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MINIMAX DEGREES OF QUASIPLANE GRAPHS WITHOUT 4-FACES

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Abstract. The *M-degree* of an edge xy in a graph is the maximum of the degrees of x and y. The *minimax degree* of a graph G is the minimum over M-degrees of its edges. In order to get upper bounds on the game chromatic number, W. He et al showed that every planar graph G without leaves and 4-cycles has minimax degree at most 8. This was improved by Borodin et al to the best possible bound 7. Answering a question by D. West, we show that every plane graph G without leaves and 4-faces has minimax degree at most 15. The bound is sharp. Similar results are obtained for graphs embeddable on the projective plane, torus and Klein bottle.

1. Introduction

By *quasiplane* graphs we mean those embedded on a surface with nonnegative Euler characteristics, i.e., the plane, the projective plane, the torus or the Klein bottle. For an edge xy in a graph G, the *maximum degree* (for short, M-degree) $M(xy)$ is the maximum of the degrees of x and y. The *minimax degree* (for short, M-degree) of a graph G is $M^*(G) = \min\{M(xy) | xy \in E(G)\}$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of a graph G, respectively.

Wernicke [11] proved that $M^*(G) \leq 6$ for every planar graph G with $\delta(G) \geq 5$. Kotzig [10] proved that $M^*(G) \leq 7$ for every planar graph G with $\delta(G) \geq 4$. Borodin [4] showed that $M^*(G) \leq 10$ for every planar graph G with $\delta(G) \geq 3$, extending Kotzig's similar result [9] on 3-polytopes. (This last upper bound was

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conjectured by P. Erdös and announced by D. Barnette (see Grünbaum [7], p. 454) to be true but seems to have never been published.) All these bounds are tight.

The M-degree of planar graphs with $\delta(G) \geq 2$ is not bounded from above. For example, $M^*(K_{2,n}) = n$. Note that every cycle in $K_{2,n}$ has length four.

By $d(v)$ denote the degree of a vertex v. An induced cycle $C = v_1v_2 \dots v_{2k}$ in a graph G is called 2-*alternating* if $d(v_1) = d(v_3) = \dots = d(v_{2k-1}) = 2$. Note that every cycle in $K_{2,n}$ is 2-alternating. Suppose G is a planar graph with $\delta(G) \geq 2$. If it has no 2-alternating 4-cycles, then $M^*(G) \leq 15$ (see [2]), and if G has no 2-alternating cycles at all, then $M^*(G) \leq 13$ (see [3]), where both bounds are tight.

He, Hou, Lih, Shao, Wang and Zhu [8] found upper bounds on M-degrees of planar graphs with $\delta(G) \geq 2$ and restrictions on girth. They used these bounds to estimate from above the game chromatic number (introduced by Bodlander [1]) and the game coloring number of such graphs. In particular, one of the main results in [8], Theorem 2.2, says that $M^*(G) \leq 8$ for planar graphs G having no leaves and 4-cycles. This result yields that every C_4-free planar graph can be decomposed into a forest and a graph with maximum degree at most 7, which in turn implies that the game chromatic number, $\chi_g(G)$, and the game coloring number of every C_4-free planar graph G are at most 11. (Indeed, the game chromatic number of a tree is at most 4, as proved in [6], and it is easy to see that if the edges of G are partitioned into a tree and a graph H, then $\chi_g(G) \leq 4+\Delta(H)$.) It is also mentioned in [8] that it is not known whether 8 is the exact bound and that the M-degree of dodecahedron is 3.

In [5], Borodin et al determined the exact upper bounds on M-degrees for C_4-free graphs G with $\delta(G) \geq 2$ embeddable into the plane and projective plane to be 7, and those embeddable into the torus and the Klein bottle to be 8.

Douglas West asked for similar upper bounds for M-degrees of quasi-planar graphs without 4-faces. The main result of this paper is

Theorem 1. *Let G be a graph without leaves. If G can be embedded into the plane or the projective plane without* 4*-faces, then $M^*(G) \leq 15$. If G can be embedded into the torus or the Klein bottle without* 4*-faces, then $M^*(G) \leq 18$. Both bounds are sharp.*

We do not allow loops and multiple edges because otherwise M^* may be arbitrarily large under the absence of 4-faces.

Let $N(S)$ denote the Euler characteristics of a surface S. Recall that $N(S) = 2$ if S is the plane, $N(S) = 1$ if S is the projective plane, and $N(S) = 0$ if S is the torus or the Klein bottle; for the other surfaces $N(S) < 0$.

In [5], we prove that any graph G without 4-cycles and leaves embedded into a surface S, where $N(S) < 0$, and having more than $-72N(S)$ edges has $M^*(G) \leq 8$.

From the proof of Theorem 1, we deduce the following fact.

Theorem 2. *Every graph G without leaves embedded without* 4*-faces into a surface S with $N(S) < 0$ and having more than $-342N(S)$ edges has $M^*(G) \leq 18$.*

Thus, a large graph without 4-faces on a fixed surface S with $N(S) < 0$ behaves in terms of M^* as a graph embedded into the torus or the Klein bottle.

2. Proof of Theorem 1

We first show that the bounds on $M^*(G)$ are sharp. Let G' be either the icosahedron graph embedded into the plane or a 6-regular triangulation of the torus or the Klein bottle. For every edge $e = ab$ in G', we add 2-vertices u and w adjacent with a and b so that e becomes the common edge of 3-faces abu and abw. Clearly, the graph G obtained has only 3- and 6-faces, and $M^*(G)$ is 15 when G' is the icosahedron and 18 otherwise.

Let G be a counterexample to Theorem 1; i.e., a graph embedded on a surface S (with $N(S) \geq 0$) without 4-faces with $M^*(G) \geq 16$ if $N(S) > 0$ and $M^*(G) \geq 19$ if $N(S) = 0$. We want to construct from G another counterexample, H, with additional structural properties.

Let B denote the set of vertices in G of degree at least $M^*(G)$. By a *B-vertex* we mean a vertex in B. By definition, every edge of G is incident with a B-vertex. A vertex is *minor* if it is neither a 2-vertex nor a B-vertex.

We first exclude all minor vertices from our counterexample G to obtain the counterexample G', as follows. Let a minor vertex v be adjacent to B-vertices $v_1, v_2, v_3, \dots, v_{d(v)}$ clockwise. Then we delete v and, for $i = 1, \dots, d(v)$, add a new 2-vertex $v_{i,i+1}$ adjacent to v_i and v_{i+1} so that in the embedding we have a new ≥ 6-face $(v_1 v_{1,2} v_2 v_{2,3} \dots v_{d(v),1})$, while the sizes of old faces do not change. Observe that in G' each edge is still incident with a B-vertex.

Next, we exclude 3-faces incident with three B-vertices. If uvw is such a 3-face, we simply insert three new 2-vertices inside the face, connecting one of them to u and v, the second to v and w, and the third to w and u. The resulting counterexample to Theorem 1 is denoted by G''.

Finally, for each nontriangular face f in G'' and each edge uv on the boundary of f with $u, v \in B$, we add a new vertex into f and connect it by edges with u and v. The resulting graph H embedded into S is still a counterexample to Theorem 1 and has the following properties:
(a) every vertex of H is either a 2-vertex or a B-vertex;
(b) every 3-face of H is incident with exactly one 2-vertex;
(c) for every nontriangular face f of H, the vertices on the boundary of f are alternately 2-vertices and B-vertices. In particular, every nontriangular face has an even size.

If H has no edges, we have nothing to prove; otherwise, we can assume that H is connected. By Euler's formula $|V(H)| - |E(H)| + |F(H)| \geq N(S)$, we have

$$(1) \qquad \sum_{x \in V(H) \cup F(H)} (d(x) - 4) = \sum_{x \in V(H) \cup F(H)} \mu(x) \leq -4N(S).$$

We will use discharging to obtain a contradiction with the properties of H. Let the *initial charge* of every $x \in V(H) \cup F(H)$ be $\mu(x) = d(x) - 4$. The vertices and faces of H discharge their initial charge by the following rules:

Rule 1. Each triangular face gets $\frac{5}{6}$ from each incident B-vertex.
Rule 2. Every face f gives every incident 2-vertex charge $\frac{2}{3}$.
Rule 3. Every B-vertex gives $\frac{1}{3}$ to each adjacent 2-vertex.

It remains to show that the *final charge* $\mu^*(y)$ is nonnegative for each $y \in V(H) \cup F(H)$, and that the final charge of every vertex of degree at least 19 is strictly positive. This yields a contradiction to (1), since the total charge does not change, and hence should be strictly negative when H is projective-plane and non-positive when G is embedded into the torus or the Klein bottle.

If y is a 3-face, then $\mu^*(y) = 3 - 4 + 2 \times \frac{5}{6} - \frac{2}{3} = 0$ by Rules 1 and 2.

Suppose y is a face with $d(y) \geq 5$. By Property (c) of H, $d(y) \geq 6$ and y has exactly $d(y)/2$ incident 2-vertices. Hence $\mu^*(y) \geq d(y) - 4 - \frac{d(y)}{2} \times \frac{2}{3} = \frac{2d(y)}{3} - 4 \geq 0$ by Rule 2.

Now suppose that y is a vertex.

If $d(y) = 2$, then it gets $\frac{2}{3}$ from the adjacent vertices by Rule 3 and $\frac{4}{3}$ from the incident faces by Rule 2. Hence, $\mu^*(y) = 0$.

Suppose y is a B-vertex. We want to estimate the maximum total expenditure of y by Rules 1 and 3. Let $f_1, f_2, f_3, \ldots, f_{d(y)}$ be consecutive faces at y. Note that 3-faces at y appear in pairs: if $f_2 = xyz$ is a triangle, then precisely one of f_1, f_3 is a triangle, too. Indeed, by Properties (a)–(c) of H, either $d(x) = 2$ and z is a B-vertex, or vice versa. Suppose f_1 is adjacent to f_2 along a BB-edge. Then by (c), f_1 is a 3-face, while f_3 cannot be a 3-face since H has no multiple edges. In turn, f_1 is incident with a 2-vertex, hence f_1 cannot be adjacent to a 3-face other than f_2.

So, let y be incident with k couples of triangular faces and l nontriangular edges. Of course, $3k + l = d(y)$. Every couple causes y the loss of charge at most $2 \times \frac{1}{3} + 2 \times \frac{5}{6} = \frac{7}{3}$ by Rules 1 and 3; i.e., each of the three edges in a couple takes $\frac{7}{9}$ away from y on the average. Recall that a nontriangular edge takes at most $1/3$ away from y. This implies that the total expenditure of y is at most $\frac{7d(y)}{9}$, so that $\mu^*(y) \geq d(y) - 4 - \frac{7d(y)}{9} = \frac{2(d(y)-18)}{9}$.

Thus, we have proved that $\mu^*(y) \geq 0$ if $d(y) \geq 18$ and $\mu^*(y) > 0$ if $d(y) \geq 19$. For $d(y) = 17$ we have $\mu^*(y) \geq 17 - 4 - 15 \times \frac{7}{9} - 2 \times \frac{1}{3} > 0$, while $\mu^*(y) \geq 16 - 4 - 15 \times \frac{7}{9} - 1 \times \frac{1}{3} = 0$ if $d(y) = 16$.

This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let H be a modified counterexample to Theorem 2 (see the proof of Theorem 1). Then, as shown a few lines above, each B-vertex v in H has $\mu^*(v) \geq \frac{2(d(v)-18)}{9} > 0$. For each B-vertex v, we distribute $\mu^*(v)$ evenly among the edges incident with v, i.e. by $\frac{2(d(y)-18)}{9d(y)} \geq \frac{2}{171}$.

Let $\nu(x)$ denote the new charge of an element $x \in V(H) \cup E(H) \cup F(H)$. Then $\nu(v) = 0$ for every $v \in V(H)$. Since the charges of faces did not change, we have $\nu(f) = \mu^*(f) \geq 0$ for every $f \in F(H)$. Note that every edge e of H is incident with a B-vertex, and hence $\nu(e) \geq \frac{2}{171}$. Now (1) implies

$$|E(H)| \times \frac{2}{171} \leq \sum_{x \in E(H) \cup F(H)} \nu(x) = \sum_{x \in V(H) \cup F(H)} \mu^*(x) \leq -4N(S),$$

which contradicts the assumption that $|E(H)| > -342N(S)$.

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