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ON METABELIAN GROUPS WITH DERIVED QUOTIENT AN ELEMENTARY ABELIAN 2-GROUP OF RANK 3

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Abstract. Necessary and sufficient conditions in terms of rank and exponent for the existence of torsion-free metabelian groups with derived quotient an elementary abelian p-group of rank k are formulated.

1. Introduction

Metabelian groups with finite derived quotients have many interesting properties. Such groups are studied in the theory of crystallographic groups and in the theory of right ordered groups. Special cases appear when the groups are torsion-free. Torsion-free crystallographic groups are called Bieberbach groups and constitute a proper theory. On the contrary, in the theory of right ordered groups the metabelian groups with finite derived quotients are used to construct counter-examples. Below we cited three examples of torsion-free metabelian groups with derived quotients of order 16. Example 1.1 is a part of an example of R.B. Mura and A.H. Rhemtulla of a right ordered group in which the center is not convex. Example 1.2, constructed by D.M. Smirnov in 1967, was the first example of a torsion free metabelian group that is not right orderable.

Torsion-free metabelian groups $M(k,p)$ with derived quotients an elementary abelian group of rank k and of prime exponent p were studied by N. Gupta and S. Sidki in [2]. They proved that $k \geq 3$ and constructed examples of such groups for every odd prime exponent and any rank $k \geq 3$. They also constructed examples of groups $M(k,2)$ for any $k \geq 4$ and raised a following conjecture:

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there does not exist a metabelian torsion-free group with derived quotient an elementary abelian 2-group of rank 3.

We confirm this conjecture by proving the following

Theorem A. *Let G be a metabelian group. If*

$$G/G' \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2, \tag{1}$$

where $\mathbf{Z}_2$ is the cyclic group of order two, then the group G is torsion.

Hence in a pair with the results of N. Gupta and S. Sidki from [2], we can formulate a necessary and sufficient conditions in terms of rank and exponent for the existence of torsion-free metabelian groups with derived quotient an elementary abelian p-group of rank k.

Corollary B. 1) *Let p be odd prime. There exists metabelian groups with derived quotient an elementary abelian p-group of rank k if and only if $k \geq 3$.*

2) *There exists metabelian groups with derived quotient an elementary abelian 2-group of rank k if and only if $k \geq 4$.*

Case 1 and the part "if" of case 2 is proved in [2]. The part "only if" of case 2 follows from Theorem A. □

To show that there are many examples of metabelian torsion-free groups with derived quotient of order 16, we cited three examples known in the theory of right-ordered groups:

Example 1.1 (Mura, Rhemtulla [4]) G_1 is a metabelian group with generators x, y and defining relations $x^4 = [y, x^2]$, $y^4 = [x, y^2]$.

Example 1.2 (Smirnov [5], see also [3]) G_2 is a metabelian group with generators a, b, c and defining relations $a^c = a^{-1}$, $b^c = b^{-1}$, $[a, b]^a = [a, b]^b = [a, b]$, $c^4 = [a, b]$.

Example 1.3 (Kopytov, Medvedev [3]) G_3 is a metabelian group with generators a_1, b_1, a_2, b_2 and defining relations $a_1^2 = b_2^2 = [b_1, a_1]$, $b_1^2 = a_2^2 = [b_2, a_2]$, $[a_1, a_2] = [b_1, b_2] = 1$.

All this groups are metabelian torsion-free and $G_1/G_1' \cong \mathbf{Z}_4 \times \mathbf{Z}_4$, $G_2/G_2' \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$, $G_3/G_3' \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

2. Notations and Preliminaries

We use common notation for conjugated elements $x^y = y^{-1}xy$ and commutators $[x, y] = x^{-1}y^{-1}xy$, $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$. We also write $x^{na+mb+\cdots}$ for $(x^n)^a(x^m)^b \dots$, $n, m \in \mathbf{Z}$.

Since all considered groups are metabelian we use additive notation for the derived subgroup. For examples, instead of $[a, b]^{-a}[a, b]^{bc}[b, c]^{-1}$ we write $-[a, b]^a + [a, b]^{bc} - [b, c]$.

Let G be metabelian group and let $u, v \in G'$, $f, g \in G$. It is well known that

$$[u, f, g] = [u, g, f];$$

$$[u + v, f] = [u, f] + [v, f]; \quad [nu, f] = n[u, f],\ n \in \mathbf{N};$$

$$[u, fv] = [u, f];$$

$$[u, fg] = [u, f] + [u, g] + [u, f, g];$$

$$[fu, gv] = [f, v] + [f, g] + [u, g] \tag{2}$$

We also need two identities

— **Jacobi's identity**, which valid for metabelian groups,

$$[x, y, z] + [y, z, x] + [z, x, y] = 0$$

or

(3) $$[x, y] - [x, y]^z + [y, z] - [y, z]^x - [x, z] + [x, z]^y = 0,$$

— and **identity:**

$$[x, y, z^2] = 0,$$

which valid for metabelian groups with property (1). The last identity has an equivalent form

(4) $$[x, y, z, z] = -2[x, y, z].$$

It is well known that torsion-free abelian groups are R-groups (i.e. $x^n = y^n$ implies $x = y$ for any elements x, y and for any integer $n \neq 0$). We need this fact in a little generalized form.

Lemma 2.1 *Let G be a torsion-free group with a normal abelian subgroup H. Then $x^n = h^n$ implies $x = h$ for any integer $n \neq 0$ and any $x \in G$, $h \in H$.*

Proof. Conjugating $x^n = h^n$ by x we obtain $(h^x)^n = h^n \in H$. Since H is R-group then $h^x = h$ and therefore $(xh^{-1})^n = x^n h^{-n} = 1$. As G is torsion-free then $xh^{-1} = 1$ and $x = h$. □

Finally we cite a result that we will use in section 4.

Theorem 2.2 (Furtwängler [1]) *Let $M = \langle s_i, i = 1, \ldots, n \rangle$ be metabelian group with relations: $s_i^{\epsilon_i} = t_i \in M'$. We set $f_i = 1 + s_i + s_i^2 + \cdots + s_i^{\epsilon_i - 1}$. Then*

$$t_i^{f_1 \cdots f_{i-1} f_{i+1} \cdots f_n} = 1.$$

□

3. Generating set for the derived subgroup

We denote with $\mathcal{M}(3, 2)$ a class of metabelian groups with property (1). Our aim is to prove that there are no torsion-free groups in $\mathcal{M}(3, 2)$. For this we study groups from $M(3, 2)$, deduce their properties, claim that there exist a torsion-free group in $M(3, 2)$, and obtain a contradiction. Let $G \in \mathcal{M}(3, 2)$. Because of (1) we can represent G/G' as $\langle \bar{a_1} \rangle \times \langle \bar{a_2} \rangle \times \langle \bar{a_3} \rangle$ where $\bar{a_1}$, $\bar{a_2}$, $\bar{a_3}$ are elements of order two. Thereupon we choose in G the preimages of $\bar{a_1}$, $\bar{a_2}$, $\bar{a_3}$ and denote its with a_1, a_2, a_3. Besides a_1, a_2, a_3 there are another four elements: $a_4 = a_1 a_2$, $a_5 = a_1 a_3$, $a_6 = a_2 a_3$, and $a_7 = a_1 a_2 a_3$ in G that have nontrivial images in G/G'. Through all the article *we use the notation $a_1, \ldots, a_7$ only for this elements.* In order to emphasize that elements a_1, a_2, a_3 are chosen as above we write $G = G(a_1, a_2, a_3)$.

Let $G(a_1, a_2, a_3) \in \mathcal{M}(3, 2)$. We denote with A a subgroup of G generated by a_1, a_2, a_3: $A = \langle a_1, a_2, a_3 \rangle$. In general case $A \neq G$ although $G/G' \cong A/A'$. Nevertheless A is densely embedded into G as we will see in Lemma 3.2 below.

Now we determine a generating set for the subgroup A'. Since A' is the derived subgroup of A, its generating elements are various commutators of elements a_1, a_2, a_3 and their conjugated. Therefore these generating elements are of the form: $[a_1, a_2]^{u_1}, [a_1, a_3]^{u_2}, [a_2, a_3]^{u_3}$, $u_1, u_2, u_3 \in A$. Since A is metabelian group, it is enough to take u_i from $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. So the elements $[a_i, a_j]$, $[a_i, a_j]^{a_k}$,

$i, j = 1, 2, 3,\ k = 1, \ldots, 7$ generate A'. Thus we have twenty four elements that is too much for further calculations. Therefore we select fourteen elements

$$\begin{gathered} b_1 = [a_1, a_2],\ b_2 = [a_1, a_2]^{a_1},\ b_3 = [a_1, a_2]^{a_2}, \\ b_4 = [a_1, a_3],\ b_5 = [a_1, a_3]^{a_1},\ b_6 = [a_1, a_3]^{a_2}, \\ b_7 = [a_1, a_3]^{a_3},\ b_8 = [a_1, a_3]^{a_4}, \\ b_9 = [a_2, a_3],\ b_{10} = [a_2, a_3]^{a_1},\ b_{11} = [a_2, a_3]^{a_2}, \\ b_{12} = [a_2, a_3]^{a_3},\ b_{13} = [a_2, a_3]^{a_4},\ b_{14} = [a_2, a_3]^{a_5}, \end{gathered} \tag{5}$$

set $B = \{b_1, \ldots, b_{14}\}$, and prove

Lemma 3.1 *The subgroup A' is generated by B .*

Proof. Taking into account (3), we obtain

$$\begin{gathered} [a_1, a_2]^{a_3} = [a_1, a_2] + [a_2, a_3] - [a_2, a_3]^{a_1} - [a_1, a_3] + [a_1, a_3]^{a_2} = \\ b_1 - b_4 + b_6 + b_9 - b_{10} \end{gathered} \tag{6}$$

Conjugating by elements a_1, a_2 and a_3 both sides, we get

$$\begin{gathered} [a_1, a_2]^{a_5} = [a_1, a_2]^{a_1 a_3} = \\ [a_1, a_2]^{a_1} + [a_2, a_3]^{a_1} - [a_2, a_3] - [a_1, a_3]^{a_1} + [a_1, a_3]^{a_1 a_2} = \\ b_2 - b_5 + b_8 - b_9 + b_{10}, \end{gathered} \tag{7}$$

$$\begin{gathered} [a_1, a_2]^{a_6} = [a_1, a_2]^{a_2 a_3} = \\ [a_1, a_2]^{a_2} + [a_2, a_3]^{a_2} - [a_2, a_3]^{a_1 a_2} - [a_1, a_3]^{a_2} + [a_1, a_3] = \\ b_3 + b_4 - b_6 + b_{11} - b_{13} \end{gathered} \tag{8}$$

and

$$[a_1, a_2] = [a_1, a_2]^{a_3} + [a_2, a_3]^{a_3} - [a_2, a_3]^{a_1 a_3} - [a_1, a_3]^{a_3} + [a_1, a_3]^{a_2 a_3}.$$

Using this equality with (6), we obtain

$$\begin{gathered} [a_1, a_3]^{a_6} = [a_1, a_3]^{a_2 a_3} = -[a_2, a_3]^{a_3} + [a_2, a_3]^{a_1 a_3} + \\ [a_1, a_3]^{a_3} - [a_2, a_3] + [a_2, a_3]^{a_1} + [a_1, a_3] - [a_1, a_3]^{a_2} = \\ b_4 - b_6 + b_7 - b_9 + b_{10} - b_{12} + b_{14}. \end{gathered} \tag{9}$$

>From $[a_1^2, a_2^2] = 1$, we obtain

$$1 = [a_1^2, a_2^2] = [a_1, a_2^2]^{a_1} + [a_1, a_2^2] = [a_1, a_2]^{a_1} + [a_1, a_2]^{a_1 a_2} + [a_1, a_2] + [a_1, a_2]^{a_2},$$

whence

$$\begin{gathered} [a_1, a_2]^{a_4} = [a_1, a_2]^{a_1 a_2} = -[a_1, a_2] - [a_1, a_2]^{a_1} - [a_1, a_2]^{a_2} = \\ -b_1 - b_2 - b_3. \end{gathered} \tag{10}$$

Similarly,

$$[a_1, a_3]^{a_5} = [a_1, a_3]^{a_1 a_3} = b_4 - b_5 - b_7, \tag{11}$$

$$[a_2, a_3]^{a_6} = [a_2, a_3]^{a_2 a_3} = -b_9 - b_{11} - b_{12}. \tag{12}$$

Conjugating both sides of (6), (11), and (12) by elements a_1a_2, a_2 and a_1 respectively, we obtain:

$$\begin{gathered}[a_1,a_2]^{a_7} = [a_1,a_2]^{a_1a_2a_3} = [a_1,a_2]^{a_1a_2}+\\ [a_2,a_3]^{a_1a_2} - [a_2,a_3]^{a_2} - [a_1,a_3]^{a_1a_2} + [a_1,a_3]^{a_1} =\\ = -b_1 - b_2 - b_3 + b_5 - b_8 - b_{11} + b_{13},\end{gathered} \tag{13}$$

$$\begin{gathered}[a_1,a_3]^{a_7} = [a_1,a_3]^{a_1a_2a_3} =\\ -[a_1,a_3]^{a_2} - [a_1,a_3]^{a_1a_2} - [a_1,a_3]^{a_2a_3} =\\ = -b_4 - b_7 - b_8 + b_9 - b_{10} + b_{12} - b_{14}.\end{gathered} \tag{14}$$

and

$$\begin{gathered}[a_2,a_3]^{a_7} = [a_2,a_3]^{a_1a_2a_3} =\\ -[a_2,a_3]^{a_1} - [a_2,a_3]^{a_1a_2} - [a_2,a_3]^{a_1a_3} = -b_{10} - b_{13} - b_{14}.\end{gathered} \tag{15}$$

Thus we expressed all another elements of B in terms of $b_1, b_2 \dots b_{14}$. □

Lemma 3.2 *Let $G(a_1,a_2,a_3) \in \mathcal{M}(3,2)$ and $A = \langle a_1,a_2,a_3\rangle$. Then $G = A \cdot 2^nG'$ for any positive integer n.*

Proof. Let $g \in G \setminus G'$. Since $G/G' \cong AG'/G$ then $g = fu$ for suitable $f \in A$, $u \in G'$. So it is sufficient to prove the lemma statement for $g \in G'$. In this case g is a sum of commutators $[g_i,g_j]$. Representing every g_i, g_j as $g_i = f_iu_i$, $g_j = f_ju_j$ and using (2), we get $[g_i,g_j] = [f_i,f_j] - [u_j,f_i] + [u_i,f_j]$. Therefore $g \in G'$ implies $g \in A' + [G',A]$ and, by induction, $g \in A' + [G',\underbrace{A,\dots,A}_{m}]$ for any $m \in N$. So g can be presented in the form $g = w_0 + w_1$, where $w_0 \in A'$ and w_1 is a sum of commutators of the form $[v,\underbrace{a_1,\dots,a_1}_{k_1},\underbrace{a_2,\dots,a_2}_{k_2},\underbrace{a_3,\dots,a_3}_{k_3}]$, $v \in G'$, $0 \le k_1,k_2,k_3$, $k_1+k_2+k_3 = m$. By setting $m = n+3$ and using (4) we obtain $w_1 \in 2^nG'$. □

Lemmata 3.1, 3.2, and property (1) give us

Corollary 3.3 *If $G(a_1,a_2,a_3) \in \mathcal{M}(3,2)$ then*

$$a_i^2 = \sum_{j=1}^{14} x_{i,j}b_j + 8d_i, \quad i = 1,2,3, \tag{16}$$

for suitable $x_{i,j} \in \mathbf{Z}$, $d_i \in G'$. □

4. Cyclic invariants

In this section we consider a subgroup $C < A$ generated by elements $c_{i,j}$, $i = 1,2,\dots 7$, $j = 1,2$, where:

$$
\begin{aligned}
c_{1,1} &= b_9 + b_{10} + b_{11} + b_{13},\\
c_{1,2} &= b_4 + b_5 + b_6 + b_8,\\
c_{2,1} &= -2b_1 - 2b_2 + b_4 + b_5 - b_6 - b_8,\\
c_{2,2} &= b_9 + b_{10} + b_{12} + b_{14},\\
c_{3,1} &= 2b_4 + 2b_7 - b_9 + b_{10} - b_{12} + b_{14},\\
c_{3,2} &= 2b_1 + 2b_3 + b_9 - b_{10} + b_{11} - b_{13},\\
c_{4,1} &= b_{11} + b_{12} + b_{13} + b_{14}\\
c_{4,2} &= -b_4 - b_5 + b_6 + b_8,\\
c_{5,1} &= -2b_2 - 2b_3 - b_4 + b_5 + b_6 - b_8 + b_9 - b_{10} - b_{11} + b_{13},\\
c_{5,2} &= b_9 - b_{10} + b_{12} - b_{14},\\
c_{6,1} &= -b_4 - b_5 + b_6 - 2b_7 - b_8 + b_9 - b_{10} + b_{12} - b_{14},\\
c_{6,2} &= -b_9 + b_{10} - b_{11} + b_{13},\\
c_{7,1} &= b_4 - b_5 - b_6 + b_8 - b_9 + b_{10} + b_{11} - b_{13},\\
c_{7,2} &= b_4 - b_5 - b_6 + b_8 - b_9 + b_{10} - b_{12} + b_{14}.
\end{aligned}
\tag{17}
$$

This subgroup has two helpful properties. At first, direct calculations show that all cyclic subgroups $\langle c_{i,j}\rangle$, $i = 1, \dots, 7$, $j = 1, 2$ are invariant with respect to conjugation by a_k, $k = 1, \dots, 7$. Here $c_{i,j}^{a_k} = c_{i,j}$ for some couples (i, k) and $c_{i,j}^{a_k} = c_{i,j}^{-1}$ for another couples. To precise this we define seven subsets

$$
\begin{gathered}
J_1 = \{1,2,4\},\quad J_2 = \{1,3,6\},\quad J_3 = \{2,3,5\},\\
J_4 = \{1,5,7\},\quad J_5 = \{2,6,7\},\quad J_6 = \{3,4,7\},\quad J_7 = \{4,5,6\}
\end{gathered}
\tag{18}
$$

and obtain the next formula

$$
c_{i,j}^{a_k} = \begin{cases} c_{i,j} & \text{if } k \in J_i\\ c_{i,j}^{-1} & \text{otherwise,}\end{cases} \qquad j = 1, 2.
\tag{19}
$$

We use (19) to prove

Lemma 4.1 *Let $G(a_1, a_2, a_3)$ be torsion-free group from $\mathcal{M}(3, 2)$. If*

$$
\sum_{i=1}^{7}(s_{i,1}c_{i,1} + s_{i,2}c_{i,2}) \equiv 0 \pmod{2^{n+2}G'},
$$

with $s_{i,j} \in \mathbf{Z}$, $n \in \mathbf{N}$ then $s_{i,1}c_{i,1} + s_{i,2}c_{i,2} \equiv 0 \pmod{2^n G'}$ for all $i = 1, 2, \dots, 7$.

Proof. Let $u = \sum\limits_{i=1}^{7}\sum\limits_{j=1}^{2} s_{i,j}c_{i,j} + 2^{n+2}v$, $v \in G'$. It follows from (18) – (19) that

$$
\begin{aligned}
u^{(1+a_1)(1+a_2)} &= 4(s_{1,1}c_{1,1} + s_{1,2}c_{1,2}) + 2^{n+2}v^{(1+a_1)(1+a_2)},\\
u^{(1+a_1)(1+a_3)} &= 4(s_{2,1}c_{2,1} + s_{2,2}c_{2,2}) + 2^{n+2}v^{(1+a_1)(1+a_3)},\\
u^{(1+a_2)(1+a_3)} &= 4(s_{3,1}c_{3,1} + s_{3,2}c_{3,2}) + 2^{n+2}v^{(1+a_2)(1+a_3)},\\
u^{(1+a_1)(1+a_6)} &= 4(s_{4,1}c_{4,1} + s_{4,2}c_{4,2}) + 2^{n+2}v^{(1+a_1)(1+a_6)},\\
u^{(1+a_4)(1+a_3)} &= 4(s_{5,1}c_{5,1} + s_{5,2}c_{5,2}) + 2^{n+2}v^{(1+a_4)(1+a_3)},\\
u^{(1+a_5)(1+a_2)} &= 4(s_{6,1}c_{6,1} + s_{6,2}c_{6,2}) + 2^{n+2}v^{(1+a_5)(1+a_2)},\\
u^{(1+a_4)(1+a_5)} &= 4(s_{7,1}c_{7,1} + s_{7,2}c_{7,2}) + 2^{n+2}v^{(1+a_4)(1+a_5)}.
\end{aligned}
$$

Because G' is R-group we divide by four and obtain desired conclusion. □

The next expressions follow from (17):

$$
(20)\qquad
\begin{aligned}
4b_1 &= -c_{2,1} + c_{3,2} - c_{4,2} + c_{5,1} + c_{6,2} + c_{7,1},\\
4b_2 &= -c_{2,1} - c_{3,2} - c_{4,2} - c_{5,1} - c_{6,2} - c_{7,1},\\
4b_3 &= c_{2,1} + c_{3,2} + c_{4,2} - c_{5,1} + c_{6,2} - c_{7,1},\\
4b_4 &= c_{1,2} + c_{3,1} - c_{4,2} + c_{5,2} + c_{6,1} + c_{7,2},\\
4b_5 &= c_{1,2} - c_{3,1} - c_{4,2} - c_{5,2} - c_{6,1} - c_{7,2},\\
4b_6 &= c_{1,2} + c_{3,1} + c_{4,2} - c_{5,2} + c_{6,1} - c_{7,2},\\
4b_7 &= -c_{1,2} + c_{3,1} + c_{4,2} + c_{5,2} - c_{6,1} - c_{7,2},\\
4b_8 &= c_{1,2} - c_{3,1} + c_{4,2} + c_{5,2} - c_{6,1} + c_{7,2},\\
4b_9 &= c_{1,1} + c_{2,2} - c_{4,1} + c_{5,2} - c_{6,2} - c_{7,1} + c_{7,2},\\
4b_{10} &= c_{1,1} + c_{2,2} - c_{4,1} - c_{5,2} + c_{6,2} + c_{7,1} - c_{7,2},\\
4b_{11} &= c_{1,1} - c_{2,2} + c_{4,1} - c_{5,2} - c_{6,2} + c_{7,1} - c_{7,2},\\
4b_{12} &= -c_{1,1} + c_{2,2} + c_{4,1} + c_{5,2} + c_{6,2} + c_{7,1} - c_{7,2},\\
4b_{13} &= c_{1,1} - c_{2,2} + c_{4,1} + c_{5,2} + c_{6,2} - c_{7,1} + c_{7,2},\\
4b_{14} &= -c_{1,1} + c_{2,2} + c_{4,1} - c_{5,2} - c_{6,2} - c_{7,1} + c_{7,2}.
\end{aligned}
$$

So $4A' \le C$ and this is the second helpful property from which we can represent (mod $32G'$) the elements $4a_i^2$, $i = 1, \dots, 7$ in terms of $c_{j,k}$, $j = 1, \dots, 7$, $k = 1, 2$. Multiplying both sides of (16) by four and replacing every occurrence of $4b_1, \dots 4b_{14}$ by their expressions in (20), we obtain:

$$4a_i^2 = \sum_{j=1}^{14} 4x_{i,j}b_j + 32d_i = \sum_{j=1}^{7}\sum_{k=1}^{2} t_{i,j,k}c_{j,k} + 32d_i, i = 1, 2, 3,$$

where $t_{i,1,1} = x_{i,9}+x_{i,10}+x_{i,11}-x_{i,12}+x_{i,13}-x_{i,14}$, $t_{i,1,2} = x_{i,4}+x_{i,5}+x_{i,6}-x_{i,7}+x_{i,8}$, $t_{i,2,1} = -x_{i,1} - x_{i,2} + x_{i,3}$, $t_{i,2,2} = x_{i,9} + x_{i,10} - x_{i,11} + x_{i,12} - x_{i,13} + x_{i,14}$, $t_{i,3,1} = x_{i,4} - x_{i,5} + x_{i,6} + x_{i,7} - x_{i,8}$, $t_{i,3,2} = x_{i,1} - x_{i,2} + x_{i,3}$, $t_{i,4,1} = -x_{i,9} - x_{i,10} + x_{i,11} + x_{i,12} + x_{i,13} + x_{i,14}$, $t_{i,4,2} = -x_{i,1} - x_{i,2} + x_{i,3} - x_{i,4} - x_{i,5} + x_{i,6} + x_{i,7} + x_{i,8}$, $t_{i,5,1} = x_{i,1} - x_{i,2} - x_{i,3}$, $t_{i,5,2} = x_{i,4} - x_{i,5} - x_{i,6} + x_{i,7} + x_{i,8} + x_{i,9} - x_{i,10} - x_{i,11} + x_{i,12} + x_{i,13} - x_{i,14}$, $t_{i,6,1} = x_{i,4} - x_{i,5} + x_{i,6} - x_{i,7} - x_{i,8}$, $t_{i,6,2} = x_{i,1} - x_{i,2} + x_{i,3} - x_{i,9} + x_{i,10} - x_{i,11} + x_{i,12} + x_{i,13} - x_{i,14}$, $t_{i,7,1} = x_{i,1} - x_{i,2} - x_{i,3} - x_{i,9} + x_{i,10} + x_{i,11} + x_{i,12} - x_{i,13} - x_{i,14}$, $t_{i,7,2} = x_{i,4} - x_{i,5} - x_{i,6} - x_{i,7} - x_{i,8} + x_{i,9} + x_{i,10} - x_{i,11} - x_{i,12} + x_{i,13} + x_{i,14}$.

We see that coefficients $t_{i,j,k}$ have big common parts, so we can simplify our representations for $4a_i^2$, and after changing variables: $y_{i,1} = t_{i,1,1} = x_{i,9} + x_{i,10} + x_{i,11}-x_{i,12}+x_{i,13}-x_{i,14}$, $y_{i,2} = t_{i,1,2} = x_{i,4}+x_{i,5}+x_{i,6}-x_{i,7}+x_{i,8}$, $y_{i,3} = t_{i,2,1} = -x_{i,1} - x_{i,2} + x_{i,3}$, $y_{i,4} = -x_{i,11} + x_{i,12} - x_{i,13} + x_{i,14}$, $y_{i,5} = -x_{i,5} + x_{i,7} - x_{i,8}$, $y_{i,6} = x_{i,1}$, $y_{i,7} = -x_{i,9} - x_{i,10} + x_{i,12} + x_{i,14}$, $y_{i,8} = -x_{i,4} - x_{i,5} + x_{i,7}$, $y_{i,9} = x_{i,1} - x_{i,3}$, $y_{i,10} = -x_{i,5} - x_{i,6} + x_{i,7} - x_{i,10} - x_{i,11} + x_{i,12}$, $y_{i,11} = -x_{i,5} - x_{i,8}$, $y_{i,12} = x_{i,1} - x_{i,9} - x_{i,11} + x_{i,12}$, $y_{i,13} = x_{i,1} - x_{i,3} - x_{i,9} + x_{i,12} - x_{i,13}$, $y_{i,14} = -x_{i,5} - x_{i,6} - x_{i,8} - x_{i,11} - x_{i,14}$, $i = 1, 2, 3$, we get:

$$
(21)\qquad
\begin{gathered}
4a_i^2 = y_{i,1}c_{1,1} + y_{i,2}c_{1,2} + y_{i,3}c_{2,1} + (y_{i,1} + 2y_{i,4})c_{2,2}+\\
(y_{i,2} + 2y_{i,5})c_{3,1} + (y_{i,3} + 2y_{i,6})c_{3,2} + (y_{i,1} + 2y_{i,7})c_{4,1}+\\
(y_{i,2} + y_{i,3} + 2y_{i,8})c_{4,2} + (y_{i,3} + 2y_{i,9})c_{5,1} + (y_{i,1} + y_{i,2} + 2y_{i,10})c_{5,2}+\\
(y_{i,2} + 2y_{i,11})c_{6,1} + (y_{i,1} + y_{i,3} + 2y_{i,12})c_{6,2}+\\
(y_{i,1} + y_{i,3} + 2y_{i,13})c_{7,1} + (y_{i,1} + y_{i,2} + 2y_{i,14})c_{7,2} + 32d_i, \quad i = 1, 2, 3.
\end{gathered}
$$

Now we obtain another representation for $4a_i^2$. By Lemma 3.2 factor-group $\tilde{G} = G/32G'$ has three generators: $\tilde{a}_1 = a_1 + 32G'$, $\tilde{a}_2 = a_2 + 32G'$, $\tilde{a}_3 = a_3 + 32G'$ and satisfies the property (1), so $\tilde{a}_i^2 \in \tilde{G}'$, $i = 1, 2, 3$. Therefore we can use Theorem 2.2 and obtain:

$$\tilde{a}_i^{2(\tilde{a}_j+1)(\tilde{a}_k+1)} = 1 \quad \text{for pairwise different } i, j, k.$$

In our calculations below we denote $b_i + 32G'$, $c_{j,k} + 32G'$ with $\tilde{b}_i$, $\tilde{c}_{j,k}$ respectively and use formulae (5) – (15) valid in group $\tilde{G}$.

$$\tilde{a}_1^{2(\tilde{a}_2+1)(\tilde{a}_3+1)} = (2\tilde{a}_1^2 + [\tilde{a}_1, \tilde{a}_2] + [\tilde{a}_1, \tilde{a}_2]^{\tilde{a}_1})^{(\tilde{a}_3+1)} =$$
$$4\tilde{a}_1^2 + 2[\tilde{a}_1, \tilde{a}_3] + 2[\tilde{a}_1, \tilde{a}_3]^{\tilde{a}_1} + [\tilde{a}_1, \tilde{a}_2] + [\tilde{a}_1, \tilde{a}_2]^{\tilde{a}_3} + [\tilde{a}_1, \tilde{a}_2]^{\tilde{a}_1} + [\tilde{a}_1, \tilde{a}_2]^{\tilde{a}_1\tilde{a}_3} =$$
$$4\tilde{a}_1^2 + 2\tilde{b}_4 + 2\tilde{b}_5 + \tilde{b}_1 + \tilde{b}_1 - \tilde{b}_4 + \tilde{b}_6 + \tilde{b}_9 - \tilde{b}_{10} + \tilde{b}_2 + \tilde{b}_2 - \tilde{b}_5 + \tilde{b}_8 - \tilde{b}_9 + \tilde{b}_{10} =$$
$$4\tilde{a}_1^2 + 2\tilde{b}_1 + 2\tilde{b}_2 + \tilde{b}_4 + \tilde{b}_5 + \tilde{b}_6 + \tilde{b}_8 = 0.$$

So we have in group G:

$$4a_1^2 = -2b_1 - 2b_2 - b_4 - b_5 - b_6 - b_8 + 32h_1, \quad h_1 \in G'.$$

It follows from (17) that $c_{1,2} - c_{2,1} - c_{4,2} = 2b_1 + 2b_2 + b_4 + b_5 + b_6 + b_8$. Finally:

$$4a_1^2 = -c_{1,2} + c_{2,1} + c_{4,2} + 32h_1, \quad h_1 \in G'. \tag{22}$$

Proceeding as above we get:

$$4a_2^2 = -c_{1,1} + c_{3,2} + c_{6,2} + 32h_2, \quad h_2 \in G', \tag{23}$$

$$4a_3^2 = c_{2,2} + c_{3,1} + c_{5,2} + 32h_3, \quad h_3 \in G'. \tag{24}$$

For our further purposes we find analogous representations of $4a_4^2 \dots a_7^2$ and $4(a_iu)^2$, $i = 1, \dots, 7$, $u = k_1b_1 + k_4b_4 + k_9b_9 \in G'$.

Using (5) – (15) and (20) we get

$$4a_4^2 = 4(a_1a_2)^2 = 4(a_1^2 + a_2^2 + [a_2, a_1])^{a_2} = 4a_1^2 + 4a_2^2 - 4b_3 = -c_{1,2} + c_{2,1}+$$
$$c_{4,2} - c_{1,1} + c_{3,2} + c_{6,2} - c_{2,1} - c_{3,2} - c_{4,2} + c_{5,1} - c_{6,2} + c_{7,1} + 32h_4 =$$
$$= -c_{1,1} - c_{1,2} + c_{5,1} + c_{7,1} + 32h_4, \quad h_4 \in G'. \tag{25}$$

In the same manner

$$4a_5^2 = c_{2,1} + c_{2,2} + c_{6,1} + c_{7,2} + 32h_5, \quad h_5 \in G', \tag{26}$$

$$4a_6^2 = c_{3,1} + c_{3,2} - c_{4,1} - c_{7,1} + c_{7,2} + 32h_6, \quad h_6 \in G', \tag{27}$$

$$4a_7^2 = -c_{4,1} + c_{4,2} + c_{5,1} + c_{5,2} + c_{6,1} - c_{6,2} + 32h_7, \quad h_7 \in G'. \tag{28}$$

Let now $u = k_1b_1 + k_4b_4 + k_9b_9$, $k_1, k_4, k_9 \in \mathbf{Z}$ then:

$$\begin{gathered} 4(a_1u)^2 = 4a_1^2 + 2k_9c_{1,1} + 2k_4c_{1,2} \\ -2k_1c_{2,1} + 2k_9c_{2,2} - 2k_9c_{4,1} - 2(k_1 + k_4)c_{4,2}, \end{gathered} \tag{29}$$

$$\begin{gathered} 4(a_2u)^2 = 4a_2^2 + 2k_9c_{1,1} + 2k_4c_{1,2} \\ +2k_4c_{3,1} + 2k_1c_{3,2} + 2k_4c_{6,1} + 2(k_1 + k_9)c_{6,2}, \end{gathered} \tag{30}$$

$$\begin{gathered} 4(a_3u)^2 = 4a_3^2 - 2k_1c_{2,1} + 2k_9c_{2,2} \\ +2k_4c_{3,1} + 2k_1c_{3,2} + 2k_1c_{5,1} + 2(k_4 + k_9)c_{5,2}, \end{gathered} \tag{31}$$

$$4(a_4u)^2 = 4a_4^2 + 2k_9c_{1,1} + 2k_4c_{1,2} \tag{32}$$
$$+2k_1c_{5,1} + 2(k_4 + k_9)c_{5,2} + 2(k_1 - k_9)c_{7,1} + 2(k_4 + k_9)c_{7,2},$$

$$4(a_5u)^2 = 4a_5^2 - 2k_1c_{2,1} + 2k_9c_{2,2} \tag{33}$$
$$+2k_4c_{6,1} + 2(k_1 - k_9)c_{6,2} + 2(k_1 - k_9)c_{7,1} + 2(k_4 + k_9)c_{7,2},$$

$$4(a_6u)^2 = 4a_6^2 + 2k_4c_{3,1} + 2k_1c_{3,2} \tag{34}$$
$$-2k_9c_{4,1} + 2(k_1 + k_4)c_{4,2} + 2(k_1 - k_9)c_{7,1} + 2(k_4 + k_9)c_{7,2},$$

$$4(a_7u)^2 = 4a_7^2 - 2k_9c_{4,1} - 2(k_1 + k_4)c_{4,2} \tag{35}$$
$$+2k_1c_{5,1} + 2(k_4 + k_9)c_{5,2} + 2k_4c_{6,1} + 2(k_1 + k_9)c_{6,2}.$$

Indeed $4(a_1u)^2 = 4(a_1ua_1u) = 4(a_1^2 + u + u^{a_1}) = 4a_1^2 + 4k_1b_1 + 4k_1b_1^{a_1} + 4k_4b_4 + 4k_4b_4^{a_1} + 4k_9b_9 + 4k_9b_9^{a_1} = 4a_1^2 - 2k_1c_{2,1} - 2k_1c_{4,2} + 2k_4c_{1,2} - 2k_4c_{4,2} + 2k_9c_{1,1} + 2k_9c_{2,2} - 2k_9c_{4,1} = 4a_1^2 + 2k_9c_{1,1} + 2k_4c_{1,2} - 2k_1c_{2,1} + 2k_9c_{2,2} - 2k_9c_{4,1} - 2(k_1 + k_4)c_{4,2}$. Then we check representations (30) – (35) in the same manner.

Now we compare representations (21) and (22) – (24) and obtain three congruences (mod $32G'$):

$$\begin{aligned}
&y_{1,1}c_{1,1} + (y_{1,2} + 1)c_{1,2} + (y_{1,3} - 1)c_{2,1} + (y_{1,1} + 2y_{1,4})c_{2,2}+\\
&(y_{1,2} + 2y_{1,5})c_{3,1} + (y_{1,3} + 2y_{1,6})c_{3,2} + (y_{1,1} + 2y_{1,7})c_{4,1}+\\
&(y_{1,2} + y_{1,3} + 2y_{1,8} - 1)c_{4,2} + (y_{1,3} + 2y_{1,9})c_{5,1} + (y_{1,1} + y_{1,2} + 2y_{1,10})c_{5,2}+\\
&(y_{1,2} + 2y_{1,11})c_{6,1} + (y_{1,1} + y_{1,3} + 2y_{1,12})c_{6,2}+\\
&(y_{1,1} + y_{1,3} + 2y_{1,13})c_{7,1} + (y_{1,1} + y_{1,2} + 2y_{1,14})c_{7,2} \equiv 0 \pmod{32G'},
\end{aligned}$$

$$\begin{aligned}
&(y_{2,1} + 1)c_{1,1} + y_{2,2}c_{1,2} + y_{2,3}c_{2,1} + (y_{2,1} + 2y_{2,4})c_{2,2}+\\
&(y_{2,2} + 2y_{2,5})c_{3,1} + (y_{2,3} + 2y_{2,6} - 1)c_{3,2} + (y_{2,1} + 2y_{2,7})c_{4,1}+\\
&(y_{2,2} + y_{2,3} + 2y_{2,8})c_{4,2} + (y_{2,3} + 2y_{2,9})c_{5,1} + (y_{2,1} + y_{2,2} + 2y_{2,10})c_{5,2}+\\
&(y_{2,2} + 2y_{2,11})c_{6,1} + (y_{2,1} + y_{2,3} + 2y_{2,12} - 1)c_{6,2}+\\
&(y_{2,1} + y_{2,3} + 2y_{2,13})c_{7,1} + (y_{2,1} + y_{2,2} + 2y_{2,14})c_{7,2} \equiv 0 \pmod{32G'},
\end{aligned}$$

$$\begin{aligned}
&y_{3,1}c_{1,1} + y_{3,2}c_{1,2} + y_{3,3}c_{2,1} + (y_{3,1} + 2y_{3,4} - 1)c_{2,2}+\\
&(y_{3,2} + 2y_{3,5} - 1)c_{3,1} + (y_{3,3} + 2y_{3,6})c_{3,2} + (y_{3,1} + 2y_{3,7})c_{4,1}+\\
&(y_{3,2} + y_{3,3} + 2y_{3,8})c_{4,2} + (y_{3,3} + 2y_{3,9})c_{5,1} + (y_{3,1} + y_{3,2} + 2y_{3,10} - 1)c_{5,2}+\\
&(y_{3,2} + 2y_{3,11})c_{6,1} + (y_{3,1} + y_{3,3} + 2y_{3,12})c_{6,2}+\\
&(y_{3,1} + y_{3,3} + 2y_{3,13})c_{7,1} + (y_{3,1} + y_{3,2} + 2y_{3,14})c_{7,2} \equiv 0 \pmod{32G'}.
\end{aligned}$$

By Lemma 4.1 this three congruences leads us to three systems of congruences (mod $8G'$):

$$\begin{cases}
y_{1,1}c_{1,1} + (y_{1,2} + 1)c_{1,2} \equiv 0\\
(y_{1,3} - 1)c_{2,1} + (y_{1,1} + 2y_{1,4})c_{2,2} \equiv 0\\
(y_{1,2} + 2y_{1,5})c_{3,1} + (y_{1,3} + 2y_{1,6})c_{3,2} \equiv 0\\
(y_{1,1} + 2y_{1,7})c_{4,1} + (y_{1,2} + y_{1,3} + 2y_{1,8} - 1)c_{4,2} \equiv 0\\
(y_{1,3} + 2y_{1,9})c_{5,1} + (y_{1,1} + y_{1,2} + 2y_{1,10})c_{5,2} \equiv 0\\
(y_{1,2} + 2y_{1,11})c_{6,1} + (y_{1,1} + y_{1,3} + 2y_{1,12})c_{6,2} \equiv 0\\
(y_{1,1} + y_{1,3} + 2y_{1,13})c_{7,1} + (y_{1,1} + y_{1,2} + 2y_{1,14})c_{7,2} \equiv 0
\end{cases} \tag{36}$$

$$(37)\quad\begin{cases} (y_{2,1}+1)c_{1,1}+y_{2,2}c_{1,2}\equiv 0\\ y_{2,3}c_{2,1}+(y_{2,1}+2y_{2,4})c_{2,2}\equiv 0\\ (y_{2,2}+2y_{2,5})c_{3,1}+(y_{2,3}+2y_{2,6}-1)c_{3,2}\equiv 0\\ (y_{2,1}+2y_{2,7})c_{4,1}+(y_{2,2}+y_{2,3}+2y_{2,8})c_{4,2}\equiv 0\\ (y_{2,3}+2y_{2,9})c_{5,1}+(y_{2,1}+y_{2,2}+2y_{2,10})c_{5,2}\equiv 0\\ (y_{2,2}+2y_{2,11})c_{6,1}+(y_{2,1}+y_{2,3}+2y_{2,12}-1)c_{6,2}\equiv 0\\ (y_{2,1}+y_{2,3}+2y_{2,13})c_{7,1}+(y_{2,1}+y_{2,2}+2y_{2,14})c_{7,2}\equiv 0 \end{cases}$$

$$(38)\quad\begin{cases} y_{3,1}c_{1,1}+y_{3,2}c_{1,2}\equiv 0\\ y_{3,3}c_{2,1}+(y_{3,1}+2y_{3,4}-1)c_{2,2}\equiv 0\\ (y_{3,2}+2y_{3,5}-1)c_{3,1}+(y_{3,3}+2y_{3,6})c_{3,2}\equiv 0\\ (y_{3,1}+2y_{3,7})c_{4,1}+(y_{3,2}+y_{3,3}+2y_{3,8})c_{4,2}\equiv 0\\ (y_{3,3}+2y_{3,9})c_{5,1}+(y_{3,1}+y_{3,2}+2y_{3,10}-1)c_{5,2}\equiv 0\\ (y_{3,2}+2y_{3,11})c_{6,1}+(y_{3,1}+y_{3,3}+2y_{3,12})c_{6,2}\equiv 0\\ (y_{3,1}+y_{3,3}+2y_{3,13})c_{7,1}+(y_{3,1}+y_{3,2}+2y_{3,14})c_{7,2}\equiv 0 \end{cases}$$

5. Proof of Theorem A.

Let $G(a_1,a_2,a_3)\in\mathcal{M}(3,2)$ is torsion-free. By corollary 3.3 there are three relations between a_i and b_j and, as we have seen in the section above, it leads to three systems of congruences (mod $8G'$) between $c_{i,j}$: (36)–(38). Although we do not know exact values of coefficients $y_{i,j}$ and so can not find all relations between $c_{i,j}$; nevertheless, as we will show in this section, we can find sufficiently many relations between $c_{i,j}$ modulo $8G'$ and knowing only the parity of coefficients $y_{i,j}$, $i,j=1,2,3$ we will be able to obtain a contradiction with the existence of a torsion-free group in $\mathcal{M}(3,2)$. So we have to examine $2^9=512$ cases accordingly to parity of $y_{i,j}$, $i,j=1,2,3$. We have done this with a help of computer and obtained a listing of over than forty pages. Then we studied this listing, found many common parties, assembled all results in four pages, verified them by hand, and posed in the article as application. Below we consider in detail only sixteen cases gathered in one block.

We use the binary code of a sequence $\bar{y}_{1,1}\bar{y}_{1,2}\bar{y}_{1,3}\bar{y}_{2,1}\bar{y}_{2,2}\bar{y}_{2,3}\bar{y}_{3,1}\bar{y}_{3,2}\bar{y}_{3,3}$, where $\bar{y}_{i,j}$ denotes the parity of $y_{i,j}$. Thus (000 000 000) does the case 0, (100 000 000) does the case 256, (111 111 111) does the case 511. When the parity of some $\bar{y}_{i,j}$ does not matter we use $*$ in the position of this variable: thus $(\bar{y}_{1,1}\bar{y}_{1,2}\bar{y}_{1,3}\ \bar{y}_{2,1}\bar{y}_{2,2}\bar{y}_{2,3}\ \bar{y}_{3,1}\bar{y}_{3,2}*)$ denotes that we obtain the same result both in the case of even $y_{3,3}$ and in the case of odd $y_{3,3}$.

0–3, 8–11, 16–19, 24–27 (000 0$**$ 0$**$) — coefficients $y_{1,1},y_{1,2},y_{1,3},y_{2,1},y_{3,1}$ are even, the others are arbitrary.

The first congruences from (36) and (37) give us after substituting $y_{1,j}=2z_{1,j}$, $j=1,2,3$, $y_{2,1}=2z_{2,1}$:

$$\begin{cases} 2z_{1,1}c_{1,1}+(2z_{1,2}+1)c_{1,2}\equiv 0\\ (2z_{2,1}+1)c_{1,1}+y_{2,2}c_{1,2}\equiv 0 \end{cases}$$

Because the determinant of this system is odd that is relatively prime with 8, then

$$(39)\qquad c_{1,1},c_{1,2}\equiv 0$$

As above, the second congruences from (36) and (38) give a system with odd determinant as well

$$\begin{cases} (2z_{1,3}-1)c_{2,1}+(2z_{1,1}+2y_{1,4})c_{2,2}\equiv 0\\ y_{3,3}c_{2,1}+(2z_{3,1}+2y_{3,4}-1)c_{2,2}\equiv 0 \end{cases}$$

So

(40) $$c_{2,1}, c_{2,2} \equiv 0$$

And the 4-th congruence from (36) gives:

(41) $$c_{4,2} \equiv 2t_{4,1}c_{4,1},$$

for suitable $t_{4,1}$. Remember that all congruences here and further are modulo $8G'$.

Comparing representations (22) with relations (39) – (41) we see that

(42) $$4a_1^2 \equiv 2t_{4,1}c_{4,1};$$

Using (39), (40) and (42) we substitute in the right side of (29): $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2} \equiv 0$, $4a_1^2 \equiv 2t_{4,1}c_{4,1}$, $k_1, k_4 = 0$, $k_9 = t_{4,1}$ and obtain

$$4(a_1u)^2 \cong 2t_{4,1}c_{4,1} - 2t_{4,1}c_{4,1} = 0,$$

so $(a_1u)^8 = w^8$, $u, w \in G'$. By Lemma 2.1 $a_1 \in G'$, it is a contradiction with (1). □

6. Application

Here we collect all cases arising from the parity of $y_{i,j}$, $i, j = 1, 2, 3$ gathered in blocks having common parts. We write in this blocks relations between $c_{i,j}$ that follow from (36) – (38) accordingly to the parity of $y_{i,j}$ and we close the blocks writing a congruence (mod $8G'$) $a_i^2 \cong 2kc_{j,s}$, where $j \in J_i$, $i = 1, \ldots, 7$, $s = 1, 2$. It is sufficient to obtain a contradiction with (1) applying suitable congruence from (29) – (35) and Lemma 2.1.

0–3, 8–11, 16–19, 24–27 (000 0∗∗ 0∗∗) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2} \equiv 0$, $c_{4,2} \equiv 2t_{4,1}c_{4,1}$, $a_1^8 \equiv 2kc_{4,1}$

4–7, 12–15, 20–23, 28–31 (000 0∗∗ 1∗∗) $c_{1,1}, c_{1,2}, c_{4,1}, c_{4,2} \equiv 0$, $c_{2,1} \equiv 2t_{2,2}c_{2,2}$, $a_1^8 \equiv 2kc_{2,2}$

32–63 (000 1∗∗ ∗∗∗) $c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0$, $c_{1,2} \equiv 2t_{1,1}c_{1,1}$, $a_1^8 \equiv 2kc_{1,1}$

64–65, 68–69, 72–73, 76–77 (001 00∗ ∗0∗) $c_{1,1}, c_{1,2}, c_{3,1}, c_{3,2} \equiv 0$, $c_{6,2} \equiv 2t_{6,1}c_{6,1}$, $a_2^8 \equiv 2kc_{6,1}$

66–67, 70–71, 74–75, 78–79 (001 00∗ ∗1∗) $c_{1,1}, c_{1,2}, c_{6,1}, c_{6,2} \equiv 0$, $c_{3,2} \equiv 2t_{3,1}c_{3,1}$, $a_2^8 \equiv 2kc_{3,1}$

80–95 (001 01∗ ∗∗∗) $c_{1,1}, c_{1,2}, c_{3,1}, c_{3,2}, c_{6,1}, c_{6,2} \equiv 0$, $a_2^8 \equiv 0$

96, 104, 112, 120 (001 1∗∗ 000) $c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0$, $c_{2,2} \equiv 2t_{2,1}c_{2,1}$, $a_3^8 \equiv 2kc_{2,1}$

97, 105 (001 10∗ 001) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0$, $a_6^8 \equiv 0$

98, 106 (001 10∗ 010) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0$, $a_7^8 \equiv 0$

99, 107, 115, 123 (001 1∗∗ 011) $c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0$, $c_{2,1} \equiv (2t_{2,2}+1)c_{2,2}$, $a_5^8 \equiv 2kc_{2,2}$

100–101, 108 - 109 (001 10∗ 10∗) $c_{1,1}, c_{1,2}, c_{3,1}, c_{3,2} \equiv 0,\ c_{6,2} \equiv 2t_{6,1}c_{6,1}, \quad a_2^8 \equiv 2kc_{6,1}$

102–103, 110 - 111 (001 10∗ 11∗) $c_{1,1}, c_{1,2}, c_{6,1}, c_{6,2} \equiv 0,\ c_{3,2} \equiv 2t_{3,1}c_{3,1}, \quad a_2^8 \equiv 2kc_{3,1}$

113, 121 (001 11∗ 001) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0, \quad a_7^8 \equiv 0$

114, 122 (001 11∗ 010) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0, \quad a_6^8 \equiv 0$

116–119, 124–127 (001 11∗ 1∗∗) $c_{1,1}, c_{1,2}, c_{3,1}, c_{3,2}, c_{6,1}, c_{6,2} \equiv 0, \quad a_2^8 \equiv 0$

128–135, 192–199 (01∗ 000 ∗∗∗) $c_{3,1}, c_{3,2}, c_{6,1}, c_{6,2} \equiv 0,\ c_{1,1}, \equiv 2t_{1,2}c_{1,2}, \quad a_2^8 \equiv 2kc_{1,2}$

136–139 (010 001 0∗∗) $c_{2,1}, c_{2,2}, c_{5,1}, c_{5,2} \equiv 0,\ c_{3,1}, \equiv 2t_{3,2}c_{3,2}, \quad a_3^8 \equiv 2kc_{3,2}$

140, 142 (010 001 1∗0) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0, \quad a_7^8 \equiv 0$

141, 143 (010 001 1∗1) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0, \quad a_6^8 \equiv 0$

144–147 (010 010 0∗∗) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2} \equiv 0,\ c_{5,2} \equiv 2t_{5,1}c_{5,1}, \quad a_3^8 \equiv 2kc_{5,1}$

148, 150 (010 010 1∗0) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0, \quad a_6^8 \equiv 0$

149, 151 (010 010 1∗1) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0, \quad a_7^8 \equiv 0$

152–159 (010 011 ∗ ∗ ∗) $c_{5,1}, c_{5,2}, c_{7,1}, c_{7,2} \equiv 0,\ c_{1,1} \equiv (2t_{1,2}+1)c_{1,2}, \quad a_4^8 \equiv 0$

160–167, 176–183 (010 1∗0 ∗∗∗) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2} \equiv 0,\ c_{5,2} \equiv 2t_{5,1}c_{5,1}, \quad a_3^8 \equiv 2kc_{5,1}$

168–175, 184–191 (010 1∗1 ∗∗∗) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2} \equiv 0,\ c_{7,2} \equiv 2t_{7,1}c_{7,1}, \quad a_5^8 \equiv 2kc_{7,1}$

200, 216 (011 0∗1 000) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0, \quad a_3^8 \equiv 0$

201, 217 (011 0∗1 001) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0, \quad a_5^8 \equiv 0$

202, 218 (011 0∗1 010) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0, \quad a_5^8 \equiv 0$

203, 219 (011 0∗1 011) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0, \quad a_3^8 \equiv 0$

204–207 (011 001 1∗∗) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2} \equiv 0,\ c_{6,1} \equiv (2t_{6,2}+1)c_{6,2}, \quad a_7^8 \equiv 2kc_{6,2}$

208–215 (011 010 ∗ ∗ ∗) $c_{5,1}, c_{5,2}, c_{7,1}, c_{7,2} \equiv 0,\ c_{1,1} \equiv (2t_{1,2}+1)c_{1,2}, \quad a_4^8 \equiv 2kc_{1,2}$

220–223 (011 011 1∗∗) $c_{4,1}, c_{4,2}, c_{6,1}, c_{6,2} \equiv 0,\ c_{5,1} \equiv (2t_{5,2}+1)c_{5,2}, \quad a_7^8 \equiv 2kc_{5,2}$

224–231 (011 100 ∗ ∗ ∗) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2} \equiv 0,\ c_{6,1} \equiv (2t_{6,2}+1)c_{6,2}, \quad a_7^8 \equiv 2kc_{6,2}$

232–239 (011 101 ∗ ∗ ∗) $c_{4,1}, c_{4,2}, c_{6,1}, c_{6,2} \equiv 0,\ c_{5,1} \equiv (2t_{5,2}+1)c_{5,2}, \quad a_7^8 \equiv 2kc_{5,2}$

240–247 (011 110 $***$) $c_{4,1}, c_{4,2}, c_{6,1}, c_{6,2} \equiv 0$, $c_{5,1} \equiv (2t_{5,2}+1)c_{5,2}$, $a_7^8 \equiv 2kc_{5,2}$

248–255 (011 111 $***$) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2} \equiv 0$, $c_{6,1} \equiv (2t_{6,2}+1)c_{6,2}$, $a_7^8 \equiv 2kc_{6,2}$

256–257, 260–261, 320–321, 324–325 (10$*$ 000 $*$0$*$) $c_{1,1}, c_{1,2}, c_{3,1}, c_{3,2} \equiv 0$, $c_{6,2} \equiv 2t_{6,1}c_{6,1}$, $a_2^8 \equiv 2kc_{6,1}$

258–259, 262–263, 322–323, 326–327 (10$*$ 000 $*$1$*$) $c_{1,1}, c_{1,2}, c_{6,1}, c_{6,2} \equiv 0$, $c_{3,2} \equiv 2t_{3,1}c_{3,1}$, $a_2^8 \equiv 2kc_{3,1}$

264–271, 328–335 (10$*$ 001 $***$) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0$, $a_1^8 \equiv 0$

272–279, 304–311 (100 $*$10 $***$) $c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0$, $c_{2,1} \equiv (2t_{2,2}+1)c_{2,2}$, $a_5^8 \equiv 2kc_{2,2}$

280–287, 312–319 (100 $*$11 $***$) $c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0$, $c_{4,1} \equiv (2t_{4,2}+1)c_{4,2}$, $a_7^8 \equiv 2kc_{4,2}$

288–289, 292–293 (100 100 $*$0$*$) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2} \equiv 0$, $c_{5,2} \equiv 2t_{5,1}c_{5,1}$, $a_3^8 \equiv 2kc_{5,1}$

290, 294 (100 100 $*$10) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0$, $a_5^8 \equiv 0$

291, 295 (100 100 $*$11) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0$, $a_7^8 \equiv 0$

296–303 (100 101 $***$) $c_{5,1}, c_{5,2}, c_{7,1}, c_{7,2} \equiv 0$, $c_{1,1} \equiv (2t_{1,2}+1)c_{1,2}$, $a_4^8 \equiv 2kc_{1,2}$

336–343 (101 010 $***$) $c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0$, $c_{2,2} \equiv 2t_{2,1}c_{2,1}$, $a_3^8 \equiv 2kc_{2,1}$

344–351 (101 011 $***$) $c_{2,1}, c_{2,2}, c_{7,1}, c_{7,2} \equiv 0$, $c_{6,1} \equiv 2t_{6,2}c_{6,2}$, $a_5^8 \equiv 2kc_{6,2}$

352–359 (101 100 $***$) $c_{5,1}, c_{5,2}, c_{7,1}, c_{7,2} \equiv 0$, $c_{1,1} \equiv (2t_{1,2}+1)c_{1,2}$, $a_4^8 \equiv 2kc_{1,2}$

360 (101 101 000) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0$, $a_3^8 \equiv 0$

361 (101 101 001) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0$, $a_6^8 \equiv 0$

362–363 (101 101 01$*$) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0$, $a_1^8 \equiv 0$

364–365 (101 101 10$*$) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0$, $a_1^8 \equiv 0$

366–367 (101 101 11$*$) $c_{4,1}, c_{4,2}, c_{6,1}, c_{6,2} \equiv 0$, $c_{5,1} \equiv (2t_{5,2}+1)c_{5,2}$, $a_7^8 \equiv 2kc_{5,2}$

368–375 (101 110 $***$) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0$, $a_6^8 \equiv 0$

376–383 (101 111 $***$) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0$, $a_3^8 \equiv 0$

384–391, 448–455 (11$*$ 000 $***$) $c_{3,1}, c_{3,2}, c_{6,1}, c_{6,2} \equiv 0$, $c_{1,1} \equiv 2t_{1,2}c_{1,2}$, $a_2^8 \equiv 2kc_{1,2}$

392–393 (110 001 00∗) $c_{2,1}, c_{2,2}, c_{5,1}, c_{5,2} \equiv 0,\ \ c_{3,1} \equiv 2t_{3,2}c_{3,2},\quad a_3^8 \equiv 2kc_{3,2}$

394–395, 398–399, 458–459, 462–463 (11∗ 001 ∗1∗) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0,$ $a_1^8 \equiv 0$

396 (110 001 100) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0,\quad a_5^8 \equiv 0$

397 (110 001 101) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0,\quad a_6^8 \equiv 0$

400–407 (110 010 ∗∗∗) $c_{3,1}, c_{3,2}, c_{4,1}, c_{4,2}, c_{7,1}, c_{7,2} \equiv 0,\quad a_6^8 \equiv 0$

408–415 (110 011 ∗∗∗) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0,\quad a_5^8 \equiv 0$

416–423 (110 100 ∗∗∗) $c_{3,1}, c_{3,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{4,1} \equiv 2t_{4,2}c_{4,2},\quad a_6^8 \equiv 2kc_{4,2}$

424–431 (110 101 ∗∗∗) $c_{6,1}, c_{6,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{2,1} \equiv (2t_{2,2}+1)c_{2,2},\quad a_5^8 \equiv 2kc_{2,2}$

432–439 (110 110 ∗∗∗) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0,\quad a_1^8 \equiv 0$

440–441 (110 111 00∗) $c_{1,1}, c_{1,2}, c_{5,1}, c_{5,2} \equiv 0,\ \ c_{7,1} \equiv 2t_{7,2}c_{7,2},\quad a_4^8 \equiv 2kc_{7,2}$

442–443 (110 111 01∗) $c_{1,1}, c_{1,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{5,1} \equiv 2t_{5,2}c_{5,2},\quad a_4^8 \equiv 2kc_{5,2}$

444–445 (110 111 10∗) $c_{1,1}, c_{1,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{5,1} \equiv 2t_{5,2}c_{5,2},\quad a_4^8 \equiv 2kc_{5,2}$

446–447 (110 111 11∗) $c_{1,1}, c_{1,2}, c_{5,1}, c_{5,2} \equiv 0,\ \ c_{7,1} \equiv 2t_{7,2}c_{7,2},\quad a_4^8 \equiv 2kc_{7,2}$

456 (111 001 000) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0,\quad a_3^8 \equiv 0$

457 (111 001 001) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0,\quad a_7^8 \equiv 0$

460–461 (111 001 10∗) $c_{2,1}, c_{2,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{6,1} \equiv 2t_{6,2}c_{6,2},\quad a_5^8 \equiv 2kc_{6,2}$

464–471 (111 010 ∗∗∗) $c_{4,1}, c_{4,2}, c_{5,1}, c_{5,2}, c_{6,1}, c_{6,2} \equiv 0,\quad a_7^8 \equiv 0$

472–479 (111 011 ∗∗∗) $c_{2,1}, c_{2,2}, c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0,\quad a_3^8 \equiv 0$

480–487 (111 100 ∗∗∗) $c_{3,1}, c_{3,2}, c_{5,1}, c_{5,2} \equiv 0,\ \ c_{2,2} \equiv 2t_{2,1}c_{2,1}\quad a_3^8 \equiv 2kc_{2,1}$

488–495 (111 101 ∗∗∗) $c_{2,1}, c_{2,2}, c_{6,1}, c_{6,2} \equiv 0,\ \ c_{7,2} \equiv 2t_{7,1}c_{7,1}\quad a_5^8 \equiv 2kc_{7,1}$

496–497 (111 110 00∗) $c_{1,1}, c_{1,2}, c_{5,1}, c_{5,2} \equiv 0,\ \ c_{7,1} \equiv 2t_{7,2}c_{7,2}\quad a_4^8 \equiv 2kc_{7,2}$

498–499 (111 110 01∗) $c_{1,1}, c_{1,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{5,1} \equiv 2t_{5,2}c_{5,2}\quad a_4^8 \equiv 2kc_{5,2}$

500–501 (111 110 10∗) $c_{1,1}, c_{1,2}, c_{7,1}, c_{7,2} \equiv 0,\ \ c_{5,1} \equiv 2t_{5,2}c_{5,2}\quad a_4^8 \equiv 2kc_{5,2}$

502–503 (111 110 11$*$) $c_{1,1}, c_{1,2}, c_{5,1}, c_{5,2} \equiv 0, \quad c_{7,1} \equiv 2t_{7,2}c_{7,2} \quad a_4^8 \equiv 2kc_{7,2}$

504–511 (111 111 $***$) $c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{4,1}, c_{4,2} \equiv 0, \quad a_1^8 \equiv 0$

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