| \\ \le\ & C \left\| |v_1(t)|^{\sigma} v_1(t) - \theta^{1+\sigma} G_0^{1+\sigma} \right\|_{\mathbf{L}^1} \\ \le\ & C(2-\sigma)\langle t \rangle^{-\frac{\sigma}{2}} + C\langle t \rangle^{-\frac{\sigma}{2} - \frac{a}{2}} \end{aligned}$$

for all $t > 0$. Hence

$$\left| \int_0^t d\tau \int_{\mathbf{R}} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \frac{\theta^{1+\sigma} t^{1-\frac{\sigma}{2}}}{(4\pi)^{\frac{\sigma}{2}} \left(1-\frac{\sigma}{2}\right)(1+\sigma)^{\frac{1}{2}}} \right|$$
$$= \left| \int_0^t d\tau \int_{\mathbf{R}} |v_1(t,x)|^{\sigma} v_1(t,x)\, dx - \zeta t^{1-\frac{\sigma}{2}} \right|$$
$$\leq C(2-\sigma) \int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}} d\tau + C \int_0^t \langle\tau\rangle^{-\frac{\sigma}{2}-\frac{a}{2}} d\tau \leq C t^{1-\frac{\sigma}{2}}$$

for all $t > 0$. Lemma 4.3 is proved.

In view of estimates (4.18), using Lemma 4.3 we see that

$$\left| h(t) - 1 - \zeta t^{1-\frac{\sigma}{2}} \right| \leq C t^{1-\frac{\sigma}{2}}$$

for all $t > 0$, where

$$\zeta = \frac{\theta^{1+\sigma}}{(4\pi)^{\frac{\sigma}{2}} \left(1-\frac{\sigma}{2}\right)(1+\sigma)^{\frac{1}{2}}}$$

is sufficiently large. Now the asymptotic formulas are proved in the same manner as in paper [14]. Theorem 1.2 is proved.

Remark 4.1. *The restriction for the value $\sigma < 2$ to be close to 2 is rather technical. It allows us to prove estimate (4.18) in the proof of Theorem 1.2. We believe that some more restrictions on the initial data are necessary to be able to generalize the method applied in this section to the case when σ is not close to 2 ($\sigma < 2$). In this connection let us refer some decay restrictions on the initial data which are well-known for the nonlinear heat equation. In paper* [8] *it was proved that if the initial data are nonnegative $u_0 \geq 0$, $u_0 \in \mathbf{L}^1$ and decay slowly at infinity as $\lim_{x\to\pm\infty} |x|^{\frac{2}{\sigma}} u_0(x) = +\infty$, then the solution of the nonlinear heat equation has the asymptotic representation $u(t,x) = t^{-\frac{1}{\sigma}} \sigma^{-\frac{1}{\sigma}} + o\left(t^{-\frac{1}{\sigma}}\right)$ as $t \to \infty$ uniformly in domains $\left\{x \in \mathbf{R}^n;\ |x| \leq C\sqrt{t}\right\}$ with any $C > 0$. In paper* [2]*, there were considered the nonnegative initial data decaying sufficiently rapidly at infinity, i.e. $0 \leq u_0(x) \leq Ce^{-bx^2}$ for all $x \in \mathbf{R}^n$, with some $b, C > 0$. Then it was shown that the main term of the asymptotic behavior of solution has a self-similar character $u(t,x) = t^{-\frac{1}{\sigma}} w_0\left(\frac{x}{\sqrt{t}}\right) + o\left(t^{-\frac{1}{\sigma}}\right)$ as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $w_0(\xi)$ is a positive solution of the elliptic equation*

$$-\Delta w - \frac{1}{2}\xi \nabla w + w^{1+\sigma} = \frac{1}{\sigma} w \tag{4.22}$$

which decays rapidly at infinity: $\lim_{|\xi|\to\infty} |\xi|^{\frac{2}{\sigma}} w_0(\xi) = 0$. This result was improved in paper [3]*, where the intermediate case was considered: if the initial data are such that $u_0 \in \mathbf{L}^1$, $u_0 \neq 0$ and $\lim_{|x|\to\infty} |x|^{\frac{2}{\sigma}} u_0(x) = \lambda > 0$, then the solutions of the nonlinear heat equation have the asymptotic representation $u(t,x) = t^{-\frac{1}{\sigma}} w_\lambda\left(\frac{x}{\sqrt{t}}\right) + o\left(t^{-\frac{1}{\sigma}}\right)$ as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $w_\lambda(\xi)$ is a positive solution of equation (4.22) such that $\lim_{|\xi|\to\infty} |\xi|^{\frac{2}{\sigma}} w_\lambda(\xi) = \lambda$. Another asymptotic behavior characterized by self-similar solutions changing a sign is possible (*[15]*). We emphasize that the $\mathbf{L}^p$ - time decay rate is the same for all the different types of the asymptotic behavior of solutions. Note that an optimal $\mathbf{L}^p$ time decay estimate* *of solutions to the Cauchy problem (1.1) was obtained in* [24] *for the subcritical case* $\sigma \in (0, 2)$ *under the condition, that the initial data decay exponentially at infinity without any restriction on the size. Certainly some decay restrictions on the initial data are necessary to remove the assumption that* σ *is close to* 2*. However we probably need some more conditions to obtain the result of Theorem 1.2 saying precisely which type of the self-similar solution can serve as an asymptotic profile for large time.*

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