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# PERIODIC GROUPS SATURATED BY THE GROUP $U_3(9)$

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Abstract. Let $\mathfrak{M}$ be a set of finite groups. A group $G$ is said to be *saturated* by $\mathfrak{M}$, if every finite subgroup of $G$ is contained in a subgroup isomorphic to a group from $\mathfrak{M}$. We prove that a periodic group saturated by set consisting of the single finite simple group $U_3(9) = PSU_3(81)$ is isomorphic to $U_3(9)$.

Let $\mathfrak{M}$ be a set of finite groups. A group $G$ is said to be *saturated* by $\mathfrak{M}$, if every finite subgroup of $G$ is contained in a subgroup isomorphic to a group from $\mathfrak{M}$.

The paper [1] contains a hypothesis that a periodic group saturated by a finite set $\mathfrak{M}$ of finite non-abelian simple groups is finite and also confirms this hypothesis for the case when centralizers of Sylow 2-subgroups of groups from $\mathfrak{M}$ do not contain elements of odd order larger than three.

In this connection, it is interesting to investigate groups saturated by one simple group (precisely, by one-element set containing a finite simple group) in which centralizer of Sylow 2-subgroup contains an element of odd order larger than three. All such groups are listed in [2].

A simple group of the least order in which centralizer of Sylow 2-subgroup contains an element of odd order larger than three is $U_3(9) \simeq SU_3(81)$, and the goal of present article is to prove the following result.

**Theorem**. *Periodic group $G$ saturated by group $U_3(9)$ is isomorphic to $U_3(9)$.*

The proof uses the folowing well-known properties of $U_3(9)$ (see for example [3] and [4]).

**Proposition 1.** *Let $U \simeq U_3(9)$.*

1. *Sylow 2-subgroup $T$ in $U$ is semi-diedral group of order 32, i.e. $T = \langle a, b | a^{16} = b^2 = 1, a^b = a^7\rangle$.*

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2. *Every involution of $U$ is a conjugate of $b$.*

3. *The centralizer $C$ of $b$ in $U$ is the direct product of a group of order 5 and a group $C_0$ isomorphic to extension of $C_1 \simeq SL_2(9)$ by a group of order 2. All elements of order 4 from $C$ are contained in $C_1$, their squares coincide with an unique involution $b$ in $C_1$. The order of any maximal cyclic subgroup of $C_1$ equals 6, 8 or 10.*

4. *If $R$ is a Sylow 3-subgroup of $U$ then $R$ is a group of exponent 3, which center $Z$ is an elementary abelian group of order 9, and the factor group by center is an elementary abelian group of order 81, $N_U(R)$ is a maximal subgroup in $U$ which is an extention of $R$ by cyclic group $D$ of order 80. Here $|C_D(Z)| = 10$ and $D/C_D(Z)$ acts transitively upon conjugation in $N_U(R)$ on a set of non-trivial elements of $Z$ and $D$ acts regularly on $R/Z$. In particular, $r^d Z = r^{-1}Z$ for any element $r \in R$ and any involution $d \in N_U(R)$. Every 3-local subgroup of $U$ is contained in a subgroup isomorphic to $N_U(R)$.*

In addition we shall need the following results ([5]).

**Proposition 2.** *Periodic group containing an involution with finite centralizer is locally finite.*

The proof will also use the following facts which can be easily checked with the help of coset enumeration algorithm (see, for example, [6]).

**Proposition 3.** 1. *The order of the group $\langle x, y, z | x^3 = y^2 = z^2 = (yz)^3 = (y^x z)^3 = (yz^x)^3\rangle$ equals 54.*

2. *Suppose $H = \langle x, y, z | x^3 = y^2 = z^2 = (xy)^4 = (yz)^p = (yz^x)^q = ((xy)^2 z)^r\rangle$, where $p, q, r \in 3, 4, 5$ and $r \neq 4$. Then $|H| \leqslant 24$. Moreover $|H| = 24$ if $(p, q) = (4, 4)$, and $|H| = 1$ in other cases.*

3. *The order of the group $\langle x, y, z | x^3 = y^2 = z^2 = (xy)^4 = (yz)^2 = (yz^x)^4 = ((xy)^2 z)^4 = (y^x yz)^4 = 1\rangle$ equals 14400.*

**Proof of Theorem**. Obviously the theorem is true for finite group $G$. Suppose $G$ is infinite. As shown in [1], every locally finite subgroup of group $G$ is finite.

By assumption $G$ contains a subgroup $U \simeq U_3(9)$ and we shall later on use notations from the statement of Proposition 1.

**Lemma 1.** $N_G(Z) = N_G(R) = N_U(R)$.

**Proof**. By Proposition 1.4, $C_G(Z)$ contains an involution $b$. If $c$ is an involution in $C_G(Z)$ then $\langle b, c, Z\rangle$ is a finite group contained in a subgroup isomorphic to $U$. By Proposition 1.4, $\langle b, c, Z\rangle/Z$ contains an elementary abelian 3-subgroup of index 2. In particular, $bc \in Z$.

Let us show that every involution $d$ from $C_G(Z)$ inverts $R/Z$. Suppose $r$ is an element from $R$. By Proposition 1.4, the coset $brZ$ contains an involution $c$. By Proposition 3.1 and previous paragraph, $\langle b, c, d, Z\rangle/Z$ is a finite group containing $rZ$. By Proposition 1.4, $r^d Z = r^{-1}Z$.

Thus, all involutions from $C_G(Z)$ generate a subgroup $N$ which is an extension of $Z$ by a group containing an elementary abelian 3-subgroup of index 2. In particular, $N$ is locally finite and hence finite. By Proposition 1.4, a subgroup of index 2 from $N$ coincides with $R$. In particular, $C_G(Z) \leqslant N_G(R)$. This implies that $N_G(Z) \leqslant N_G(R)$. By Proposition 1.4, $C_G(R) \leqslant R$, hence $N_G(R)$ is a finite subgroup isomorphic to a subgroup of Ё§ $U$. This implies that $N_G(R) = N_U(R)$. The proof is completed.

Suppose $b$ is an involution in $C_U(Z)$, $H = C_G(b)$ and $\overline{H} = H/\langle b\rangle$. Denote the image of any element $h \in H$ and any subset $M \subseteq H$ by $\overline{h}$ and $\overline{M}$, respectively.

**Lemma 2.** *Every finite subgroup of $H$ is isomorphic to some subgroup of $C = C_U(b)$. If $\overline{a}$ and $\overline{c}$ are involutions conjugated to an involution from $\overline{C_1}$ then $(\overline{ac})^m = 1$ for $m = 3, 4$ or $5$. If, in addition, $\overline{ac}$ is an element of order $4$ then $(\overline{ac})^2$ is conjugated to an involution from $C_1$.*

**Proof**. If $F$ is a finite subgroup of $H$ then $\langle b, F\rangle$ is contained in subgroup $V$ isomorphic to $U$ and $F \leqslant C_V(b) \simeq C_U(b)$.

If $\overline{a}$ and $\overline{c}$ are conjugated to an involution from $\overline{C_1}$ then, by Proposition 1.3, $a$ and $c$ are elements of order 4 and $a^2 = c^2 = b$. Finite subgroup $\langle a, c\rangle$ is isomorphic to some subgroup of $C_1$ and order of $ac$ divides one of the numbers 6, 8 or 10. In addition, if this order is even then $b \in \langle ac\rangle$ and hence equality $(\overline{ac})^m = 1$ holds, where $m = 3, 4$ or $5$. If in addition the order of $\overline{ac}$ equals 4 then $ac$ is an element of order 8, $(ac)^2$ is an element of order 4 and, by Proposition 1.3, $(ac)^2$ is conjugated in $H$ to an element of $C_1$. The proof is completed.

**Lemma 3.** *If $\overline{a}$ is an involution from $\overline{H}$ not lying in $\overline{C_1}$ and conjugated to an involution from $\overline{C_1}$, and $\overline{c}$ is an involution from $\overline{C_1}$ then $(\overline{ac})^4 = 1$.*

**Proof**. The subgroup $\overline{C_1}$ is isomorphic to $A_6$ and hence contains a subgroup $\overline{S} \simeq S_4$. The group $\overline{S}$ is generated by an element $x$ of order 3 and an involution $y$ whose product is an element of order 4. Since all involutions of $A_6$ are conjugated in $A_6$, without loss of the generality, we can assume that $\overline{c} = (xy)^2$. Denote $\overline{a}$ with $z$. By Lemma 2, there exist $p, q, r \in 3, 4, 5$ such that $(yz)^p = (yz^x)^q = ((xy)^2z)^r = 1$. Since $|\langle x, y, z\rangle| > 24$, Proposition 3.2 implies that $r = 4$. The proof is completed.

**Lemma 4.** *$C_1 \lhd H$.*

**Proof**. Suppose the contrary. Since $\overline{C_1}$ is generated by involutions, there exists an involution $z \in \overline{H}$ not contained in $\overline{C_1}$ and conjugated to an involution from $\overline{C_1}$. Suppose again that $\overline{S}$ is a subgroup of $\overline{C_1}$ isomorphic to $S_4$, and $x, y$ are elements of $\overline{S}$ generating $\overline{S}$ such that $x^3 = y^2 = (xy)^4 = 1$. By Lemma 3, $(yz)^4 = (yz^x)^4 = ((xy)^2z)^4 = 1$.

If $(yz)^2 = 1$ then supplementary $(y^x \cdot yz)^4 = 1$ and $\langle x, y, z\rangle$ is finite by Proposition 3.3. Since it is generated by elements of order 2 conjugated with elements of $\overline{C_1}$, $\langle x, y, z\rangle$ is isomorphic to a subgroup of $\overline{C_1} \simeq A_6$. Since every subgroup of $A_6$ isomorphic to $S_4$ is maximal in $A_6$, $\overline{C_2} = \langle x, y, z\rangle \simeq A_6$. Obviously $\overline{C_1} \cap \overline{C_2} = \overline{S}$.

If now $z_1$ is an involution from $\overline{C_2} \simeq A_6$ such that the order of $yz_1$ equals 5 then $z_1$ is conjugated with $y$ and not lying in $\overline{S}$, hence $z_1 \notin C_1$ and, by Lemma 3, $(yz_1)^4 = 1$ which contradicts the choice of $z_1$. Thus, $(yz)^2 \neq 1$.

Let $t = (yz)^2$. Then $t$ is an involution conjugated by Lemma 2 with an involution from $\overline{C_1}$. If $t \notin \overline{C_1}$, then whereas all involutions from $\overline{C_1}$ are conjugated in $\overline{C_1}$ we may replace $y$ with $t$. After this replacing, the equality $(yz)^2 = 1$ will hold and equalities $(yz)^4 = (yz^x)^4 = ((xy)^2z)^4 = (y^x \cdot yz)^4 = 1$ will be kept. As indicated above, these equalities bring us to a contradiction. The proof is completed.

**Lemma 5.** *$H$ is a finite group.*

**Proof**. By the choice of the involution $b$, $Z$ is contained in $C_1$ as a Sylow 3-subgroup. Hence $H = C_1N_G(Z)$. By Lemma 1, $N_G(Z)$ is a finite group. The proof is completed.

Now, by Proposition 2, $G$ is a locally finite and hence finite group. This implies that $G$ is isomorphic to $U$. Theorem is proved.

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