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DECOMPOSING A PLANAR GRAPH INTO A FOREST AND A SUBGRAPH OF RESTRICTED MAXIMUM DEGREE

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Abstract. We disprove the conjecture of He, Hou, Lih, Shao, Wang and Zhu that every plane graph G can be edge-partitioned into a forest and a subgraph of the maximum degree at most $\lceil \Delta(G)/2 \rceil + 1$.

1. Introduction

He, Hou, Lih, Shao, Wang and Zhu [1] proved several results on decomposing the edges of planar graphs of certain types into a forest and a subgraph whose maximum degree is restricted from above. They used these results to derive upper bounds on the game chromatic number and the game coloring number of the corresponding planar graphs.

In particular, He et al [1] proved that a planar graph with girth 11 or more can be decomposed into a forest and a matching, which implies that the game chromatic number and the game coloring number of such graphs are at most 5. Kleitman et al [2] proved the same statement for planar graphs with girth at least 10, which restriction on girth was further improved to 9 by Borodin, Kostochka, Sheikh, and Yu [3].

One of the main results in [1] says that every C_4-free planar graph can be decomposed into a forest and a graph with maximum degree at most 7, which in turn implies that the game chromatic number and the game coloring number of such graphs are at most 11. These 7 and 11 were lowered by 1 in [4].

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Let $\Delta(G)$ denote the maximum degree of graph G. He et al [1] proved that every planar graph G has an edge-partition into a forest and a graph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + 5$ and made the following

Conjecture 1. *Every planar graph G has an edge-partition into a forest and a graph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + 1$.*

The main purpose of our note is to disprove Conjecture 1 as follows:

Theorem 1. *For every integer $\Delta \geq 12$, there is a planar graph G with $\Delta(G) = \Delta$ such that in any decomposition of G into a forest and a subgraph H, we have $\Delta(H) \geq \lceil \Delta/2 \rceil + 2$.*

This gives rise to the following

Problem 1. *Find the least integer λ such that every planar graph G has a decomposition into a forest and a subgraph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + \lambda$.*

For the C_3-free planar graphs, He et al [1] proved that every such graph G has an edge-partition into a forest and a graph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + 2$. For these graphs, we have similar lower bound and further problem:

Theorem 2. *For every integer $\Delta \geq 6$, there is a C_3-free planar graph G with $\Delta(G) = \Delta$ such that any decomposition of G into a forest and a subgraph H yields $\Delta(H) \geq \lceil \Delta/2 \rceil - 1$.*

Problem 2. *Find the least integer λ' such that every planar graph G without 3-cycles splits into a forest and a subgraph H with $\Delta(H) \leq \lceil \Delta(G)/2 \rceil + \lambda'$.*

It follows from [1] and our present results that $2 \leq \lambda \leq 5$ and $-1 \leq \lambda' \leq 2$.

2. Proof of Theorem 1

We first take two concentric 6-cycles, $x_1x_2 \ldots x_6$ and $y_1y_2 \ldots y_6$, and join every x_i with y_i and y_{i+1} (henceforth, addition modulo 6). Now join all x_i's with a new vertex x, and all y_i's with a vertex y. Put a 3-vertex into every triangle of the graph obtained. Next, join the ends of each of the 7 edges: xx_1, yy_1, and x_iy_i, where $2 \leq i \leq 6$, by $\Delta - 12$ disjoint paths of length 2. Finally, join the ends of each of the following 12 edges by a 2-path: x_ix_{i+1} and y_iy_{i+1}, where $1 \leq i \leq 6$. Denote the resulting graph by G_1.

It is easy to see that G_1 has 14 vertices of degree Δ and $6+12+6 = 24$ vertices of degree 3, while the number of 2-vertices in G_1 is $12 + (\Delta - 12) \times 7 = 7\Delta - 72$. So, $|V(G_1)| = 7\Delta - 34$, and

$$|E(G_1)| = \frac{14\Delta + 24 \times 3 + (7\Delta - 72) \times 2}{2} = 14\Delta - 36.$$

Now suppose $E(G_1)$ is partitioned into a forest F' and a subgraph H' with $\Delta(H') = k$.

Claim 1. *The edges of G_1 can be partitioned into a forest F and a graph H with $\Delta(H) \leq k$ so that every vertex of G_1 is incident with at least one edge in F.*

Indeed, among all the partitions of $E(G_1)$ into a forest F and a subgraph H with $\Delta(H) \leq k$, we take one in which F has the largest number of edges. Now if a vertex v were not incident with F-edges, then putting one of its incident edges into F would neither create F-cycles nor increase $\Delta(H)$, contrary to the choice of F and H.

Since $|E(F)| \leq |V(G_1)| - 1 = 7\Delta - 35$, we have $|E(H)| \geq 7\Delta - 1$. On the other hand, Claim 1 yields

$$|E(H)| \leq \frac{14k + 24 \times 2 + (7\Delta - 72) \times 1}{2} = 7k + \frac{7\Delta}{2} - 12.$$

It follows that

$$7\Delta - 1 \leq 7k + \frac{7\Delta}{2} - 12,$$

or

$$k \geq \frac{\Delta}{2} + \frac{11}{7}.$$

Since k is an integer, this implies that

$$k \geq \left\lceil \frac{\Delta}{2} \right\rceil + 2,$$

as desired.

3. Proof of Theorem 2

We follow the lines of the proof of Theorem 1. Let G_2 denote the graph, without 3-cycles, obtained from G_1 by:

(1) deleting all edges between the Δ-vertices of G_1,

(2) deleting the six 2-vertices adjacent to x_{2i} and x_{2i+1} or to y_{2i} and y_{2i+1}, where $1 \leq i \leq 3$, and

(3) making all the bunches of $\Delta - 12$ disjoint paths of length 2 into bunches with $\Delta - 6$ paths.

It is easy to see that G_2, as well as G_1, has 14 vertices of degree Δ and 24 vertices of degree 3, while the number of 2-vertices in G_2 is $6 + (\Delta - 6) \times 7 = 7\Delta - 36$. Thus, $|V(G_2)| = 7\Delta + 2$, and

$$|E(G_2)| = \frac{14\Delta + 24 \times 3 + (7\Delta - 36) \times 2}{2} = 14\Delta.$$

As well as in the proof of Theorem 1, we have $|E(F)| \leq |V(G_2)| - 1 = 7\Delta + 1$, so that $|E(H)| \geq 7\Delta - 1$. On the other hand,

$$|E(H)| \leq \frac{14k + 24 \times 2 + (7\Delta - 36) \times 1}{2} = 7k + \frac{7\Delta}{2} + 6.$$

It follows that

$$7\Delta - 1 \leq 7k + \frac{7\Delta}{2} + 6,$$

or

$$k \geq \frac{\Delta}{2} - 1,$$

i.e.,

$$k \geq \left\lceil \frac{\Delta}{2} \right\rceil - 1.$$

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