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THE WHITEHEAD CONJECTURE – AN OVERVIEW

STEPHAN ROSEBROCK

Abstract. These notes are an elaboration of a talk held November 3, 2006 at the "Metzler Fest" in honour of Wolfgang Metzler's 65-th birthday at the university of Frankfurt. The aim is to give an overview of results concerning Whitehead's asphericity conjecture.

1. Introduction to the Whitehead Conjecture

A 2-complex K is called *aspherical* if its second homotopy group is trivial ($\pi_2(K) = 0$). This means, every continuous map $f\colon S^2 \to K$ is homotopy equivalent to the trivial map where the 2-sphere S^2 is mapped to a single point.

Whitehead Conjecture [1941]:

(WH): Let L be an aspherical 2-complex. Then $K \subset L$ is also aspherical.

Whitehead posed this 1941 as a question (see [17]). A more algebraic introduction to the Whitehead Conjecture than the one given here may be found in [3].

Motivation at the time was the asphericity of knot complements:
Given any knot $k \subset S^3$, the space $S^3 - k$ is the *knot complement*. What is $\pi_2(S^3 - k)$?

Theorem 1.1. *(WH) implies the asphericity of knot complements.*

Proof. Glue a (thickened) meridian disk into $S^3 - k$ to get a 3-ball which collapses to an aspherical 2-complex. So if (WH) were true than the subcomplex which collapses from $S^3 - k$ has to be aspherical and the asphericity of knot complements were

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shown. □

The asphericity of knot complements was shown 1957 by Papakyriakopoulos with 3-manifold techniques but the Whitehead Conjecture remains open.

Equivalent to the Whitehead question is the following: Can we create π_2 by taking away a 2-cell?

Closely related is intuitively the following observation: We can make π_2 "bigger" by taking away a 2-cell:

Take $K = S^2 \vee S^1$ and let $L = K \cup_{S^1} D^2$

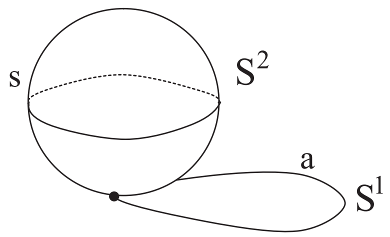

Рис. 1. $K = S^2 \vee S^1$

Let s be the element of $\pi_2(K)$ generated by the 2-sphere and a the element of $\pi_1(K)$ generated by S^1. Then $(a-1)[s]$ is nontrivial in $\pi_2(K)$ but trivial in $\pi_2(L)$. The inclusion $K \subset L$ does not induce an injective homomorphism $\pi_2(K) \to \pi_2(L)$.

Another elementary and interesting example is the following: Define K to be the projective plane modeled on the presentation $< x \,|\, x^2 >$. The second homotopy group of K is non-trivial and generated by $g = (1-x)[r]$ where r corresponds to the relator x^2. Define L to be the 2-complex modeled on $< x \,|\, x^2, x >$. K is a subcomplex of L. The element g and therefore the whole second homotopy group of K is killed in L. This is certainly not a counterexample to (WH) since $\pi_2(L) \neq 0$. New elements of the second homotopy group arise when the old ones are killed.

2. Elementary Results

Throughout this section we assume $K \subset L^2$ and $\pi_2(L) = 0$.

Theorem 2.1. *(WH) is true if:*

- *K has at most one 2-cell (Cockcroft 1954)*
- *$\pi_1(L)$ is finite and non-trivial, abelian or free*

(WH) is homologically true: Certainly $H_2(L) = 0 \Rightarrow H_2(K) = 0$ because L is 2-dimensional.

More general: The Hurewicz homomorphism $\pi_2(K) \to H_2(K)$ is trivial, because $\pi_2(K) \to H_2(K) \rightarrowtail H_2(L)$ factors through $\pi_2(L) = 0$.

This means, every spherical map $f\colon S^2 \to K$ is homologically trivial. Any 2-complex K with this property is called *Cockcroft*.

Theorem 2.2. *(Cockcroft 1954* [4]*): If* $\pi_1(K) \to \pi_1(L)$ *is injective then* $\pi_2(K) = 0$.

Proof. Let $\widetilde{L}$ be the universal cover of L and $p\colon \widetilde{L} \to L$ the corresponding covering projection. K is a subcomplex of L, so we can consider $\overline{K}$, a component of $p^{-1}(K)$.

$$\begin{array}{ccc} \overline{K} & \subset & \widetilde{L} \\ \downarrow & & \downarrow p \\ K & \subset & L \end{array}$$

First observe: $\overline{K}$ is a regular covering of K. If $p_\sharp\colon \pi_1(\overline{K}) \to \pi_1(K)$ is the corresponding map for the fundamental group, then

$$p_\sharp(\pi_1(\overline{K})) = ker(\pi_1(K) \to \pi_1(L)) = 0$$

since any loop in the kernel survives as loop in $\overline{K}$. This implies that $\overline{K} \to K$ is the universal cover.

The fundamental group of the universal cover is always trivial, so we have $\pi_2(K) \cong H_2(\overline{K}) < H_2(\widetilde{L}) \cong \pi_2(L) = 0$. □

So it is natural to ask: When is $\pi_1(K) \to \pi_1(L)$ injective? There we have a connection to equations over groups.

There is a stronger result:

Theorem 2.3. *(Howie 1979* [7]*): If* $ker(\pi_1(K) \to \pi_1(L))$ *has no nontrivial perfect subgroups then* $\pi_2(K) = 0$.

Let $\overline{K} \to K$ be the covering corresponding to the commutator subgroup. Then it is easy to see that $H_2(\overline{K}) = 0$. This result can be sharpened in the following way:

A 2-complex K^* is called *acyclic*, if $H_2(K^*) = H_1(K^*) = 0$.

Theorem 2.4. *(Adams 1955* [1]*): There is an acyclic regular covering:*

$$K^* \to \overline{K} \to K.$$

So a counterexample to (WH) can be covered by an acyclic complex but not by a contractible one.

3. Is the Whitehead Conjecture false?

There are some hints that (WH) may be false. One is from Geometric Group Theory:

The *cohomological dimension* $cd(G)$ of a group G is the smallest number n such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to \mathbb{Z} \to 0$$

over the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. Each P_i is a finitely generated projective $\mathbb{Z}G$-module.

The *geometric dimension* $gd(G)$ of a group G is the smallest number n such there exists a n-dimensional Eilenberg-McLane space $K(G, 1)$. The Eilenberg-McLane space $K = K(G, 1)$ satisfies $\pi_1(K) = G$ and $\pi_i(K) = 0$ for all $i > 1$.

It is easy to see that $gd(G) \geq cd(G)$ because the universal cover of a $K(G, 1)$ gives rise to a projective resolution.

Now the *Eilenberg-Ganea Conjecture* states $gd(G) = cd(G)$. This is only open for $cd(G) = 2$.

Theorem 3.1. *(Bestvina/Brady 1997* [2]*): Either (WH) or the Eilenberg-Ganea Conjecture is false.*

Another result of Howie characterizes Whitehead Counterexamples.

Observe first, if we had a Whitehead Counterexample $K \subset L^2$ and $\pi_2(L) = 0$ it can be assumed:

- L is obtained from K by attaching 2-cells
  since we can always add the 1-skeleton of L to K without changing its asphericity.
- K is finite.
  If K would be infinite and non-aspherical, then restrict K to the image of a nontrivial S^2-map.
- L is contractible.
  If L is not contractible take instead of L the universal cover $\tilde{L}$. We know $\pi_2(L) = \pi_2(\tilde{L})$.

Theorem 3.2. *(Howie 1983* [8]*): If (WH) is false then there exists a counterexample $K \subset L$ such that either:*

(a): *L is finite and contractible and $K = L - e$ for one 2-cell e.*

(b): *L is the union of an infinite chain of finite nonaspherical subcomplexes*

$$K = K_0 \subset K_1 \subset K_2 \subset \ldots$$

where each $K_i \subset K_{i+1}$ is nullhomotopic.

This is easy to prove, if the word "contractible" is replaced by "aspherical":

Proof. Assume (WH) is false and

1. L is finite. Add to K 2-cells of L as long as it stays non-aspherical. Define a new 2-complex, for simplicity also called L having only one 2-cell more. Then $K = L - e$.

2. Now assume (WH) is false but
(*) connected subcomplexes of finite aspherical 2-complexes are aspherical.

Select $Y = Y^2$ aspherical and $X \subset Y$ connected and non-aspherical. Let $\widetilde{Y}$ be the universal cover of Y and $p\colon \widetilde{Y} \to Y$ the corresponding covering projection. X is a subcomplex of Y, so we can consider $\overline{X}$, a component of $p^{-1}(X)$.

$$\begin{array}{ccc} \overline{X} & \subset & \widetilde{Y} \\ \downarrow & & \downarrow p \\ X & \subset & Y \end{array}$$

Since $\overline{X}$ is a component of $p^{-1}(X)$ we know $\pi_2(\overline{X}) \neq 0$. So we have a $K_0 \subset \overline{X}$ which is non-aspherical and finite.

Because $\widetilde{Y}$ is contractible it follows $K_0 \subset \widetilde{Y}$ is nullhomotopic. So the cone CK_0 over K_0 is finite and there is a map $CK_0 \to \widetilde{Y}$. The image K_1 under this map is a finite connected 2-complex and $K_0 \subset K_1$ is nullhomotopic.
(*) and $\pi_2(K_0) \neq 0$ implies $\pi_2(K_1) \neq 0$.

Repeat the arguments with K_1 instead of K_0. End up with:

$$K = K_0 \subset K_1 \subset K_2 \subset \dots$$

and define $L = \bigcup K_i$. $\square$

This result was strengthened by Luft ([13]):

Theorem 3.3. *(Luft 1996): If (WH) is false then there is a counterexample of type (b) of theorem 3.2.*

4. Labelled Oriented Graphs

Let $P = < x_1, \dots, x_n \mid R_1, \dots, R_m >$ be a finite presentation where each relator is of the form $x_i x_j = x_j x_k$, i.e. is a Wirtinger relation. Such a presentation is called a *labelled oriented graph presentation*, or short, *LOG-presentation* because it is represented by a *labelled oriented graph* T_P in the following way: For each generator x_i of P, T_P has a vertex labelled x_i and for each relator $x_i x_j = x_j x_k$ (or, equivalently, $x_i = x_j x_k x_j^{-1}$), T_P has an oriented edge from the vertex x_i to the vertex x_k labelled by x_j. If T_P is a tree we call it a *labelled oriented tree* or *LOT* and P a *LOT-presentation.* The 2-complex modeled on P will be called a LOG (or LOT)-complex.

LOTs are of importance for the Whitehead Conjecture because of the following theorem (see [8]):

Theorem 4.1. *(Howie 1983): Let L be a finite 2-complex and $e \subset L$ a 2-cell. If $L \overset{3}{\nearrow\searrow} *$ (i.e. L 3-deforms to a single vertex) then $L - e \overset{3}{\nearrow\searrow} K$ and K is a LOT complex.*

As an example we consider the presentation
$< a, b, c, d, e \mid ac = cb, bd = dc, db = bc, da = ae >$ which encodes to

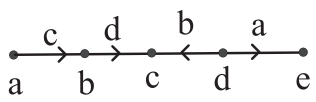

Рис. 2. Example of a LOT

Andrews-Curtis Conjecture (AC): Let L be a finite, contractible 2-complex. Then $L \stackrel{3}{\nearrow\!\searrow} *$.

Corollary 4.2. *If (AC) is true and LOTs are aspherical then there is no counterexample of type (a) of theorem 3.2 to (WH).*

On the other hand, any nonaspherical LOT is a counterexample to (WH):
A LOT is a subcomplex of an aspherical 2-complex because if one adds $x_1 = 1$ as a relator then one has a balanced presentation of the trivial group which 3-deforms to a point. Hence LOTs are interesting for (WH).

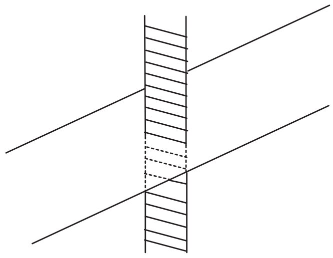

Рис. 3. A ribbon singularity

A *ribbon-disc* is a proper embedding $f\colon D^2 \to D^4$ which can be immersed in the boundary $i\colon D^2 \to S^3$ having only ribbon singularities (see Figure 3). The complement of a properly embedded ribbon-disc in the 4-ball $D^4 - i(D^2)$ collapses to a LOT-complex. Each LOT-complex can be realized as a spine of a complement of a properly embedded ribbon-disc in the 4-ball.

Ribbon-disc Conjecture: ribbon-disc complements (and hence LOT-complexes) are aspherical.

If K is a 2-complex and $g \in \pi_1(K)$ a nontrivial element of finite order, then $\pi_2(K) \neq 0$. If $g^k = 1$ and g^k is trivialized by r in $\pi_1(K)$ then $(1-g)[r]$ is nontrivial

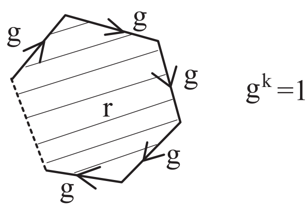

Рис. 4. An element of finite order

in $\pi_2(K)$.

Is there a LOT-group with a nontrivial element of finite order? This also is still open.

Several classes of aspherical LOTs are known:

(1) Wirtinger presentations of knots are LOTs. They are aspherical by the result of Papakyriakopulos cited in chapter 1.

(2) Howie 1985 [9]: LOTs of Diameter ≤ 3 are aspherical.

(3) LOT-complexes, which satisfy some kind of curvature condition like the small cancellation conditions C(4), T(4) or the weight test (Gersten 1987 [5]) or the cycle test (Huck/Rosebrock 1992 [10]) are aspherical.

(4) A LOT is called *injective* if each generator occurs at most once as an edge label (this corresponds to alternating knots).
A LOT is called *compressed* if every relator contains 3 different generators.
A LOT is called *reducible* if there is a generator that occurs exactly once upon the set of relators and *reduced* otherwise.
Any LOT can be homotoped into a compressed reduced LOT.

Theorem 4.3. *(Huck/Rosebrock 2001* [11]*): If a LOT is compressed and injective and does not contain a reducible Sub-LOT then it is aspherical.*

(5) Generalized knot-theory (Harlander/Rosebrock 2003 [6]):

A knot-projection on a 2-sphere leads to a LOT via its Wirtinger-presentation. An arc between undercrossing and undercrossing leads to a generator. Any crossing gives rise to a relator (see Figure 5).

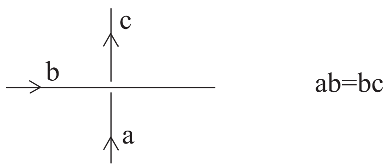

Рис. 5. A crossing

It is also possible to realize a given LOT as a knot-projection but only on orientable surfaces instead of a 2-sphere. First, a LOT can be homotoped to a LOI (labelled oriented interval, where the corresponding tree is just an interval). Any LOI can then be realized on an oriented surface such that reading off its Wirtinger presentation gives back the LOI. This projection can be realized as a projection from an embedding of an arc in a singular 3-manifold:

Let P be a LOI and K_P its corresponding 2-complex. There is a singular 3-manifold $X = F \times [0,1]/F \times \{1\}$ (F is an orientable surface) and a link $L \subset X$, such that $X - L \searrow K_P$.

Let F' be the cell decomposition of F dual to the one induced by $\bar{L}$, where $\bar{L}$ is the projection of L on F. The definition of a prime knot is generalized from the classical case:

Definition 4.4. *An alternating link-projection $\bar{L}$ on an orientable surface F is* prime *if the 1-skeleton of F' does not contain cycles of length shorter than four except for cycles made up of boundary edges.*

Theorem 4.5. *(Harlander/Rosebrock 2003* [6]*) If L has a prime and alternating link projection on F then K_P is aspherical.*

(6) LOTs of complexity two (Rosebrock [16]):

Given a LOT $P = < x_1, \dots, x_n \mid R_1, \dots, R_{m-1} >$, we say that P has *complexity* n, provided there is a subset $S = \{x_{i_1}, \dots, x_{i_n}\}$ of the set of generators X consisting of n elements, such that the following inductive process defines every generator of P to be good and there is no such set consisting of $n-1$ elements:

(1) The elements of S are good.
(2) If $xy = yz$ or $zy = yx$ is a relator of P and x, y are good then so is z.

We say that P is *derived by S*.

Theorem 4.6. *(Rosebrock 2007* [16]*) LOTs of complexity 2 are aspherical.*

The idea of the proof is as follows: For a LOT of complexity 2 it is shown that it can be transformed into a 1-relator presentation without changing the homotopy-type of the corresponding 2-complex. Since LOT-groups abelianize to the infinite cyclic group, one can show that the corresponding 2-complex is aspherical.

There are some other classes of aspherical LOTs which are not mentioned here. See for example [15] or [12].

5. Spherical Diagrams

$f\colon C \to K^2$ is a *spherical diagram* over the 2-complex K^2, if C is a cell decomposition of the 2-sphere and open cells are mapped homeomorphically. If K is non-aspherical then there exists a spherical diagram which realizes a nontrivial element of $\pi_2(K)$. So in order to check whether a 2-complex is aspherical or not it is enough to check spherical diagrams.

A spherical diagram $f\colon C \to K^2$ is *reducible*, if there is a pair of 2-cells in C with a common edge t, such that both 2-cells are mapped to K by folding over t.

A 2-complex K is said to be *diagrammatically reducible* (DR), if each spherical diagram over K is reducible. Certainly if K is (DR) then K is aspherical since any spherical diagram over a DR 2-complex can be reduced to the trivial diagram. So there was the hope to show that any LOT can be homotoped to a diagrammatically reducible LOT, but:

Theorem 5.1. *(Rosebrock 1990* [14]*) There are reduced and compressed LOT-complexes, which are aspherical but not DR.*

Let $P =< x_1, \dots, x_n \mid R_1, \dots, R_m >$ be any finite presentation and K_P the corresponding standard 2-complex.

The *Whitehead graph* W_P of K_P is the boundary of a regular neighborhood of the only vertex of K_P. It is, in general, a non-oriented graph consisting of a pair of vertices x_i^+ and x_i^- for each generator x_i of P, which correspond to the beginning and the end of the oriented loop labelled x_i in K_P. The edges of W_P, also called *corners*, are the intersections of the polygonal 2-cells with the boundary of a regular neighborhood of the vertex of K_P. The *left graph* $L_P \subset W_P$ is the full subgraph on the vertices $x_1^+, \ldots, x_n^+$, the *right graph* $R_P \subset W_P$ is the full subgraph on the vertices $x_1^-, \ldots, x_n^-$.

The following theorem was observed by several people and gives a new class of aspherical LOTs:

Theorem 5.2. *If the left graph or the right graph of a LOT does not contain a cycle then the corresponding 2-complex is DR.*

Proof. If G is the fundamental group of a LOT-complex K then G abelianizes to the infinite cyclic group. So the levels of the corresponding covering space $\bar{K}$ may be enumerated by the integers. Any spherical diagram $f\colon C \to K$ lifts into $\bar{K}$. The maximum level in the image of this map $\bar{f}\colon C \to \bar{K}$ comes from a sink in the preimage. Since there is no cycle in the left graph a reduction must be possible at such a sink. The same happens with the minimum level reached by the image and the right graph. □

This theorem is generalized to a wider class of LOTs in [12]. There the left and the right graph may contain cycles but some subtile conditions on the Whitehead graph still guarantee asphericity.

Let $P = < x_1, \ldots, x_n \mid r_1, \ldots, r_m >$ be a LOT. To *change the orientation* of the edge corresponding to one of its relations $r_t : \; x_i = x_j x_k x_j^{-1}$ will mean to replace r_t by $r_t' : \; x_k = x_j x_i x_j^{-1}$. This is the same as changing the orientation of the corresponding edge in T_P. An *orientation* of a LOT P is a LOT which arises from P by changing the orientations of a (possibly empty) subset of edges of T_P.

Theorem 5.3. *(Huck/Rosebrock 2001* [11]*) For any LOT P there is an orientation Q of P such that K_Q is diagrammatically reducible.*

In order to prove this, for any LOT P orientations are chosen such that the left graph is a tree and then Theorem 5.2 gives the desired result.

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Stephan Rosebrock
Fakultaet III (Mathematik /Informatik),
Paedagogische Hochschule Karlsruhe,
Bismarckstr. 10,
D-76133, Karlsruhe, Germany
E-mail address: nsheikh@math.uiuc.edu