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SHEAVES AND $\mathfrak{Ta}$-BICOMPACTIFICATIONS OF MAPPINGS

V. M. ULYANOV

Abstract. The paper is devoted to an investigation of relations between bicompactifications of mappings and sheaves of algebras. Bicompactifications of mappings are a generalization of compactifications of topological spaces, and sheaves of algebras take place of algebras of continuous bounded functions on topological spaces.

The first section contains a historical review of main constructions and notions used in the paper as well as a short introduction to the theory of bicompactifications of mappings. In particular, we state here basic definitions and recall some statements about bicompactifications of mappings that were obtained earlier.

In the second section some new topological properties of the fan product and the inverse limit are proved.

The third section contains important constructions which are used for an upbuilding of bicompactifications of mappings. Several new properties of these constructions are proved.

The fourth section is devoted to a definition and an investigation of algebras of functions on mappings. In this section a natural topology on these algebras is defined; the class of globally completely regular mappings is singled out for which such algebras play a role similar to that of algebras of continuous bounded functions on completely regular spaces; a functor from the category of mappings to the category of perfect globally completely regular mappings is constructed which preserves algebras of continuous "bounded" functions on mappings; a correspondence between "mappings" of mappings and homomorphisms of their algebras is investigated.

In the fifth section sheaves of algebras connected with mappings are defined and investigated.

The sixth section contains a proof of the main result of the paper: there exists a one-to-one correspondence preserving the order between the set of all $\mathfrak{Ta}$-bicompactifications of a given mapping and the set of all sheaves of a special kind.

In the seventh section we define maximal closed ideals of sheaves of algebras; relations between these ideals and points of $\mathfrak{Ta}$ of a given mapping are investigated.

Ulyanov V.M., Sheaves and $\mathfrak{Ta}$-bicompactifications of mappings.

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§ 1. Basic constructions and notions

1.1. This section contains a historical review of basic constructions and notions used in the paper.

The term "mapping" will mean "continuous map". No axioms of separability will be assumed. The symbol $[A]_X$ stands for the closure of the set A in the topological space X.

For mappings we write subscripts and superscripts on the left rather then on the right, that is, we write ${}^{\mathfrak{A}}_{\alpha}\pi$ instead of $\pi^{\mathfrak{A}}_{\alpha}$ and so on. This is somewhat unusual but more convenient since we can write, for example, ${}^{\mathfrak{A}}_{\alpha}\pi^{\#}$ and ${}^{\mathfrak{A}}_{\alpha}\pi^{-1}$ instead of $(\pi^{\mathfrak{A}}_{\alpha})^{\#}$ and $(\pi^{\mathfrak{A}}_{\alpha})^{-1}$ (see [35]). Analogously, $[A \setminus B]_X$ is shorter than $Cl_X(A \setminus B)$.

A. **Constructions**

1.2. The fan product of topological spaces relative to given mappings is a topological version of the well-known fibred product in the theory of categories (see, for example, [61], the item 1.5.4). The fan product have been described, for example, in the book [3] (§2 of Supplement to Chapter I), but for our purposes its discussion there is not sufficiently detailed, so that we shall investigate this construction in §2. We shall also discuss some properties of the well-known inverse limit (see, for example, [3], §1 of Supplement to Chapter I).

1.3. In the item 3.1 a construction is described which have been investigated in the papers [43] and [46]. This construction was used for an upbuilding of the absolutes and compactifications of topological spaces and their mappings, for an upbuilding of completely regular spaces which have not compactifications of special kinds.

Two partial cases of this construction were known earlier: first, the partial topological product which was investigated in the paper [35] and can be obtained if $G_\alpha = O_\alpha$ for all $\alpha \in \mathfrak{A}$ (see the item 3.1); the partial topological product was used for an upbuilding of universal spaces in dimension theory (see, for example, [35], [41] or [59]); second, the construction which was described in the paper [49] and can be obtained if $|G_\alpha| = 1$ for all $\alpha \in \mathfrak{A}$; this construction was used for an upbuilding of a great number of Hausdorff compact spaces with "pathological" properties in dimension theory and in the theory of cardinal-valued topological invariants.

B. **Properties of mappings**

1.4. **Definition.** A class $\mathfrak{C}$ of topological spaces will be called *closed* if the following conditions are fulfilled:

1) there exists $Z \in \mathfrak{C}$ such that $|Z| = 1$;
2) if $Z_\alpha \in \mathfrak{C}$ for all $\alpha \in A$ then $\prod\{Z_\alpha : \alpha \in A\} \in \mathfrak{C}$;
3) if $Z \in \mathfrak{C}$ and $Z' \subseteq Z$ then $Z' \in \mathfrak{C}$.

Further on the symbol "$\mathfrak{C}$" will always denote a closed class of topological spaces.

1.5. **Definition.** A family $\mathfrak{a}$ of locally closed subsets of a space Y will be called *closed* if the following conditions are fulfilled:

1) $\varnothing \in \mathfrak{a}$;
2) if $G_1, G_2 \in \mathfrak{a}$ then $(G_1 \cup G_2) \setminus (G_1^* \cup G_2^*) \in \mathfrak{a}$ where $G^* = [G]_Y \setminus G$ for $G \subseteq Y$;
3) if $G \subseteq Y$ is a locally closed subset such that for each point $y \in G$ there exist a neighborhood $Uy \subseteq Y$ and a set $G_y \in \mathfrak{a}$ satisfying the condition $G \cap Uy \subseteq G_y$ then $G \in \mathfrak{a}$.

Particularly, if $G \in \mathfrak{a}$ and $G' \subseteq G$ is a locally closed subset then $G' \in \mathfrak{a}$.

Further on the symbol "$\mathfrak{a}$" will always denote a closed family of locally closed subsets of a topological space Y.

1.6. **Definition.** We shall say that a mapping $f\colon X \to Y$ *has the property* $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ if for an arbitrary point $x \in X$ in each of the following two cases

a) for every point $x' \in f^{-1}fx \setminus \{x\}$ and
b) for every neighborhood $Ux \subseteq X$

there exist a neighborhood $Ofx \subseteq Y$, a set $G \in \mathfrak{a}$, a space $Z \in \mathfrak{C}$ and mappings $g\colon Ofx \setminus G \to Z$ and $\tilde{g}\colon f^{-1}Ofx \to Z$ such that $[G]_Y \cap Ofx = G$, $\tilde{g}|_{f^{-1}(Ofx\setminus G)} = gf|_{f^{-1}(Ofx\setminus G)}$ and, respectively,

a) $\tilde{g}x' \neq \tilde{g}x$ or
b) $\tilde{g}x \notin [\tilde{g}(f^{-1}Ofx \setminus Ux)]_Z$.

$$\begin{tikzcd}
f^{-1}(Ofx \setminus G) \arrow[rr, "\subseteq"] \arrow[d, "f"'] \arrow[dr, "\tilde{g}"] & & f^{-1}Ofx \arrow[r, "\subseteq"] \arrow[dl, "\tilde{g}"'] \arrow[d, "f"] & X \arrow[d, "f"] \\
& Z & & \\
Ofx \setminus G \arrow[rr, "\subseteq"] \arrow[ur, "g"] & & Ofx \arrow[r, "\subseteq"] & Y
\end{tikzcd}$$

1.7. If $\mathfrak{C}$ is the class of all completely regular spaces then we shell write $\mathfrak{T}\mathfrak{a}$ instead of $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$. In this case we can always take $Z = \mathbb{R}$ (the space of real numbers) or $Z = [0, 1]$ in Definition 1.6. If $\mathfrak{a}$ is a family of all discrete (in itself) locally closed subsets of the space Y then we shall write $\mathfrak{T}^{\mathfrak{C}}$ instead of $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$. In this case we can always suppose that $|G| \leqslant 1$ in Definition 1.6. If the above assumptions are both fulfilled, we shall write simply $\mathfrak{T}$.

Definition 1.6 is more general than the corresponding definition of the paper [43], but all statements and their proves remain valid (it is possible to omit the operators of the closure in Lemma 5 in [43]).

The property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ and the construction described in the item 3.1 are connected. Namely, the following two statements are valid.

1.8. **Assertion** ([43], Lemma 5)**.** *The mapping* ${}^{\mathfrak{A}}\pi\colon Y_{\mathfrak{A}} \xrightarrow{\text{onto}} Y$ *constructed in the item 3.1 has the property* $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$*, where* $\mathfrak{C}$ *is any closed class of topological spaces containing* $\{Z_\alpha : \alpha \in \mathfrak{A}\}$ *and* $\mathfrak{a}$ *is any closed family of locally closed subsets of the space* Y *containing* $\{G_\alpha : \alpha \in \mathfrak{A}\}$.

1.9. **Assertion** (a consequence of Lemma 6 of the paper [43])**.** *If a mapping* $f\colon X \to Y$ *has the property* $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ *then there exist a mapping* ${}^{\mathfrak{A}}\pi\colon Y_{\mathfrak{A}} \xrightarrow{\text{onto}} Y$ *and a homeomorphic embedding* $f_{\mathfrak{A}}\colon X \to Y_{\mathfrak{A}}$ *such that* $f = {}^{\mathfrak{A}}\pi f_{\mathfrak{A}}$*, where* $Y_{\mathfrak{A}} = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha\}, \alpha \in \mathfrak{A})$*,* $Z_\alpha \in \mathfrak{C}$ *and* $G_\alpha \in \mathfrak{a}$ *for all* $\alpha \in \mathfrak{A}$ (see 3.1–3.2).

1.10. Mappings with the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ have been defined in the paper [43] and they have been investigated in the papers [45], [47] and [42]. The property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ is an analog of $\mathfrak{C}$-regularity of topological spaces ([57]). An analog of the $\mathfrak{C}$-compactness is defined for mappings in the paper [5].

Mappings with the property $\mathfrak{T}\mathfrak{a}$ are analogous to completely regular spaces. These mappings admit a great deal of structures which exist in completely regular spaces. For example, in the paper [42] the notion of a normal base is studied, in the paper [25] the concept of a subordination on a mapping is defined, in the papers [6], [53] and [56] uniformities on mappings are discussed. The weakest property $\mathfrak{T}\mathfrak{a}$ can be obtained if $\mathfrak{a}$ is the family of all locally closed subsets of the space Y. Mappings with this property have been called *Tychonoff mappings* in the paper [34], where a great number of properties of mappings has been defined which are analogous to properties of topological spaces (see also [52]). Some of them are included in the book [58][1] (without direct references).

Earlier, in the paper [40], the property $\mathfrak{T}$ has been defined for mappings of completely regular spaces. The paper [44] is connected with the paper [40] and is devoted to related properties. In the paper [27] subordinations on mappings with the property $\mathfrak{T}$ have been defined.

Some of earlier defined properties of mappings are equivalent to properties $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ for suitable $\mathfrak{C}$ and $\mathfrak{a}$. For example, the following two statements are valid.

[1]The term "fibrewise topological space" in [58] corresponds to the term "mapping" in [34] and so on.

1.11. **Assertion.** *A mapping* $f\colon X \to Y$ *is dividing* ([9], Definition 1) *iff it has the property* $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$ *where* $\mathfrak{E} = \{Z : Z$ *is completely regular and* $\operatorname{ind} Z = 0\}$ *and* $\mathfrak{a} = \{G \subseteq Y : G$ *is locally closed*$\}$.

1.12. **Assertion** ([44] - for regular X and Y)**.** a) *If a* T_3*-mapping* ([34]) $f\colon X \to \to Y$ *is completely closed* ([49]) *then the mapping* f *is closed and has the property* $\mathfrak{T}^{\mathfrak{E}}$ *where* $\mathfrak{E}$ *is the class of all topological spaces, and the set* $Y \setminus fX$ *is discrete and clopen in* Y.

b) *If a mapping* $f\colon X \to Y$ *is closed and has the property* $\mathfrak{T}^{\mathfrak{E}}$ *where* $\mathfrak{E}$ *is the class of all topological spaces, and the set* $Y \setminus fX$ *is discrete and clopen in* Y*, then the mapping* f *is completely closed.*

1.13. *Remark.* a) If $\mathfrak{a} = \{\varnothing\}$ or $|Z| \leqslant 1$ for all $Z \in \mathfrak{E}$ then each mapping $f\colon X \to Y$ with the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$ is a homeomorphic embedding.

b) If $\mathfrak{a}$ is a family of all locally closed subsets of the space Y and $\mathfrak{E}$ is the class of all topological spaces, then each mapping $f\colon X \to Y$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$.

c) If $\mathfrak{E}_1 \subseteq \mathfrak{E}_2$ and $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$, then every mapping with the property $\mathfrak{T}^{\mathfrak{E}_1}\mathfrak{a}_1$ has the property $\mathfrak{T}^{\mathfrak{E}_2}\mathfrak{a}_2$.

d) If a mapping $f\colon X \to Y$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$, $X' \subseteq X$, $fX' \subseteq Y' \subseteq Y$, $G \cap Y' \in \mathfrak{a}'$ for all $G \in \mathfrak{a}$ where $\mathfrak{a}'$ is a closed family of locally closed subsets of Y', then the mapping $f' = f|_{X'}\colon X' \to Y'$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}'$.

1.14. **Definition** ([55])**.** A mapping $f\colon X \to Y$ will be called *separable* if any two distinct points $x_1, x_2 \in X$ such that $fx_1 = fx_2$ have disjoint neighborhoods in X.

1.15. **Lemma** ([43])**.** *If every space* $Z \in \mathfrak{E}$ *is Hausdorff, then each mapping with the property* $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$ *is separable.*

C. Compactifications of mappings

1.16. **Definition** ([63])**.** Let $f\colon X \to Y$ be a mapping such that $[fX]_Y = Y$. A mapping $f_v\colon v_f X \to Y$ will be called *a compactification of the mapping* f if the following conditions are fulfilled:

1) the mapping f_v is perfect;
2) $X \subseteq v_f X$;
3) $f_v|_X = f$;
4) $[X]_{v_f X} = v_f X$.

1.17. It has been proved in the paper [63] that a mapping of a Hausdorff locally compact space onto another such space has a compactification. An analogous statement has been proved in the paper [54] for mappings of completely regular spaces onto regular spaces. The statement that any mapping has a compactification is a partial case of results of the paper [43]. The problem on the existence of separable compactifications of mappings has been studied in the paper [26].[2] Various problems on compactifications of mappings in the sense of Definition 1.16 have been considered in the papers [11]–[15], [18], [21]–[23], [29]–[34], [36]–[39], [56], [60].

It is possible to obtain a definition of an extension of a mapping if one replace the condition 1) in Definition 1.16 by another suitable condition. Such extensions have been studied in the papers [4], [5], [11]–[13], [20].

It should be mentioned that the notion of an extension of a topological space can be considered as a partial case of an extension of a mapping (a mapping $X \to \{*\}$ onto the one-point space corresponds to the topological space $X \neq \varnothing$).

[2] V.A.Matveev asserts that his proof of Corollary 1 in the paper [26] is incomplete, but a counter-example is not known. A correct condition can be found in his thesis “Структуры подчинений, связанных с отображениями” (Москва, 1990).

D. **Covers of topological spaces**

1.18. **Definition** ([64])**.** *A cover of a topological space* Y is a perfect irreducible mapping $f\colon X \xrightarrow{\text{onto}} Y$.

Usually it is convenient to say that the space X is *the cover of the space* Y.

1.19. The most important cover of a topological space is its absolute. Absolutes for all topological spaces have been constructed in the paper [43]. Other covers have been studied too (see, for example, the papers [1], [10], [16], [17], [62]). Different general constructions of covers can be found in the papers [1], [2], [16], [17], [62], [64]. The paper [24] contains a method to construct all separable (in the sense of Definition 1.14) and all Tychonoff (in the sense of the paper [34]) covers of an arbitrary topological space.

It is noted in the paper [64] that the notions of an extension and of a cover of a topological space are analogous. However, the notions of a compactification of a mapping and a cover of a topological space are much more similar (compare, for example, the papers [26] and [24]). All these notions are partial cases of an extension of a mapping which can be obtained if we replace the conditions 1) and 5) of Definition 1.21 by other suitable conditions. Definition 1.21 has been formulated in the paper [43] to unify the notions of a compactification of a mapping (and, particularly, of a topological space) and of a cover of a topological space. This is, probably, the best version; the condition 5) could be replaced by other conditions to obtain bicompactifications with special properties.

E. $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$**-bicompactifications of mappings**

1.20. **Definition** ([43])**.** A mapping $f\colon X \xrightarrow{\text{onto}} Y$ will be called *irreducible modulo* $X' \subseteq X$ if every closed set $F \subseteq X$, which satisfies the conditions $X' \subseteq F$ and $fF = Y$, coincides with X (or, that is equivalent, if for each non-empty open set $U \subseteq X$ the set $(U \cap X') \cup f^{\#}U$ is non-empty too[3]).

In a usual way we can prove that if a mapping $f\colon X \xrightarrow{\text{onto}} Y$ and a set $X' \subseteq X$ are given such that for each $y \in Y \setminus f[X']_X$ the space $f^{-1}y$ is compact then the mapping f can be reduced modulo X', that is, there is a closed set $F \subseteq X$ such that $fF = Y$, $X' \subseteq F$ and the mapping $f|_F$ is irreducible modulo X'.

1.21. **Definition** ([43])**.** A mapping $f_v\colon v_fX \xrightarrow{\text{onto}} Y$ will be called *a* $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$*-bicompactification of a mapping* $f\colon X \to Y$ if the following conditions are fulfilled:

1) the mapping f_v is perfect;
2) $X \subseteq v_fX$;
3) $f_v|_X = f$;
4) the mapping f_v is irreducible modulo X;
5) the mapping f_v has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$.

$$\begin{tikzcd} X \arrow[rr, "\subseteq"] \arrow[dr, "f"'] & & v_fX \arrow[dl, "f_v"] \\ & Y & \end{tikzcd}$$

1.22. **Definition** ([43])**.** Let $f_v\colon v_fX \xrightarrow{\text{onto}} Y$ and $f_w\colon w_fX \xrightarrow{\text{onto}} Y$ be $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$-bicompactifications of a mapping $f\colon X \to Y$. We shall write $f_v \geqslant f_w$ if there is a mapping ${}^{v}_{w}\varphi\colon v_fX \to w_fX$ such that $f_v = f_w\,{}^{v}_{w}\varphi$ and ${}^{v}_{w}\varphi x = x$ for all $x \in X$.

[3]Let us recall that $f^{\#}U = \{y \in fX : f^{-1}y \subseteq U\}$.

$$\begin{tikzcd}
v_f X \arrow[rr, "{}^{v}_{w}\varphi"] \arrow[ddr, "f_v"'] & & w_f X \arrow[ddl, "f_w"] \\
& X \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] \arrow[d, "f"] & \\
& Y &
\end{tikzcd}$$

1.23. **Definition.** $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications $f_v\colon v_fX \xrightarrow{\text{onto}} Y$ and $f_w\colon w_fX \xrightarrow{\text{onto}} Y$ of a mapping $f\colon X \to Y$ will be called *equivalent* if there exists a homeomorphism ${}^{v}_{w}\varphi\colon v_fX \xrightarrow{\text{onto}} w_fX$ such that $f_v = f_w\, {}^{v}_{w}\varphi$ and ${}^{v}_{w}\varphi x = x$ for all $x \in X$.

In general the mapping ${}^{v}_{w}\varphi$ in Definitions 1.22 and 1.23 is not unique, and there are non-equivalent $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications f_v and f_w such that $f_v \geqslant f_w$ and $f_w \geqslant f_v$, but this is impossible in the case of separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications (for example, if all spaces of the class $\mathfrak{C}$ are Hausdorff).

1.24. The existence of $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications has been considered in the paper [43]. Properties of the largest separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications have been investigated in the paper [45]. Constructions of all $\mathfrak{T}$-bicompactifications and $\mathfrak{T}\mathfrak{a}$-bicompactifications by means of subordinations have been described in the papers [27] and [25]. The main results of the papers [43] and [45] about $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications are following.

F. The existence of $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications

1.25. **Theorem.** *If each space $Z \in \mathfrak{C}$ has a compactification $vZ \in \mathfrak{C}$ then each mapping $f\colon X \to Y$ with the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ has a $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification $f_v\colon v_fX \xrightarrow{\text{onto}} Y$.*

1.26. **Corollary.** *If each space $Z \in \mathfrak{C}$ has a Hausdorff compactification $vZ \in \mathfrak{C}$ then each mapping $f\colon X \to Y$ with the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ has a separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification $f_v\colon v_fX \xrightarrow{\text{onto}} Y$.*

1.27. **Lemma.** *Let $f_v\colon v_fX \xrightarrow{\text{onto}} Y$ be a $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification and $f_w\colon w_fX \xrightarrow{\text{onto}} Y$ be a separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification of a mapping $f\colon X \to Y$ such that $f_v \geqslant f_w$. Then the mapping ${}^{v}_{w}\varphi\colon v_fX \xrightarrow{\text{onto}} w_fX$ satisfying the conditions $f_v = f_w\, {}^{v}_{w}\varphi$ and ${}^{v}_{w}\varphi x = x$ for all $x \in X$, is perfect, "onto", irreducible and is determined by these conditions uniquely.*

1.28. **Corollary.** *Let $f_v\colon v_fX \xrightarrow{\text{onto}} Y$ and $f_w\colon w_fX \xrightarrow{\text{onto}} Y$ be separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications of a mapping $f\colon X \to Y$ such that $f_v \geqslant f_w$ and $f_w \geqslant f_v$. Then the mappings ${}^{v}_{w}\varphi$ and ${}^{w}_{v}\varphi$ are mutually inverse homeomorphisms, and the $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications f_v and f_w are equivalent.*

1.29. **Assertion.** *Let $\{f_\alpha : \alpha \in \mathfrak{A}\}$ be any non-empty set of (separable) $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications $f_\alpha\colon v_\alpha X \xrightarrow{\text{onto}} Y$, $\alpha \in \mathfrak{A}$, of a mapping $f\colon X \to Y$. Then there is a (separable) $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification $f_v\colon v_fX \xrightarrow{\text{onto}} Y$ of the mapping f such that $f_v \geqslant f_\alpha$ for all $\alpha \in \mathfrak{A}$.*

1.30. **Proposition.** *For every mapping $f\colon X \to Y$ there exists the set $\mathfrak{C}(f)$ of all pairwise non-equivalent separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications of this mapping. The relation "$\geqslant$" is a partial order on the set $\mathfrak{C}(f)$.*

Of course, it is possible that the set $\mathfrak{C}(f)$ is empty.

G. The largest $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactifications and $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-absolutes

1.31. **Theorem.** *If the mapping $f\colon X \to Y$ has at least one separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification then it has the largest separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification $f_\beta\colon \beta_f X \xrightarrow{\text{onto}} Y$ (of course, f_β is unique).*

1.32. **Corollary.** *If each space $Z \in \mathfrak{C}$ has a Hausdorff compactification $vZ \in \in \mathfrak{C}$ then every mapping $f\colon X \to Y$ with the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ has the largest $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification $f_\beta\colon \beta_f X \xrightarrow{\text{onto}} Y$ (obviously, f_β is separable).*

1.33. Let us consider a mapping $f\colon \varnothing \to Y$ with the empty domain. Obviously, the identical mapping $i_Y\colon Y \xrightarrow{\text{onto}} Y$ is a separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification of the mapping f for any closed $\mathfrak{C}$ and $\mathfrak{a}$. Hence, the mapping f has the largest separable $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification $\mathfrak{p}\colon \mathfrak{a}_{\mathfrak{C}}Y \xrightarrow{\text{onto}} Y$. The space $\mathfrak{a}_{\mathfrak{C}}Y$ is called *the $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-absolute of the space Y*.

$\mathfrak{T}\mathfrak{a}$-absolutes of topological spaces have been studied in the paper [62].

Obviously, each $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-bicompactification of the mapping $f\colon \varnothing \to Y$ is a cover of the space Y; hence, we get the following statement.

1.34. **Corollary.** *Each topological space Y has the largest separable cover $\mathfrak{p}\colon \mathfrak{a}_{\mathfrak{C}}Y \xrightarrow{\text{onto}} Y$ with the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$.*

1.35. **Assertion.** *If a family $\mathfrak{a}$ contains all boundaries of regular closed subsets of a space Y, and there is a space $Z \in \mathfrak{C}$ such, that there exists an open subset $U \subseteq Z$ satisfying the condition $\varnothing \neq U \neq Z$, then the $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$-absolute of the space Y coincides with the absolute of the space Y.*

1.36. **Assertion** ([62]). *If $\mathfrak{a}$ is the smallest closed family containing all nowhere dense zero-sets of a completely regular space Y, then the $\mathfrak{T}\mathfrak{a}$-absolute of the space Y coincides with the sequential absolute oY* ([10]) *of the space Y.*

1.37. **Theorem** ([45]). *Let a mapping $f_1\colon X_1 \to Y_1$ has the largest separable $\mathfrak{T}^{\mathfrak{C}_1}\mathfrak{a}_1$-bicompactification $f_{1\beta}\colon \beta_{f_1}X_1 \xrightarrow{\text{onto}} Y_1$, $f_2\colon X_2 \xrightarrow{\text{onto}} Y_2$ be a perfect separable mapping with the property $\mathfrak{T}^{\mathfrak{C}_2}\mathfrak{a}_2$, $h_1\colon X_1 \to X_2$ and $h_2\colon Y_1 \to Y_2$ be mappings such that $h_2 f_1 = f_2 h_1$ and $h_2^{-1}G \in \mathfrak{a}_1$ for all $G \in \mathfrak{a}_2$, $\mathfrak{C}_2 \subseteq \mathfrak{C}_1$. Then there exists a mapping $h\colon \beta_{f_1}X_1 \to X_2$ such that $f_2 h = h_2 f_{1\beta}$ and $h|_{X_1} = h_1$. Moreover,*

1) *if the mapping h_2 is perfect or separable then the mapping h is, respectively, perfect or separable too;*
2) *if for each $G \in \mathfrak{a}_1$ the set $G \setminus [f_1X_1]_{Y_1}$ is nowhere dense in Y_1 then the mapping h is unique.*

$$
\begin{tikzcd}
X_1 \arrow[rr, "h_1"] \arrow[dr, "\subseteq"] \arrow[dd, "f_1"'] & & X_2 \arrow[dd, "f_2"] \\
& \beta_{f_1}X_1 \arrow[ur, "h"] \arrow[dl, "f_{1\beta}"] & \\
Y_1 \arrow[rr, "h_2"] & & Y_2
\end{tikzcd}
$$

1.38. **Corollary.** *Let $\mathfrak{p}_1\colon \mathfrak{a}_{1\mathfrak{C}_1}Y_1 \xrightarrow{\text{onto}} Y_1$ and $\mathfrak{p}_2\colon \mathfrak{a}_{2\mathfrak{C}_2}Y_2 \xrightarrow{\text{onto}} Y_2$ be the largest separable covers with the properties $\mathfrak{T}^{\mathfrak{C}_1}\mathfrak{a}_1$ and $\mathfrak{T}^{\mathfrak{C}_2}\mathfrak{a}_2$ respectively, $\mathfrak{C}_2 \subseteq \mathfrak{C}_1$, and $h\colon Y_1 \to Y_2$ be a mapping such that $h^{-1}G \in \mathfrak{a}_1$ for all $G \in \mathfrak{a}_2$. Then there is a mapping $\tilde{h}\colon \mathfrak{a}_{1\mathfrak{C}_1}Y_1 \to \mathfrak{a}_{2\mathfrak{C}_2}Y_2$ such that $\mathfrak{p}_2\tilde{h} = h\mathfrak{p}_1$. Moreover,*

1) *if the mapping h is perfect or separable then the mapping $\tilde{h}$ is, respectively, perfect or separable;*

2) *if each set $G \in \mathfrak{a}_1$ is nowhere dense in Y_1 then the mapping $\tilde{h}$ is unique.*

$$\begin{tikzcd} \mathfrak{a}_{1\mathfrak{C}_1} Y_1 \arrow[r, "\tilde{h}"] \arrow[d, "\mathfrak{p}_1"'] & \mathfrak{a}_{2\mathfrak{C}_2} Y_2 \arrow[d, "\mathfrak{p}_2"'] \\ Y_1 \arrow[r, "h"] & Y_2 \end{tikzcd}$$

1.39. **Corollary** ([50]). *Let qY_1 and qY_2 be absolutes of topological spaces Y_1 and Y_2 respectively, $q_1 \colon qY_1 \xrightarrow{\text{onto}} Y_1$ and $q_2 \colon qY_2 \xrightarrow{\text{onto}} Y_2$ be their projections, $h \colon Y_1 \to Y_2$ be a mapping. Then there exists a mapping $\tilde{h} \colon qY_1 \to qY_2$ such that $q_2\tilde{h} = hq_1$. Moreover,*

1) *if the mapping h is perfect or separable then the mapping $\tilde{h}$ is, respectively, perfect or separable;*
2) *if for each regular open set $U \subseteq Y_2$ the set $h^{-1}Fr_{Y_2}U$ is nowhere dense[4] in Y_1 then the mapping $\tilde{h}$ is unique.*

1.40. **Corollary** (it seems to be new). *Let oY_1 and oY_2 be sequential absolutes of completely regular spaces Y_1 and Y_2 respectively, $o_1 \colon oY_1 \xrightarrow{\text{onto}} Y_1$ and $o_2 \colon oY_2 \xrightarrow{\text{onto}} Y_2$ be their projections, $h \colon Y_1 \to Y_2$ be a mapping such that for each nowhere dense zero-set $G \subseteq Y_2$ the set $h^{-1}G$ is nowhere dense in Y_1. Then there exists a unique mapping $\tilde{h} \colon oY_1 \to oY_2$ such that $o_2\tilde{h} = ho_1$. Moreover, if the mapping h is perfect or separable then the mapping $\tilde{h}$ is, respectively, perfect or separable.*

1.41. *Remark.* It is possible to eliminate the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ from statements 1.25, 1.27–1.32, 1.34 using Remark 1.13 b).

H. **Sheaves**

1.42. In §4 and §5 we construct and investigate an object connected with a given mapping which corresponds to the algebra of continuous bounded functions on a given topological space. An analogous problem has considered in the papers [19], [18].

A required object is a sheaf. Unfortunately, a usual sheaf over a topological space is not convenient to describe $\mathfrak{T}\mathfrak{a}$-bicompactifications of a given mapping with the property $\mathfrak{T}\mathfrak{a}$, therefore we have to use a more general definition. We re-formulate Definition 0.31 of the book [8] in a convenient way for our special purposes.

The symbol "T" will denote further on a partially ordered set. We shall denote the relation of the partial order by the symbol "$\subseteq$". We shall also suppose that for each $t_1, t_2 \in T$ there exists $\min\{t_1, t_2\} \in T$ which will be denoted by $t_1 \cap t_2$.

1.43. **Definition.** We shall say that *a Grothendieck pretopology* is given *on the set T* if for each $t \in T$ a family $P(t)$ of subsets of T is given satisfying the following conditions:

1) if $t \in T$, $\gamma \in P(t)$ and $t' \in \gamma$, then $t' \subseteq t$;
2) if $t \in T$, then $\{t\} \in P(t)$;
3) if $t, t' \in T$, $t' \subseteq t$ and $\{t_\alpha : \alpha \in A\} \in P(t)$, then $\{t_\alpha \cap t' : \alpha \in A\} \in P(t')$;
4) if $t \in T$, $\{t_\alpha : \alpha \in A\} \in P(t)$ and $\{t_{\alpha\beta} : \beta \in B_\alpha\} \in P(t_\alpha)$ for all $\alpha \in A$, then $\{t_{\alpha\beta} : \beta \in B_\alpha, \alpha \in A\} \in P(t)$.

Elements of $P(t)$ are called *coverings of the element $t \in T$*.

[4]Let us recall that $Fr_X A = [A]_X \cap [X \setminus A]_X$ is a boundary of a set $A \subseteq X$.

1.44. **Definition.** We shall say that *a presheaf C of sets* is given *on the set T* if for each $t \in T$ a set $C(t)$ is given, and for each $t_1, t_2 \in T$ such that $t_1 \subseteq t_2$ a (*restriction*) map ${}^{t_2}_{t_1}h\colon C(t_2) \to C(t_1)$ is given satisfying the following conditions:

1) ${}^{t}_{t}h\colon C(t) \to C(t)$ is an identity map for every $t \in T$;
2) if $t_1, t_2, t_3 \in T$ and $t_1 \subseteq t_2 \subseteq t_3$ then ${}^{t_3}_{t_1}h = {}^{t_2}_{t_1}h\,{}^{t_3}_{t_2}h$.

1.45. **Definition** ([61], the item 4.5.2)**.** Let C be a presheaf on the set T and let $\gamma \subseteq T$ and $g_t \in C(t)$ for all $t \in \gamma$. The set $\{g_t : t \in \gamma\}$ will be called *compatible* if for each $t_1, t_2 \in \gamma$ the equality ${}^{\;\;t_1}_{t_1 \cap t_2}hg_{t_1} = {}^{\;\;t_2}_{t_1 \cap t_2}hg_{t_2}$ holds.

1.46. **Definition.** A presheaf $\mathcal{C}$ on the set T with a given Grothendieck pretopology $\{P(t) : t \in T\}$ will be called *a sheaf* if for each element $t_0 \in T$, a covering $\gamma \in P(t_0)$ and a compatible set $\{g_t : t \in \gamma\}$ there is a unique element $g \in \mathcal{C}(t_0)$ such that ${}^{t_0}_{t}hg = g_t$ for all $t \in \gamma$.

1.47. **Example.** Let Y be a topological space and T be the set of all open subsets of the space Y (that is, T is the topology of the space Y). For each $U \in T$ let $P(U) = \{\gamma \subseteq T : \bigcup \gamma = U\}$. It is easy to verify that $\{P(U) : U \in T\}$ is a Grothendieck pretopology and that $\mathcal{C}$ is a sheaf on the set T with this pretopology iff $\mathcal{C}$ is a sheaf over the space Y (see [61], Definition 4.5.1, or [8], Definition 0.23).

1.48. We shall consider sheaves of topological algebras. In this case restriction maps are supposed to be continuous homomorphisms.

In §6 we shall show that there is an order isomorphism of the set $\mathfrak{C}(f)$ of all $\mathfrak{Ta}$-bicompactifications of a given mapping $f\colon X \to Y$ with the property $\mathfrak{Ta}$ onto a set of sheaves with special properties.

In §7 we shall consider closed maximal ideals of sheaves of topological algebras of continuous functions on $\mathfrak{Ta}$-bicompactifications of mappings.

§ 2. The fan product and the inverse limit

A. **The fan product**

2.1. Let mappings ${}^{\alpha}\pi\colon Y_\alpha \to Y$, $\alpha \in \mathfrak{A}$, be given. *The fan product of the spaces Y_α relative to the mappings ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$*, is the set $Y_{\mathfrak{A}} = \prod_Y(\{Y_\alpha\}, \{{}^{\alpha}\pi\}, \alpha \in \mathfrak{A}) = \{\{y_\gamma : \gamma \in \mathfrak{A}\} \in \prod\{Y_\gamma : \gamma \in \mathfrak{A}\} : {}^{\alpha}\pi y_\alpha = {}^{\beta}\pi y_\beta$ for all $\alpha, \beta \in \mathfrak{A}\}$, equipped with the topology of the subspace of the product $\prod\{Y_\alpha, \alpha \in \mathfrak{A}\}$. Let ${}^{\mathfrak{A}}_{\alpha}\pi\colon Y_{\mathfrak{A}} \to Y_\alpha$ be the restriction of the projection ${}^{\alpha}p\colon \prod\{Y_\gamma : \gamma \in \mathfrak{A}\} \to Y_\alpha$ of the product to its factor for each $\alpha \in \mathfrak{A}$. Due to the definition of the fan product the equality ${}^{\alpha}\pi\,{}^{\mathfrak{A}}_{\alpha}\pi = {}^{\beta}\pi\,{}^{\mathfrak{A}}_{\beta}\pi$ holds for all $\alpha, \beta \in \mathfrak{A}$. Therefore the equality ${}^{\mathfrak{A}}\pi = {}^{\alpha}\pi\,{}^{\mathfrak{A}}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, defines the mapping ${}^{\mathfrak{A}}\pi\colon Y_{\mathfrak{A}} \to Y$ correctly. The mapping ${}^{\mathfrak{A}}\pi$ will be called *the fan product of the mappings ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$*. We shall write ${}^{\mathfrak{A}}\pi = \prod_Y\{{}^{\alpha}\pi : \alpha \in \mathfrak{A}\}$.

It is convenient to use the following coordinate representation of the fan product (it follows from Proposition of the book [3], §2 of Supplement to Chapter I): $Y_{\mathfrak{A}} = \prod_Y(\{Y_\alpha\}, \{{}^{\alpha}\pi\}, \alpha \in \mathfrak{A}) = \{\{y, z_\alpha : \alpha \in \mathfrak{A}\} : y \in Y, z_\alpha \in {}^{\alpha}\pi^{-1}y$ for all $\alpha \in \mathfrak{A}\}$. Then we have the equalities ${}^{\mathfrak{A}}\pi\{y, z_\alpha : \alpha \in \mathfrak{A}\} = y$ and ${}^{\mathfrak{A}}_{\beta}\pi\{y, z_\alpha : \alpha \in \mathfrak{A}\} = z_\beta \in {}^{\beta}\pi^{-1}y \subseteq Y_\beta$ for all $\beta \in \mathfrak{A}$ and $\{y, z_\alpha : \alpha \in \mathfrak{A}\} \in Y_{\mathfrak{A}}$.

For each $\mathfrak{B} \subseteq \mathfrak{A}$ let us define a mapping ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi\colon Y_{\mathfrak{A}} \to Y_{\mathfrak{B}} = \prod_Y(\{Y_\alpha\}, \{{}^{\alpha}\pi\}, \alpha \in \mathfrak{B})$ by the equality ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi\{y, z_\alpha : \alpha \in \mathfrak{A}\} = \{y, z_\alpha : \alpha \in \mathfrak{B}\}$ for all $\{y, z_\alpha : \alpha \in \mathfrak{A}\} \in Y_{\mathfrak{A}}$. Of course, ${}^{\mathfrak{A}}\pi = {}^{\mathfrak{B}}\pi\,{}^{\mathfrak{A}}_{\mathfrak{B}}\pi$ and ${}^{\mathfrak{A}}_{\alpha}\pi = {}^{\mathfrak{B}}_{\alpha}\pi\,{}^{\mathfrak{A}}_{\mathfrak{B}}\pi$ for all $\alpha \in \mathfrak{B}$.

$$\begin{tikzcd}
\prod\{Y_\alpha:\alpha\in\mathfrak{A}\} \arrow[dr, "{}^{\alpha}p"'] & & Y_{\mathfrak{A}} \arrow[ll, "\supseteq"'] \arrow[dl, "{}^{\mathfrak{A}}_{\alpha}\pi"'] \arrow[dd, "{}^{\mathfrak{A}}\pi"] \arrow[dr, "{}^{\mathfrak{A}}_{\beta}\pi"] \arrow[drr, "{}^{\mathfrak{A}}_{\mathfrak{B}}\pi"] & & \\
& Y_\alpha \arrow[dr, "{}^{\alpha}\pi"'] & & Y_\beta \arrow[dl, "{}^{\beta}\pi"] & Y_{\mathfrak{B}} \arrow[l, "{}^{\mathfrak{B}}_{\beta}\pi"'] \arrow[dll, "{}^{\mathfrak{B}}\pi"] \\
& & Y & &
\end{tikzcd}$$

Let us note that the space $Y_{\{\alpha\}}$ is naturally homeomorphic to the space Y_α for each $\alpha\in\mathfrak{A}$; we shall identify these spaces and corresponding mappings ${}^{\{\alpha\}}\pi$ and ${}^{\alpha}\pi$.

If $\mathfrak{B}\subseteq\mathfrak{A}$ then we shall call the mapping ${}^{\;\;\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi$ *parallel to the mapping* ${}^{\mathfrak{B}}\pi$.

$$\begin{tikzcd}
Y_{\mathfrak{A}} \arrow[r, "{}^{\mathfrak{A}}_{\mathfrak{B}}\pi"] \arrow[d, "{}^{\;\;\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi"'] \arrow[dr, "{}^{\mathfrak{A}}\pi"] & Y_{\mathfrak{B}} \arrow[d, "{}^{\mathfrak{B}}\pi"] \\
Y_{\mathfrak{A}\setminus\mathfrak{B}} \arrow[r, "{}^{\mathfrak{A}\setminus\mathfrak{B}}\pi"] & Y
\end{tikzcd}$$

Further on we shall assume that $Y_{\mathfrak{A}}$ is the fan product, without specifying it each time.

The following statements 2.2–2.5 can be proved by a comparison of corresponding sets and topologies.

2.2. **Proposition** ([3]). *For each point $y\in Y$ the space ${}^{\mathfrak{A}}\pi^{-1}y$ is homeomorphic to the space $\prod\{{}^{\alpha}\pi^{-1}y:\alpha\in\mathfrak{A}\}$. In particular, if ${}^{\alpha}\pi Y_\alpha=Y$ for all $\alpha\in\mathfrak{A}$, then ${}^{\mathfrak{A}}\pi Y_{\mathfrak{A}}=Y$.*

2.3. **Proposition.** *Let $z\in Y_{\mathfrak{A}\setminus\mathfrak{B}}$ and $y={}^{\mathfrak{A}\setminus\mathfrak{B}}\pi z$. Then the mapping ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi$ maps the space ${}^{\;\;\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}z$ onto the space ${}^{\mathfrak{B}}\pi^{-1}y$ homeomorphically. In particular, if ${}^{\mathfrak{B}}\pi Y_{\mathfrak{B}}=y$ then ${}^{\;\;\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi Y_{\mathfrak{A}}=Y_{\mathfrak{A}\setminus\mathfrak{B}}$.*

2.4. **Proposition.** *Let A be a family of pairwise disjoint subsets of the set $\mathfrak{A}$ such that $\bigcup\{\mathfrak{B}:\mathfrak{B}\in A\}=\mathfrak{A}$. Then the fan products $Y_{\mathfrak{A}}=\prod_Y(\{Y_\alpha\},\{{}^{\alpha}\pi\},\alpha\in\mathfrak{A})$ and $Y_A=\prod_Y(\{Y_{\mathfrak{B}}\},\{{}^{\mathfrak{B}}\pi\},\mathfrak{B}\in A)$ are naturally homeomorphic and (if we identify $Y_{\mathfrak{A}}$ and Y_A) ${}^{A}\pi={}^{\mathfrak{A}}\pi$, ${}^{A}_{\mathfrak{B}}\pi={}^{\mathfrak{A}}_{\mathfrak{B}}\pi$ for each $\mathfrak{B}\in A$.*

2.5. **Proposition.** *Let A be a family of subsets of the set $\mathfrak{A}$ such that $\bigcup\{\mathfrak{B}:\mathfrak{B}\in\in A\}=\mathfrak{A}$, and let A be directed by the relation "$\subseteq$". Let $S=\{Y_{\mathfrak{B}},{}^{\mathfrak{B}}_{\mathfrak{B}'}\pi:\mathfrak{B},\mathfrak{B}'\in\in A,\mathfrak{B}'\subseteq\mathfrak{B}\}$ be an inverse spectrum. Let $Y_S=\varprojlim S$, ${}^{S}\pi=\varprojlim{}^{\mathfrak{B}}\pi$ and ${}^{S}_{\mathfrak{B}}\pi$ be the projection of the space Y_S to $Y_{\mathfrak{B}}$, $\mathfrak{B}\in A$* (for the definitions see [51], §2.5). *Then the fan product $Y_{\mathfrak{A}}$ and the space Y_S are naturally homeomorphic and (if we identify $Y_{\mathfrak{A}}$ and Y_S) ${}^{S}\pi={}^{\mathfrak{A}}\pi$, ${}^{S}_{\mathfrak{B}}\pi={}^{\mathfrak{A}}_{\mathfrak{B}}\pi$ for each $\mathfrak{B}\in A$.*

2.6. In the following seven items we shall prove statements about the existence of mappings connected with the fan product; also we shall prove that the fan product preserves the following properties of mappings: to be perfect, or separable, or uniquely reducible, or to have the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$. Analogous statements will be proved for the inverse limit of mappings. Other properties of the inverse limit can be found in the books [3] and [51].

2.7. **Proposition.** *Let mappings $f\colon X\to Y$ and $f_\alpha\colon X\to Y_\alpha$, $\alpha\in\mathfrak{A}$, be given such that ${}^{\alpha}\pi f_\alpha=f$ for all $\alpha\in\mathfrak{A}$. Then there is a unique map $f_{\mathfrak{A}}\colon X\to Y_{\mathfrak{A}}$ such that ${}^{\mathfrak{A}}_{\alpha}\pi f_{\mathfrak{A}}=f_\alpha$ for all $\alpha\in\mathfrak{A}$. The map $f_{\mathfrak{A}}$ is continuous and satisfies the condition ${}^{\mathfrak{A}}\pi f_{\mathfrak{A}}=f$.*

$$\begin{tikzcd}
 & & Y_{\mathfrak{A}} \arrow[dl, "{}^{\mathfrak{A}}_{\alpha}\pi"'] \arrow[dd, "{}^{\mathfrak{A}}\pi"] \\
X \arrow[urr, "f_{\mathfrak{A}}"] \arrow[r, "f_\alpha"'] \arrow[drr, "f"'] & Y_\alpha \arrow[dr, "{}^{\alpha}\pi"] & \\
 & & Y
\end{tikzcd}$$

Proof. Obviously, the map $f_{\mathfrak{A}}$ has to be defined by the equality $f_{\mathfrak{A}}x = \{fx, f_\alpha x : \alpha \in \mathfrak{A}\}$ for all $x \in X$, and it satisfies the required conditions. The continuity of the map $f_{\mathfrak{A}}$ follows from the definition of the topology of the fan product (see [3], §2 of Supplement to Chapter I, and [51], Proposition 2.3.6). $\square$

2.8. **Proposition.** *Let $X_{\mathfrak{B}} = \prod_X(\{X_\beta\}, \{{}^{\beta}p\}, \beta \in \mathfrak{B})$, ${}^{\mathfrak{B}}p = \prod_X\{{}^{\beta}p, \beta \in \mathfrak{B}\}$ and $Y_{\mathfrak{A}} = \prod_Y(\{Y_\alpha\}, \{{}^{\alpha}\pi\}, \alpha \in \mathfrak{A})$, ${}^{\mathfrak{A}}\pi = \prod_Y\{{}^{\alpha}\pi, \alpha \in \mathfrak{A}\}$. Let $h\colon \mathfrak{A} \to \mathfrak{B}$ be a map, and let mappings $f\colon X \to Y$ and ${}_{\alpha}f\colon X_{h\alpha} \to Y_\alpha$, $\alpha \in \mathfrak{A}$, satisfy the equality ${}^{\alpha}\pi\, {}_{\alpha}f = f\, {}^{h\alpha}p$ for all $\alpha \in \mathfrak{A}$. Then there exists a unique map ${}_{\mathfrak{A}}f\colon X_{\mathfrak{B}} \to Y_{\mathfrak{A}}$ such that ${}^{\mathfrak{A}}_{\alpha}\pi\, {}_{\mathfrak{A}}f = {}_{\alpha}f\, {}^{\mathfrak{B}}_{h\alpha}p$ for all $\alpha \in \mathfrak{A}$. The map ${}_{\mathfrak{A}}f$ is continuous and satisfies the equality ${}^{\mathfrak{A}}\pi\, {}_{\mathfrak{A}}f = f\, {}^{\mathfrak{B}}p$.*

$$\begin{tikzcd}
X_{\mathfrak{B}} \arrow[rrr, "{}_{\mathfrak{A}}f"] \arrow[dr, "{}^{\mathfrak{B}}_{h\alpha}p"] \arrow[dd, "{}^{\mathfrak{B}}p"'] & & & Y_{\mathfrak{A}} \arrow[dl, "{}^{\mathfrak{A}}_{\alpha}\pi"'] \arrow[dd, "{}^{\mathfrak{A}}\pi"] \\
 & X_{h\alpha} \arrow[r, "{}_{\alpha}f"] \arrow[dl, "{}^{h\alpha}p"] & Y_\alpha \arrow[dr, "{}^{\alpha}\pi"] & \\
X \arrow[rrr, "f"] & & & Y
\end{tikzcd}$$

Proof. We can define the mappings $f\, {}^{\mathfrak{B}}p\colon X_{\mathfrak{B}} \to Y$ and ${}_{\alpha}f\, {}^{\mathfrak{B}}_{h\alpha}p\colon X_{\mathfrak{B}} \to Y_\alpha$, $\alpha \in \mathfrak{A}$, and use Proposition 2.7. $\square$

2.9. **Theorem.** a) *If all mappings ${}^{\alpha}\pi\colon Y_\alpha \to Y$, $\alpha \in \mathfrak{A}$, are separable, then the mapping ${}^{\mathfrak{A}}\pi$ is separable too.*

b) *If $\mathfrak{B} \subseteq \mathfrak{A}$ and the mapping ${}^{\mathfrak{B}}\pi$ is separable then the parallel mapping ${}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi$ is separable too.*

Proof. a) Let $x_1, x_2 \in Y_{\mathfrak{A}}$ be points such that $x_1 \neq x_2$ but ${}^{\mathfrak{A}}\pi x_1 = {}^{\mathfrak{A}}\pi x_2$. Then there exists $\alpha \in \mathfrak{A}$ such that ${}^{\mathfrak{A}}_{\alpha}\pi x_1 \neq {}^{\mathfrak{A}}_{\alpha}\pi x_2$. Since the mapping ${}^{\alpha}\pi$ is separable, there exist disjoint neighborhoods $U\,{}^{\mathfrak{A}}_{\alpha}\pi x_1, U\,{}^{\mathfrak{A}}_{\alpha}\pi x_2 \subseteq Y_\alpha$. Their preimages under the mapping ${}^{\mathfrak{A}}_{\alpha}\pi$ are disjoint neighborhoods of the points x_1 and x_2.

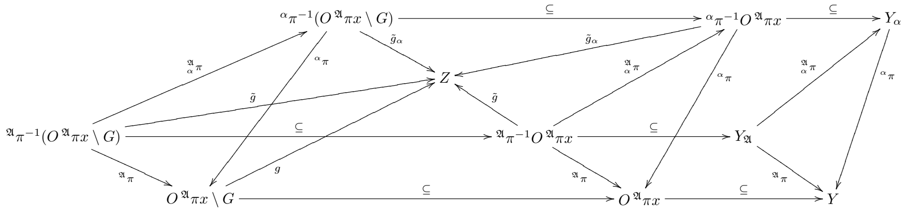
b) Let $x_1, x_2 \in Y_{\mathfrak{A}}$ be distinct points such that ${}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi x_1 = {}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi x_2$. By Proposition 2.3 points $y_1 = {}^{\mathfrak{A}}_{\mathfrak{B}}\pi x_1$ and $y_2 = {}^{\mathfrak{A}}_{\mathfrak{B}}\pi x_2$ are distinct but ${}^{\mathfrak{B}}\pi y_1 = {}^{\mathfrak{B}}\pi y_2$. Since the mapping ${}^{\mathfrak{B}}\pi$ is separable, there are disjoint neighborhoods $Uy_1, Uy_2 \subseteq Y_{\mathfrak{B}}$. Then the sets ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi^{-1}Uy_1$ and ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi^{-1}Uy_2$ are disjoint neighborhoods of the point x_1 and x_2. $\square$

2.10. **Theorem.** a) *If each mapping ${}^{\alpha}\pi\colon Y_\alpha \to Y$, $\alpha \in \mathfrak{A}$, has the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ then the mapping ${}^{\mathfrak{A}}\pi$ has the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ too.*

b) *Let $\mathfrak{B} \subseteq \mathfrak{A}$ and the mapping ${}^{\mathfrak{B}}\pi$ has the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$, and let $\mathfrak{a}'$ be a closed family of locally closed subsets of the space $Y_{\mathfrak{A}\setminus\mathfrak{B}}$ such that ${}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}G \in \mathfrak{a}'$ for all $G \in \mathfrak{a}$. Then the parallel mapping ${}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi$ has the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}'$.*

Proof. a) We have to consider the two cases of Definition 1.6.

Let $x, x' \in Y_{\mathfrak{A}}$ be distinct points such that ${}^{\mathfrak{A}}\pi x = {}^{\mathfrak{A}}\pi x'$. There exists an index $\alpha \in \mathfrak{A}$ such that ${}^{\mathfrak{A}}_{\alpha}\pi x \neq {}^{\mathfrak{A}}_{\alpha}\pi x'$. Since the mapping ${}^{\alpha}\pi$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$, there are a neighborhood $O^{\mathfrak{A}}\pi x \subseteq Y$, a set $G \in \mathfrak{a}$ a space $Z \in \mathfrak{E}$ and mappings $g\colon O^{\mathfrak{A}}\pi x \setminus G \to Z$ and $\tilde{g}_\alpha\colon {}^{\alpha}\pi^{-1}O^{\mathfrak{A}}\pi x \to Z$ such that $[G]_Y \cap O^{\mathfrak{A}}\pi x = G$, $\tilde{g}_\alpha|_{{}^{\alpha}\pi^{-1}(O^{\mathfrak{A}}\pi x \setminus G)} = g\,{}^{\alpha}\pi|_{{}^{\alpha}\pi^{-1}(O^{\mathfrak{A}}\pi x \setminus G)}$ and $\tilde{g}_\alpha {}^{\mathfrak{A}}_{\alpha}\pi x \neq \tilde{g}_\alpha {}^{\mathfrak{A}}_{\alpha}\pi x'$. Then the mapping $\tilde{g} = \tilde{g}_\alpha {}^{\mathfrak{A}}_{\alpha}\pi|_{{}^{\mathfrak{A}}\pi^{-1}O^{\mathfrak{A}}\pi x}$ has all necessary properties.



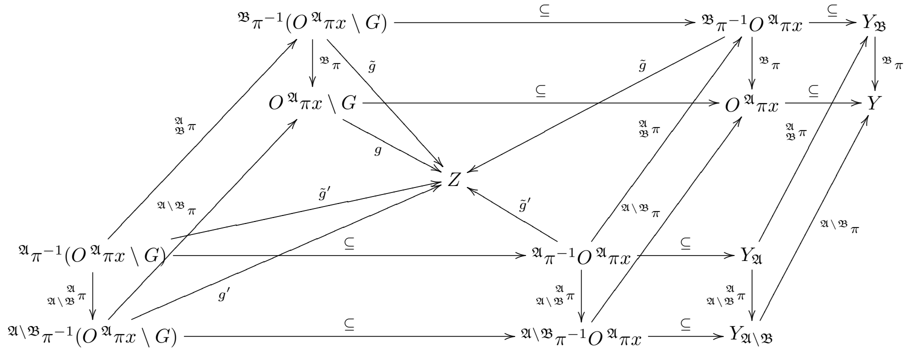
Let $x \in Y_{\mathfrak{A}}$ be any point and $Ux \subseteq Y_{\mathfrak{A}}$ be its neighborhood. By the definition of the fan product there are a finite set $\mathfrak{B} \subseteq \mathfrak{A}$ and neighborhoods $U{}^{\mathfrak{A}}_{\alpha}\pi x \subseteq Y_\alpha$, $\alpha \in \mathfrak{B}$, such that $x \in \bigcap\{{}^{\mathfrak{A}}_{\alpha}\pi^{-1}U{}^{\mathfrak{A}}_{\alpha}\pi x : \alpha \in \mathfrak{B}\} \subseteq Ux$. Since each mapping ${}^{\alpha}\pi$, $\alpha \in \mathfrak{B}$, has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$, for every $\alpha \in \mathfrak{B}$ there exist a neighborhood $O_\alpha{}^{\mathfrak{A}}\pi x \subseteq Y$, a set $G_\alpha \in \mathfrak{a}$, a space $Z_\alpha \in \mathfrak{E}$ and mappings $g_\alpha\colon O_\alpha{}^{\mathfrak{A}}\pi x \setminus G \to Z_\alpha$ and $\tilde{g}_\alpha\colon {}^{\alpha}\pi^{-1}O_\alpha{}^{\mathfrak{A}}\pi x \to Z_\alpha$ such that $[G_\alpha]_Y \cap O_\alpha{}^{\mathfrak{A}}\pi x = G_\alpha$, $\tilde{g}_\alpha|_{{}^{\alpha}\pi^{-1}(O_\alpha{}^{\mathfrak{A}}\pi x \setminus G)} = g_\alpha{}^{\alpha}\pi|_{{}^{\alpha}\pi^{-1}(O_\alpha{}^{\mathfrak{A}}\pi x \setminus G)}$ and $\tilde{g}_\alpha{}^{\mathfrak{A}}_{\alpha}\pi x \notin [\tilde{g}_\alpha({}^{\alpha}\pi^{-1}O_\alpha{}^{\mathfrak{A}}\pi x \setminus U{}^{\mathfrak{A}}_{\alpha}\pi x)]_{Z_\alpha}$. Then we can take $O^{\mathfrak{A}}\pi x = \bigcap\{O_\alpha{}^{\mathfrak{A}}\pi x : \alpha \in \mathfrak{B}\}$, $G = \bigcup\{G_\alpha \cap O^{\mathfrak{A}}\pi x : \alpha \in \mathfrak{B}\}$, $Z = \prod\{Z_\alpha : \alpha \in \mathfrak{B}\}$, $g = \Delta\{g_\alpha|_{O^{\mathfrak{A}}\pi x \setminus G} : \alpha \in \mathfrak{B}\}\colon O^{\mathfrak{A}}\pi x \setminus G \to Z$ and $\tilde{g} = \Delta\{\tilde{g}_\alpha{}^{\mathfrak{A}}_{\alpha}\pi|_{{}^{\mathfrak{A}}\pi^{-1}O^{\mathfrak{A}}\pi x} : \alpha \in \mathfrak{B}\}\colon {}^{\mathfrak{A}}\pi^{-1}O^{\mathfrak{A}}\pi x \to Z$ (the diagonal mapping; see [51], §2.3).

Obviously, $[G]_Y \cap O^{\mathfrak{A}}\pi x = G$ and $\tilde{g}|_{{}^{\mathfrak{A}}\pi^{-1}(O^{\mathfrak{A}}\pi x \setminus G)} = g\,{}^{\mathfrak{A}}\pi|_{{}^{\mathfrak{A}}\pi^{-1}(O^{\mathfrak{A}}\pi x \setminus G)}$.

Let $V_\alpha = Z_\alpha \setminus [\tilde{g}_\alpha({}^{\alpha}\pi^{-1}O_\alpha{}^{\mathfrak{A}}\pi x \setminus U{}^{\mathfrak{A}}_{\alpha}\pi x)]_{Z_\alpha}$, $\alpha \in \mathfrak{B}$; then $\tilde{g}_\alpha^{-1}V_\alpha \subseteq U{}^{\mathfrak{A}}_{\alpha}\pi x$ is a neighborhood of the point ${}^{\mathfrak{A}}_{\alpha}\pi x$ for all $\alpha \in \mathfrak{B}$. Therefore the set $V = \bigcap\{{}^{\mathfrak{A}}_{\alpha}\pi^{-1}\tilde{g}_\alpha^{-1}V_\alpha : \alpha \in \mathfrak{B}\} \subseteq \bigcap\{{}^{\mathfrak{A}}_{\alpha}\pi^{-1}U{}^{\mathfrak{A}}_{\alpha}\pi x : \alpha \in \mathfrak{B}\} \subseteq Ux$ is a neighborhood of the point x. We have the equality $V = \bigcap\{{}^{\mathfrak{A}}_{\alpha}\pi^{-1}\tilde{g}_\alpha^{-1}V_\alpha : \alpha \in \mathfrak{B}\} = \tilde{g}^{-1}\prod\{V_\alpha : \alpha \in \mathfrak{B}\}$, hence, $\tilde{g}x \in \prod\{V_\alpha : \alpha \in \mathfrak{B}\} \subseteq \prod\{Z_\alpha : \alpha \in \mathfrak{B}\} \setminus [\tilde{g}({}^{\mathfrak{A}}\pi^{-1}O^{\mathfrak{A}}\pi x \setminus Ux)]_Z$, that is, the mapping ${}^{\mathfrak{A}}\pi$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$.

b) We have to consider the two cases of Definition 1.6

Let $x, x' \in Y_{\mathfrak{A}}$ be distinct points such that ${}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi x = {}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi x'$. By Proposition 2.3 we have ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi x \neq {}^{\mathfrak{A}}_{\mathfrak{B}}\pi x'$. Since the mapping ${}^{\mathfrak{B}}\pi$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$, there exist a neighborhood $O^{\mathfrak{A}}\pi x \subseteq Y$, a set $G \in \mathfrak{a}$, a space $Z \in \mathfrak{E}$ and mappings $g\colon O^{\mathfrak{A}}\pi x \setminus G \to Z$ and $\tilde{g}\colon {}^{\mathfrak{B}}\pi^{-1}O^{\mathfrak{A}}\pi x \to Z$ such that $[G]_Y \cap O^{\mathfrak{A}}\pi x = G$, $\tilde{g}|_{{}^{\mathfrak{B}}\pi^{-1}(O^{\mathfrak{A}}\pi x \setminus G)} = g\,{}^{\mathfrak{B}}\pi|_{{}^{\mathfrak{B}}\pi^{-1}(O^{\mathfrak{A}}\pi x \setminus G)}$ and $\tilde{g}\,{}^{\mathfrak{A}}_{\mathfrak{B}}\pi x \neq \tilde{g}\,{}^{\mathfrak{A}}_{\mathfrak{B}}\pi x'$. Then the sets $O' = O\,{}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi x = {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}O^{\mathfrak{A}}\pi x$, $G' = {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}G$ and the mappings $g' = g\,{}^{\mathfrak{A}\setminus\mathfrak{B}}\pi|_{O' \setminus G'}$ and $\tilde{g}' = \tilde{g}\,{}^{\mathfrak{A}}_{\mathfrak{B}}\pi|_{{}^{\mathfrak{A}}_{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}O'}$ have all necessary properties.



Let $x \in Y_{\mathfrak{A}}$ be any point and $Ux \subseteq Y_{\mathfrak{A}}$ be its neighborhood. By Proposition 2.4 and by the definition of the fan product there are neighborhoods $U\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x \subseteq Y_{\mathfrak{A}\setminus\mathfrak{B}}$ and $U\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x \subseteq Y_{\mathfrak{B}}$ such that $x \in {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}U\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x \cap {}_{\mathfrak{B}}^{\mathfrak{A}}\pi^{-1}U\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x \subseteq Ux$. Since the mapping ${}^{\mathfrak{B}}\pi$ has the property $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$, there exist a neighborhood $O\,{}^{\mathfrak{A}}\pi x \subseteq Y$, a set $G \in \mathfrak{a}$, a space $Z \in \mathfrak{C}$ and mappings $g\colon O\,{}^{\mathfrak{A}}\pi x \setminus G \to Z$ and $\tilde{g}\colon {}^{\mathfrak{B}}\pi^{-1}O\,{}^{\mathfrak{A}}\pi x \to Z$ such that $[G]_Y \cap O\,{}^{\mathfrak{A}}\pi x = G$, $\tilde{g}|_{{}^{\mathfrak{B}}\pi^{-1}(O\,{}^{\mathfrak{A}}\pi x\setminus G)} = g\,{}^{\mathfrak{B}}\pi|_{{}^{\mathfrak{B}}\pi^{-1}(O\,{}^{\mathfrak{A}}\pi x\setminus G)}$ and $\tilde{g}\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x \notin$ $\notin [\tilde{g}({}^{\mathfrak{B}}\pi^{-1}O\,{}^{\mathfrak{A}}\pi x \setminus U\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x)]_Z$. Then the sets $O\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x = U\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x \cap {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}O\,{}^{\mathfrak{A}}\pi x$, $G' = U\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x \cap {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}G \in \mathfrak{a}$ and the mappings $g' = g\,{}^{\mathfrak{A}\setminus\mathfrak{B}}\pi|_{O\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x\setminus G'}$ and $\tilde{g}' = \tilde{g}\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi|_{{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}O\,{}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi x}$ have all necessary properties. $\square$

2.11. **Theorem.** a) *If $\mathfrak{B} \subseteq \mathfrak{A}$ and the mapping ${}^{\mathfrak{B}}\pi$ is perfect, then its parallel mapping ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi$ is perfect too.*

b) *If all mappings ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, are perfect then the mapping ${}^{\mathfrak{A}}\pi$ is perfect too.*

Proof. The mapping ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi$ is compact due to Proposition 2.3. Let us take any point $z \in Y_{\mathfrak{A}\setminus\mathfrak{B}}$ and prove that the mapping ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi$ is closed at the point z. Let us denote $\Phi = {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}z$, $y = {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi z$, $F = {}^{\mathfrak{B}}\pi^{-1}y (= {}_{\mathfrak{B}}^{\mathfrak{A}}\pi\Phi)$. Note that the set ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi Y_{\mathfrak{A}} = {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}\,{}^{\mathfrak{B}}\pi Y_{\mathfrak{B}}$ is closed since the mapping ${}^{\mathfrak{B}}\pi$ is perfect. Hence, if $z \notin {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi Y_{\mathfrak{A}}$ then there exists a neighborhood $Uz \subseteq Y_{\mathfrak{A}\setminus\mathfrak{B}}$ such that $Uz \cap {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi Y_{\mathfrak{A}} =$ $= \varnothing$, therefore the mapping ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi$ is closed at the point z.

Let us suppose that $z \in {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi Y_{\mathfrak{A}}$ and let $U\Phi \subseteq Y_{\mathfrak{A}}$ be any neighborhood. We have to find a neighborhood $Uz \subseteq Y_{\mathfrak{A}\setminus\mathfrak{B}}$ such that ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}Uz \subseteq U\Phi$.

By the definition of the fan product, for each point $x \in \Phi$ there are neighborhoods $U_x z \subseteq Y_{\mathfrak{A}\setminus\mathfrak{B}}$ and $U\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x \subseteq Y_{\mathfrak{B}}$ such that $x \in Ux = {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}U_x z \cap {}_{\mathfrak{B}}^{\mathfrak{A}}\pi^{-1}U\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x \subseteq$ $\subseteq U\Phi$. The set $\{Ux : x \in \Phi\}$ is an open covering of the compact set Φ. Let $\{Ux_i : i = 1, 2, \ldots, n\}$ be a finite subcovering, $V_1 = \bigcap\{U_{x_i}z : i = 1, 2, \ldots, n\}$, $V_2 = \bigcup\{U\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi x_i : i = 1, 2, \ldots, n\}$. Then $z \in V_1$, $F \subseteq V_2$ and $\Phi \subseteq {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}V_1 \cap$ $\cap\, {}_{\mathfrak{B}}^{\mathfrak{A}}\pi^{-1}V_2 \subseteq U\Phi$. Since the mapping ${}^{\mathfrak{B}}\pi$ is perfect, there exists a neighborhood $Vy \subseteq Y$ such that $F \subseteq {}^{\mathfrak{B}}\pi^{-1}Vy \subseteq V_2$. Then the set $Uz = V_1 \cap {}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}Vy$ satisfies the condition $\Phi \subseteq {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}Uz = {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}V_1 \cap {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}\,{}^{\mathfrak{A}\setminus\mathfrak{B}}\pi^{-1}Vy = {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}V_1 \cap$ $\cap\, {}_{\mathfrak{B}}^{\mathfrak{A}}\pi^{-1}\,{}^{\mathfrak{B}}\pi^{-1}Vy \subseteq {}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi^{-1}V_1 \cap {}_{\mathfrak{B}}^{\mathfrak{A}}\pi^{-1}V_2 \subseteq U\Phi$, that is, the mapping ${}_{\mathfrak{A}\setminus\mathfrak{B}}\overset{\mathfrak{A}}{}\pi$ is closed at the point z.

b) The mapping ${}^{\mathfrak{A}}\pi$ is compact due to Proposition 2.2. Note that the set ${}^{\mathfrak{A}}\pi Y_{\mathfrak{A}} =$ $= \bigcap\{{}^{\alpha}\pi Y_{\alpha} : \alpha \in \mathfrak{A}\}$ is closed since all mappings ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, are closed, therefore the mapping ${}^{\mathfrak{A}}\pi$ is closed at any point $y \in Y \setminus {}^{\mathfrak{A}}\pi Y_{\mathfrak{A}}$. Let us take any point $y \in {}^{\mathfrak{A}}\pi Y_{\mathfrak{A}}$ and prove that the mapping ${}^{\mathfrak{A}}\pi$ is closed at the point y.

Let $\Phi = {}^{\mathfrak{A}}\pi^{-1}y$ and $U\Phi \subseteq Y_{\mathfrak{A}}$ be any neighborhood. For each point $x \in \Phi$ there exist a finite set $\mathfrak{B}_x \subseteq \mathfrak{A}$ and neighborhoods $U\,{}_{\alpha}^{\mathfrak{A}}\pi x \subseteq Y_{\alpha}$, $\alpha \in \mathfrak{B}_x$, such that $x \in Ux = \bigcap\{{}_{\alpha}^{\mathfrak{A}}\pi^{-1}U\,{}_{\alpha}^{\mathfrak{A}}\pi x : \alpha \in \mathfrak{B}_x\} \subseteq U\Phi$. The family $\{Ux : x \in \Phi\}$ is an open covering of the compact set Φ. Let $\{Ux_i : i = 1, 2, \ldots, n\}$ be a finite subcovering, $V = \bigcup\{Ux_i : i = 1, 2, \ldots, n\}$, $\mathfrak{B} = \bigcup\{\mathfrak{B}_{x_i} : i = 1, 2, \ldots, n\}$. Note that $V = {}_{\mathfrak{B}}^{\mathfrak{A}}\pi^{-1}\,{}_{\mathfrak{B}}^{\mathfrak{A}}\pi V$ and $\mathfrak{B}$ is a finite set, $\Phi \subseteq V \subseteq U\Phi$.

It follows from the statement a) and Proposition 5a) of §10 of Chapter I of the book [7] that the fan product of two perfect mappings is perfect; by Proposition 2.4, the fan product of any finite family of perfect mappings is perfect. Therefore the mapping ${}^{\mathfrak{B}}\pi$ is perfect. Hence there is a neighborhood $Uy \subseteq Y$ such that ${}^{\mathfrak{B}}\pi^{-1}Uy \subseteq \subseteq {}^{\mathfrak{A}}_{\mathfrak{B}}\pi V$, that is, $\Phi \subseteq {}^{\mathfrak{A}}\pi^{-1}Uy = {}^{\mathfrak{A}}_{\mathfrak{B}}\pi^{-1}\,{}^{\mathfrak{B}}\pi^{-1}Uy \subseteq {}^{\mathfrak{A}}_{\mathfrak{B}}\pi^{-1}\,{}^{\mathfrak{A}}_{\mathfrak{B}}\pi V = V \subseteq U\Phi$. This means that the mapping ${}^{\mathfrak{A}}\pi$ is closed at the point y. □

2.12. **Proposition.** *If $X \subseteq Y_{\mathfrak{A}}$ is a subset such that the set $O_0 = Y \setminus {}^{\mathfrak{A}}\pi[X]_{Y_{\mathfrak{A}}}$ is open, the mapping ${}^{\mathfrak{A}}\pi_0 = {}^{\mathfrak{A}}\pi|_{{}^{\mathfrak{A}}\pi^{-1}O_0} \colon {}^{\mathfrak{A}}\pi^{-1}O_0 \xrightarrow{\text{onto}} O_0$ is perfect,*[5] *and for every $\alpha \in \mathfrak{A}$ the mapping ${}^{\alpha}\pi$ can be reduced modulo ${}^{\mathfrak{A}}_{\alpha}\pi X$ in a unique way, then the mapping ${}^{\mathfrak{A}}\pi$ can be reduced modulo X in a unique way* (*that is, there exists a unique closed subset $Y^r_{\mathfrak{A}} \subseteq Y_{\mathfrak{A}}$ such that $X \subseteq Y^r_{\mathfrak{A}}$, ${}^{\mathfrak{A}}\pi Y^r_{\mathfrak{A}} = Y$ and the mapping ${}^{\mathfrak{A}}\pi_r = {}^{\mathfrak{A}}\pi|_{Y^r_{\mathfrak{A}}}$ is irreducible modulo X*).

Proof. We can assume that the mapping ${}^{\alpha}\pi$ is irreducible modulo ${}^{\mathfrak{A}}_{\alpha}\pi X$ for every $\alpha \in \mathfrak{A}$.

We shall assume that the set O_0 is non-empty; otherwise we could use $Y^r_{\mathfrak{A}} = = [X]_{Y_{\mathfrak{A}}}$.

Let us denote by $\mathfrak{B}$ the family of all open dense subsets of the set O_0. For every $U \in \mathfrak{B}$ let $F_U = [{}^{\mathfrak{A}}\pi^{-1}U]_{Y_{\mathfrak{A}}}$ and let $F_0 = \bigcap\{F_U : U \in \mathfrak{B}\}$. The set F_0 is closed in $Y_{\mathfrak{A}}$. We shall prove that $O_0 \subseteq {}^{\mathfrak{A}}\pi F_0$.

Let $U_1, U_2, \dots, U_n \in \mathfrak{B}$. The set $\bigcap\{U_i : i = 1, 2, \dots, n\}$ is open and dense in O_0. Therefore ${}^{\mathfrak{A}}\pi\bigcap\{F_{U_i} : 1 \leqslant i \leqslant n\} = {}^{\mathfrak{A}}\pi\bigcap\{[{}^{\mathfrak{A}}\pi^{-1}U_i]_{Y_{\mathfrak{A}}} : 1 \leqslant i \leqslant \leqslant n\} \supseteq {}^{\mathfrak{A}}\pi[\bigcap\{{}^{\mathfrak{A}}\pi^{-1}U_i : 1 \leqslant i \leqslant n\}]_{Y_{\mathfrak{A}}} \supseteq {}^{\mathfrak{A}}\pi_0[\bigcap\{{}^{\mathfrak{A}}\pi^{-1}U_i : 1 \leqslant i \leqslant n\}]_{{}^{\mathfrak{A}}\pi^{-1}O_0} = = [{}^{\mathfrak{A}}\pi_0{}^{\mathfrak{A}}\pi^{-1}\bigcap\{U_i : 1 \leqslant i \leqslant n\}]_{O_0} = [\bigcap\{U_i : 1 \leqslant i \leqslant n\}]_{O_0} = O_0$ since the mapping ${}^{\mathfrak{A}}\pi_0$ is closed.

Therefore for each $y \in O_0$ the family $\{{}^{\mathfrak{A}}\pi^{-1}y \cap F_U : U \in \mathfrak{B}\}$ is a centered family of closed subsets of the compact space ${}^{\mathfrak{A}}\pi^{-1}y$, hence $\bigcap\{F_U \cap {}^{\mathfrak{A}}\pi^{-1}y : U \in \mathfrak{B}\} \neq \varnothing$ and ${}^{\mathfrak{A}}\pi F_0 \supseteq O_0$.

Let us prove that the mapping ${}^{\mathfrak{A}}\pi_1 = {}^{\mathfrak{A}}\pi|_{F_0 \cap {}^{\mathfrak{A}}\pi^{-1}O_0} \colon F_0 \cap {}^{\mathfrak{A}}\pi^{-1}O_0 \xrightarrow{\text{onto}} O_0$ is irreducible. To this end, let us take any point $z_0 \in F_0 \cap {}^{\mathfrak{A}}\pi^{-1}O_0$ and prove that

$$(1) \qquad {}^{\mathfrak{A}}\pi^{\#}Uz_0 \neq \varnothing \text{ for any neighborhood } Uz_0 \subseteq Y_{\mathfrak{A}}.$$

Let $Uz_0 \subseteq Y_{\mathfrak{A}}$ be any neighborhood. By the definition of the topology of the fan product there exist a finite subset $\mathfrak{A}_0 \subseteq \mathfrak{A}$ and open subsets $U^0_\alpha \subseteq {}^{\alpha}\pi^{-1}O_0$, $\alpha \in \mathfrak{A}_0$, such that $z_0 \in \bigcap\{{}^{\mathfrak{A}}_{\alpha}\pi^{-1}U^0_\alpha : \alpha \in \mathfrak{A}_0\} \subseteq Uz_0 \cap {}^{\mathfrak{A}}\pi^{-1}O_0$. For each $\alpha \in \mathfrak{A}_0$ let $U^1_\alpha = {}^{\alpha}\pi^{-1}O_\alpha \setminus [U^0_\alpha]_{Y_\alpha}$ and $U_\alpha = {}^{\alpha}\pi^{\#}U^0_\alpha \cup {}^{\alpha}\pi^{\#}U^1_\alpha$.

The sets U_α, $\alpha \in \mathfrak{A}_0$, are dense open subsets of O_0, since for all $\alpha \in \mathfrak{A}$ the mappings ${}^{\alpha}\pi|_{{}^{\alpha}\pi^{-1}O_0} \colon {}^{\alpha}\pi^{-1}O_0 \xrightarrow{\text{onto}} O_0$, are closed and irreducible.

It suffices to prove that $\bigcap\{{}^{\alpha}\pi^{\#}U^0_\alpha : \alpha \in \mathfrak{A}_0\} \neq \varnothing$, since

$${}^{\mathfrak{A}}\pi^{\#}Uz_0 \supseteq {}^{\mathfrak{A}}\pi^{\#}\bigcap\{{}^{\mathfrak{A}}_{\alpha}\pi^{-1}U^0_\alpha : \alpha \in \mathfrak{A}_0\} = \bigcap\{{}^{\alpha}\pi^{\#}U^0_\alpha : \alpha \in \mathfrak{A}_0\}.$$

Obviously, ${}^{\alpha}\pi^{\#}U^0_\alpha \cap {}^{\alpha}\pi^{\#}U^1_\alpha = \varnothing$. Therefore $U = \bigcap\{U_\alpha : \alpha \in \mathfrak{A}_0\} \subseteq (\bigcap\{{}^{\alpha}\pi^{\#}U^0_\alpha : \alpha \in \mathfrak{A}_0\}) \cup (\bigcup\{{}^{\alpha}\pi^{\#}U^1_\alpha : \alpha \in \mathfrak{A}_0\})$ and U is an open dense subset of O_0. If we assume that $\bigcap\{{}^{\alpha}\pi^{\#}U^0_\alpha : \alpha \in \mathfrak{A}_0\} = \varnothing$, then the set $\bigcup\{{}^{\alpha}\pi^{\#}U^1_\alpha : \alpha \in \mathfrak{A}_0\}$ is an open dense subset of O_0, and due to the construction of the set F_0 we have $z_0 \in [{}^{\mathfrak{A}}\pi^{-1}\bigcup\{{}^{\alpha}\pi^{\#}U^1_\alpha : \alpha \in \mathfrak{A}_0\}]_{Y_{\mathfrak{A}}} = \bigcup\{[{}^{\mathfrak{A}}\pi^{-1}\,{}^{\alpha}\pi^{\#}U^1_\alpha]_{Y_{\mathfrak{A}}} : \alpha \in \mathfrak{A}_0\} \subseteq \subseteq \bigcup\{[{}^{\mathfrak{A}}_{\alpha}\pi^{-1}U^1_\alpha]_{Y_{\mathfrak{A}}} : \alpha \in \mathfrak{A}_0\}$.

But the latter is impossible due to the choice of the sets U^1_α, $\alpha \in \mathfrak{A}_0$. Hence, the mapping ${}^{\mathfrak{A}}\pi_1$ is irreducible. Moreover, it is obvious that if a set $F \subseteq {}^{\mathfrak{A}}\pi^{-1}O_0$ is closed in ${}^{\mathfrak{A}}\pi^{-1}O_0$, ${}^{\mathfrak{A}}\pi F = O_0$ and the mapping ${}^{\mathfrak{A}}\pi|_F \colon F \xrightarrow{\text{onto}} O_0$ is irreducible, then the set F contains all points $z_0 \in {}^{\mathfrak{A}}\pi^{-1}O_0$ which satisfy the condition (1).

[5]Hence, all mappings ${}^{\alpha}\pi|_{{}^{\alpha}\pi^{-1}O_0} \colon {}^{\alpha}\pi^{-1}O_0 \xrightarrow{\text{onto}} O_0$, $\alpha \in \mathfrak{A}$, are perfect due to Proposition 5b) of §10 of Chapter I of the book [7].

Therefore $F = F_0 \cap {}^{\mathfrak{A}}\pi^{-1}O_0$, that is, the mapping ${}^{\mathfrak{A}}\pi|_{{}^{\mathfrak{A}}\pi^{-1}O_0}$ can be reduced in a unique way.

To conclude the proof it suffices to let $Y^r_{\mathfrak{A}} = [X]_{Y_{\mathfrak{A}}} \cup F_0$. $\square$

2.13. **Corollary.** *If $X \subseteq Y_{\mathfrak{A}}$ and all mappings ${}^{\alpha}\pi\colon Y_\alpha \xrightarrow{\text{onto}} Y$, $\alpha \in \mathfrak{A}$, are perfect and irreducible modulo ${}^{\mathfrak{A}}_{\alpha}\pi X$ (or can be reduced modulo ${}^{\mathfrak{A}}_{\alpha}\pi X$ in a unique way) then the mapping ${}^{\mathfrak{A}}\pi$ can be reduced modulo X in a unique way.*

B. **The inverse limit**

2.14. Further let $S = \{Y_\alpha, {}^{\alpha}_{\beta}\pi : \alpha, \beta \in \mathfrak{A}, \beta \leqslant \alpha\}$ be an inverse spectrum, ${}^{\alpha}\pi\colon Y_\alpha \to Y$, $\alpha \in \mathfrak{A}$, be mappings such that ${}^{\alpha}\pi = {}^{\beta}\pi\, {}^{\alpha}_{\beta}\pi$ for all $\alpha, \beta \in \mathfrak{A}, \beta \leqslant \alpha$. Let $Y_S = \varprojlim S$, ${}^{S}\pi = \varprojlim {}^{\alpha}\pi$ and ${}^{S}_{\alpha}\pi\colon Y_S \to Y_\alpha$ be the projection of the space Y_S to Y_α, $\alpha \in \mathfrak{A}$ (see [51], §2.5).

We can assume that the space Y_S is a subspace of the fan product $Y_{\mathfrak{A}} = \prod_Y(\{Y_\alpha\}, \{{}^{\alpha}\pi\}, \alpha \in \mathfrak{A})$:

$$Y_S = \{\{y, z_\alpha : \alpha \in \mathfrak{A}\} \in Y_{\mathfrak{A}} : z_\beta = {}^{\alpha}_{\beta}\pi z_\alpha \text{ for all } \alpha, \beta \in \mathfrak{A}, \beta \leqslant \alpha\}.$$

To prove that the limit topology of the space Y_S coincides with the topology of the subspace of the fan product $Y_{\mathfrak{A}}$ it suffices to observe that $Y_S \subseteq Y_{\mathfrak{A}} \subseteq \prod\{Y_\alpha : \alpha \in \mathfrak{A}\}$.

Note that ${}^{S}\pi = {}^{\mathfrak{A}}\pi|_{Y_S}$ were ${}^{\mathfrak{A}}\pi = \prod_Y\{{}^{\alpha}\pi : \alpha \in \mathfrak{A}\}$, ${}^{S}_{\alpha}\pi = {}^{\mathfrak{A}}_{\alpha}\pi|_{Y_S}$ for all $\alpha \in \mathfrak{A}$, but ${}^{\mathfrak{A}}_{\beta}\pi \neq {}^{\alpha}_{\beta}\pi\, {}^{\mathfrak{A}}_{\alpha}\pi$ for $\alpha, \beta \in \mathfrak{A}, \beta \leqslant \alpha$, in general.

$$\begin{tikzcd}
Y_S \arrow[rrrr, "\subseteq"] \arrow[dr, "{}^{S}_{\beta}\pi"] \arrow[drrr, "{}^{S}_{\alpha}\pi"] \arrow[ddrr, "{}^{S}\pi"'] & & & & Y_{\mathfrak{A}} \arrow[dl, "{}^{\mathfrak{A}}_{\alpha}\pi"'] \arrow[ddll, "{}^{\mathfrak{A}}\pi"] \\
& Y_\beta \arrow[dr, "{}^{\beta}\pi"] & & Y_\alpha \arrow[ll, "{}^{\alpha}_{\beta}\pi"] \arrow[dl, "{}^{\alpha}\pi"'] & \\
& & Y & &
\end{tikzcd}$$

2.15. **Proposition.** *Let all mappings ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, be separable. Then*

1) *the mappings ${}^{\alpha}_{\beta}\pi$, $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, are separable;*
2) *the mappings ${}^{S}\pi$ and ${}^{S}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, are separable;*
3) *the space Y_S is a closed subspace of the space $Y_{\mathfrak{A}}$.*

Proof. 1) This statement is a consequence of the following fact: if $f\colon X \to Y$, $g\colon X \to Z$ and $h\colon Z \to Y$ are mappings such that $f = hg$ and the mapping f is separable then the mapping g is separable too.

$$\begin{tikzcd}
& X \arrow[dl, "g"'] \arrow[dr, "f"] & \\
Z \arrow[rr, "h"] & & Y
\end{tikzcd}$$

2) This follows from Theorem 2.9 a).

3) For each $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, let $F_{\alpha\beta} = \{\{y, z_\gamma : \gamma \in \mathfrak{A}\} \in Y_{\mathfrak{A}} : z_\beta = {}^{\alpha}_{\beta}\pi z_\alpha\}$.

Then $Y_S = \bigcap\{F_{\alpha\beta} : \alpha, \beta \in \mathfrak{A}, \beta \leqslant \alpha\}$. Let us prove that the set $F_{\alpha\beta}$ is closed in $Y_{\mathfrak{A}}$ for every $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$.

Let $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, $x = \{y, z_\gamma : \gamma \in \mathfrak{A}\} \in Y_{\mathfrak{A}} \setminus F_{\alpha\beta}$. This means that $z_\beta = {}^{\mathfrak{A}}_{\beta}\pi x \neq z'_\beta = {}^{\alpha}_{\beta}\pi z_\alpha = {}^{\alpha}_{\beta}\pi\, {}^{\mathfrak{A}}_{\alpha}\pi x$. Since the mapping ${}^{\beta}\pi$ is separable and ${}^{\beta}\pi z_\beta = {}^{\beta}\pi z'_\beta = y$, there exist neighborhoods $Uz_\beta, Uz'_\beta \subseteq Y_\beta$ such that $Uz_\beta \cap Uz'_\beta = \varnothing$. Let $Uz_\alpha = {}^{\alpha}_{\beta}\pi^{-1}Uz'_\beta$; obviously, Uz_α is a neighborhood of the point $z_\alpha \in Y_\alpha$.

Therefore the set $Ux = {}^{\mathfrak{A}}_{\alpha}\pi^{-1}Uz_\alpha \cap {}^{\mathfrak{A}}_{\beta}\pi^{-1}Uz_\beta$ is a neighborhood of the point $x \in Y_{\mathfrak{A}}$. Moreover, if $x' \in Ux$ is an arbitrary point, then ${}^{\mathfrak{A}}_{\beta}\pi x' \in Uz_\beta$ and ${}^{\alpha}_{\beta}\pi\, {}^{\mathfrak{A}}_{\alpha}\pi x' \in Uz'_\beta$, hence, ${}^{\mathfrak{A}}_{\beta}\pi x' \neq {}^{\alpha}_{\beta}\pi\, {}^{\mathfrak{A}}_{\alpha}\pi x'$. This means that $Ux \cap F_{\alpha\beta} = \varnothing$, that is, the set $F_{\alpha\beta}$ is closed. Hence, the set Y_S is closed too. □

2.16. **Corollary.** *If all mappings ${}^{\alpha}_{\beta}\pi$, $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, are separable, then all projections ${}^{S}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, are separable too.*

2.17. **Theorem.** *If each mapping ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$, then the mapping ${}^{S}\pi$ has the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$ too.*

Proof follows from Theorem 2.10 a) and Proposition 2.15 3). □

2.18. **Theorem.** *If each mapping ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, is separable and perfect (and "onto") then the mapping ${}^{S}\pi$ is also separable and perfect (and "onto"). Moreover, the mappings ${}^{S}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, and ${}^{\alpha}_{\beta}\pi$, $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, are separable and perfect too.*

Proof. The mapping ${}^{S}\pi$ is separable and perfect by Proposition 2.15 and Theorem 2.11. The mappings ${}^{S}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, and ${}^{\alpha}_{\beta}\pi$, $\alpha, \beta \in A$, $\beta \leqslant \alpha$, are separable by Proposition 2.15 and perfect by Lemma 8 of the paper [43].

Let all mappings ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, be "onto". Let us take any point $y \in Y$ and prove that the set ${}^{S}\pi^{-1}y = Y_S \cap {}^{\mathfrak{A}}\pi^{-1}y$ is not empty.

Let $\alpha \in \mathfrak{A}$, and let $F_\alpha = \{\{y, z_\gamma : \gamma \in \mathfrak{A}\} : z_\beta = {}^{\alpha}_{\beta}\pi z_\alpha \text{ for all } \beta \in \mathfrak{A}, \beta \leqslant \alpha\}$.

Note that $F_\alpha = \bigcap\{F_{\alpha\beta} : \beta \in \mathfrak{A}, \beta \leqslant \alpha\}$, where the closed sets $F_{\alpha\beta}$, $\alpha, \beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, were defined in the proof of the statement 3) of Proposition 2.15. Hence, the set F_α is closed. Moreover, the set $F_{y\alpha} = F_\alpha \cap {}^{\mathfrak{A}}\pi^{-1}y$ is non-empty, since we can define a point $\{y, z_\gamma : \gamma \in \mathfrak{A}\} \in F_{y\alpha}$, if we choose any $z_\alpha \in {}^{\alpha}\pi^{-1}y$, put $z_\beta = {}^{\alpha}_{\beta}\pi z_\alpha$ for $\beta \in \mathfrak{A}$, $\beta \leqslant \alpha$, and choose arbitrary elements $z_\gamma \in {}^{\gamma}\pi^{-1}y \neq \varnothing$ for $\gamma \in \mathfrak{A}$ such that the inequality $\gamma \leqslant \alpha$ does not hold.

Since $F_{y\alpha} \subseteq F_{y\beta}$ for all $\alpha, \beta \in A$, $\beta \leqslant \alpha$, the family $\{F_{y\alpha} : \alpha \in \mathfrak{A}\}$ is a centered family of closed subsets of the compact space ${}^{\mathfrak{A}}\pi^{-1}y$. Therefore we have ${}^{S}\pi^{-1}y = \bigcap\{F_{y\alpha} : \alpha \in \mathfrak{A}\} \neq \varnothing$. Hence, ${}^{S}\pi Y_S = Y$. □

2.19. **Corollary.** *If all mappings ${}^{\alpha}_{\beta}\pi$, $\alpha, \beta \in A$, $\beta \leqslant \alpha$, are separable and perfect (and "onto") then all mappings ${}^{S}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, are separable and perfect (and "onto").*

2.20. **Assertion.** *Let ${}^{S}\pi Y_S = Y$ and $X \subseteq Y_S$ be a subset such that each mapping ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, is irreducible modulo ${}^{S}_{\alpha}\pi X$. Then the mapping ${}^{S}\pi$ is irreducible modulo X.*

Proof. Let $U \subseteq Y_S$ be an arbitrary non-empty open set such that $U \cap X = \varnothing$. We must prove that there is a point $y \in Y$ such that ${}^{S}\pi^{-1}y \subseteq U$.

Let us choose some point $x \in U$. By Proposition 2.2.5 of the book [51] there are an index $\alpha \in \mathfrak{A}$ and a neighborhood $U\,{}^{S}_{\alpha}\pi x \subseteq Y_\alpha$ such that ${}^{S}_{\alpha}\pi^{-1}U\,{}^{S}_{\alpha}\pi x \subseteq U$. Since $U\,{}^{S}_{\alpha}\pi x \cap {}^{S}_{\alpha}\pi X = \varnothing$ and the mapping ${}^{\alpha}\pi$ is irreducible modulo ${}^{S}_{\alpha}\pi X$, there exists a point $y \in Y$ such that ${}^{\alpha}\pi^{-1}y \subseteq U\,{}^{S}_{\alpha}\pi x$. Then we have ${}^{S}\pi^{-1}y = {}^{S}_{\alpha}\pi^{-1}\,{}^{\alpha}\pi^{-1}y \subseteq {}^{S}_{\alpha}\pi^{-1}U\,{}^{S}_{\alpha}\pi x \subseteq U$. Hence, the mapping ${}^{S}\pi$ is irreducible modulo X. □

2.21. **Corollary.** *Let $X \subseteq Y_S$ be a subset and ${}^{\alpha}\pi$, $\alpha \in \mathfrak{A}$, be perfect separable mappings onto Y which can be reduced modulo ${}^{S}_{\alpha}\pi X$ in a unique way. Then the mapping ${}^{S}\pi$ can be reduced modulo X in a unique way.*

§ 3. Some topological constructions

3.1. *Construction.* Let non-empty topological spaces Y and Z_α, open sets $O_\alpha \subseteq Y$, sets $G_\alpha \subseteq O_\alpha$, satisfying the condition $[G_\alpha]_Y \cap O_\alpha = G_\alpha$, and mappings $g_\alpha : O_\alpha \setminus G_\alpha \to Z_\alpha$, $\alpha \in \mathfrak{A}$, be given.

For each $\alpha \in \mathfrak{A}$ let

$$Y_\alpha = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha\}, \alpha \in \{\alpha\}) = (Y \setminus G_\alpha) \dot{\cup} (G_\alpha \times Z_\alpha)$$

and define the maps ${}^\alpha\pi\colon Y_\alpha \xrightarrow{\text{onto}} Y$ and ${}^\alpha\psi\colon {}^\alpha\pi^{-1}O_\alpha \to Z_\alpha$ as follows:

$$
{}^\alpha\pi z = \begin{cases} z \text{ for } z \in Y \setminus G_\alpha, \\ y \text{ for } z = (y,t) \in G_\alpha \times Z_\alpha, \end{cases} \qquad {}^\alpha\psi z = \begin{cases} gz \text{ for } z \in O_\alpha \setminus G_\alpha, \\ t \text{ for } z = (y,t) \in G_\alpha \times Z_\alpha. \end{cases}
$$

Let us equip Y_α with the smallest topology with respect to which the maps ${}^\alpha\pi$ and ${}^\alpha\psi$ are continuous. Thus all sets of the form

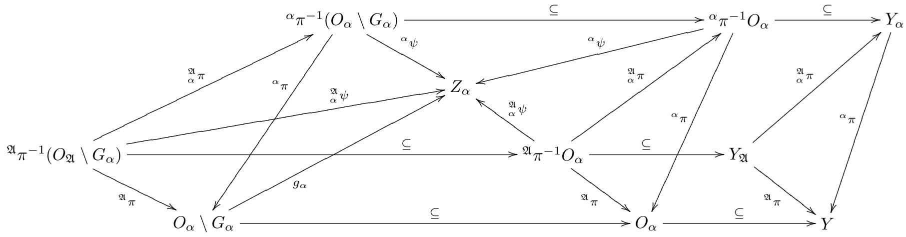
$$V(U) = {}^\alpha\pi^{-1}U \text{ and } V(U, U_\alpha) = {}^\alpha\pi^{-1}U \cap {}^\alpha\psi^{-1}U_\alpha,$$

where $U \subseteq Y$ and $U_\alpha \subseteq Z_\alpha$ are open subsets, constitute a base for the topology of Y_α. It is easily seen that

$${}^\alpha\psi|_{{}^\alpha\pi^{-1}(O_\alpha\setminus G_\alpha)} = g_\alpha {}^\alpha\pi|_{{}^\alpha\pi^{-1}(O_\alpha\setminus G_\alpha)}.$$

$$
\begin{tikzcd}
{}^\alpha\pi^{-1}(O_\alpha \setminus G_\alpha) \arrow[rr, "\subseteq"] \arrow[dr, "{}^\alpha\psi"] \arrow[dd, "{}^\alpha\pi"'] & & {}^\alpha\pi^{-1}O_\alpha \arrow[r, "\subseteq"] \arrow[dl, "{}^\alpha\psi"'] \arrow[dd, "{}^\alpha\pi"] & Y_\alpha \arrow[dd, "{}^\alpha\pi"] \\
& Z_\alpha & & \\
O_\alpha \setminus G_\alpha \arrow[ur, "g_\alpha"] \arrow[rr, "\subseteq"] & & O_\alpha \arrow[r, "\subseteq"] & Y
\end{tikzcd}
$$

Let us define a space $Y_{\mathfrak{A}} = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha\}, \alpha \in \mathfrak{A})$ as the fan product of the spaces Y_α relative to the mappings ${}^\alpha\pi$, $\alpha \in \mathfrak{A}$. The mappings ${}^{\mathfrak{A}}\pi$, ${}^{\mathfrak{A}}_{\mathfrak{B}}\pi$, $\mathfrak{B} \subseteq \mathfrak{A}$, and ${}^{\mathfrak{A}}_{\alpha}\pi$, $\alpha \in \mathfrak{A}$, were defined in the item 2.1. For each $\alpha \in \mathfrak{A}$ let ${}^{\mathfrak{A}}_{\alpha}\psi\colon {}^{\mathfrak{A}}_{\alpha}\pi^{-1}O_\alpha \to Z_\alpha$ be the mapping defined by the equality ${}^{\mathfrak{A}}_{\alpha}\psi = {}^\alpha\psi\, {}^{\mathfrak{A}}_{\alpha}\pi$. It is easily seen that this construction coincides with the construction of the paper [43], §1.



It is convenient to use the following coordinate representation of the space $Y_{\mathfrak{A}}$ ([43], §1; for $y \in Y$ let $\mathfrak{A}(y) = \{\alpha \in \mathfrak{A} : y \in G_\alpha\}$):

$$Y_{\mathfrak{A}} = \{\{y, z_\alpha : \alpha \in \mathfrak{A}(y)\} : y \in Y, z_\alpha \in Z_\alpha \text{ for all } \alpha \in \mathfrak{A}(y)\}.$$

3.2. *Construction.* Let us suppose, in addition, that for all $\alpha \in \mathfrak{A}$ the map g_α is defined for all points $y \in G_\alpha$ too, but it is not necessarily continuous at these points (in other words, the map $g_\alpha\colon O_\alpha \to Z_\alpha$ is given, and $g_\alpha|_{O_\alpha\setminus G_\alpha}$ is continuous).

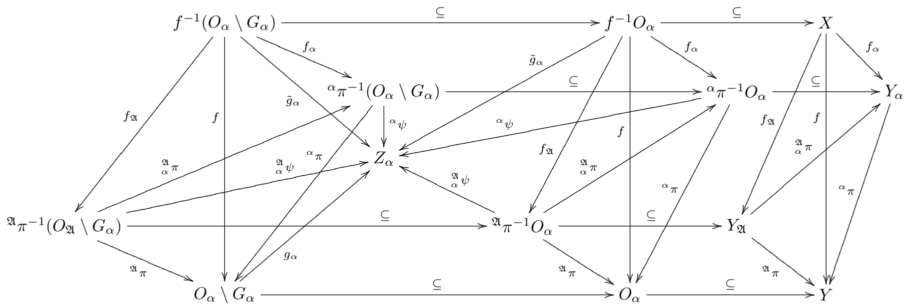
Also let a topological space X and mappings $f\colon X \to Y$ and $\tilde{g}_\alpha\colon f^{-1}O_\alpha \to Z_\alpha$, satisfying the condition $\tilde{g}_\alpha|_{f^{-1}(O_\alpha\setminus G_\alpha)} = g_\alpha f|_{f^{-1}(O_\alpha\setminus G_\alpha)}$ for all $\alpha \in \mathfrak{A}$, be given.

For each $\alpha \in \mathfrak{A}$ let us define the maps $\varphi_\alpha\colon Y \to Y_\alpha$ and $f_\alpha\colon X \to Y_\alpha$ as follows:

$$
\varphi_\alpha y = \begin{cases} y \text{ for } y \in Y \setminus G_\alpha, \\ (y, g_\alpha y) \text{ for } y \in G_\alpha, \end{cases} \qquad f_\alpha x = \begin{cases} fx \text{ for } x \in f^{-1}(Y \setminus G_\alpha), \\ (fx, \tilde{g}_\alpha x) \text{ for } x \in f^{-1}G_\alpha. \end{cases}
$$

Obviously, $\tilde{g}_\alpha = {}^\alpha\psi f_\alpha|_{f^{-1}O_\alpha}$, ${}^\alpha\psi\varphi_\alpha|_{O_\alpha} = g_\alpha$ and ${}^\alpha\pi\varphi_\alpha y = y$ for all $y \in Y$ and $\alpha \in \mathfrak{A}$. It is easily seen that the map f_α is continuous for every $\alpha \in \mathfrak{A}$.

Using Proposition 2.7 we get the mapping $f_{\mathfrak{A}}\colon X \to Y_{\mathfrak{A}}$ which satisfies the conditions ${}^{\mathfrak{A}}\pi f_{\mathfrak{A}} = f$ and $\tilde{g}_\alpha = {}^{\mathfrak{A}}_{\alpha}\psi f_{\mathfrak{A}}|_{f^{-1}O_\alpha}$ for every $\alpha \in \mathfrak{A}$ (see also [43], Lemma 6).



For every $\alpha\in\mathfrak{A}$ let $X_\alpha=[f_\alpha X\cup\varphi_\alpha Y]_{Y_\alpha}\subseteq Y_\alpha$, and let $X_{\mathfrak{A}}$ be the fan product of the spaces X_α relative to the mappings $p_\alpha={}^\alpha\pi|_{X_\alpha}$, $\alpha\in\mathfrak{A}$. It is easily seen that the space $X_{\mathfrak{A}}$ is a closed subset of $Y_{\mathfrak{A}}$, $f_{\mathfrak{A}}X\subseteq X_{\mathfrak{A}}$ and ${}^{\mathfrak{A}}\pi X_{\mathfrak{A}}=Y$; the projections of $X_{\mathfrak{A}}$ onto Y and X_α coincide with ${}^{\mathfrak{A}}\pi|_{X_{\mathfrak{A}}}$ and ${}^{\mathfrak{A}}_{\alpha}\pi|_{X_{\mathfrak{A}}}$, $\alpha\in\mathfrak{A}$, respectively.

3.3. **Proposition.** *If for some $\alpha\in\mathfrak{A}$ every point $y\in G_\alpha$ has a neighborhood $Uy\subseteq O_\alpha$ such that the set $[\tilde g_\alpha f^{-1}Uy\cup g_\alpha Uy]_{Z_\alpha}$ is compact then the mapping p_α is perfect. If it is true for all $\alpha\in\mathfrak{A}$ then the mapping ${}^{\mathfrak{A}}\pi|_{X_{\mathfrak{A}}}$ is perfect too.*

Proof. The second statement follows from the first one and Theorem 2.11 b). Therefore we have to prove only the first statement.

It is obvious that the mapping p_α is perfect at all points $y\in Y\setminus G_\alpha$.

Let $y\in G_\alpha$. There exists a neighborhood $Uy\subseteq O_\alpha$ such that the set $Z'_\alpha=$ $=[\tilde g_\alpha f^{-1}Uy\cup g_\alpha Uy]_{Z_\alpha}$ is compact. Let us consider the space

$$Z=\mathfrak{P}(Uy,\{Z'_\alpha\},\{G_\alpha\cap Uy\},\{Uy\},\{g_\alpha|_{Uy\setminus G_\alpha}\},\alpha\in\{\alpha\})$$

and the projection ${}^\alpha\pi'\colon Z\xrightarrow{\text{onto}}Uy$. The mapping ${}^\alpha\pi'$ is perfect by Theorem 1 of the paper [43]. It is clear that $p_\alpha^{-1}Uy$ is a closed subset of Z and Z is a closed subset of ${}^\alpha\pi^{-1}Uy$, ${}^\alpha\pi'={}^\alpha\pi|_Z$ and $p_\alpha|_{p_\alpha^{-1}Uy}={}^\alpha\pi'|_{p_\alpha^{-1}Uy}$. Therefore the mapping p_α is perfect. □

3.4. **Proposition.** *If for each $\alpha\in\mathfrak{A}$ the space Z_α is compact and the set $G_\alpha\setminus$ $\setminus[fX]_Y$ is nowhere dense in Y then the mapping ${}^{\mathfrak{A}}\pi\colon Y_{\mathfrak{A}}\xrightarrow{\text{onto}}Y$ can be reduced modulo $f_{\mathfrak{A}}X$ in a unique way (that is, there exists a unique closed subset $Y^r_{\mathfrak{A}}\subseteq Y_{\mathfrak{A}}$ such that $f_{\mathfrak{A}}X\subseteq Y^r_{\mathfrak{A}}$, ${}^{\mathfrak{A}}\pi Y^r_{\mathfrak{A}}=Y$ and the mapping ${}^{\mathfrak{A}}\pi|_{Y^r_{\mathfrak{A}}}$ is irreducible modulo $f_{\mathfrak{A}}X$).*

Proof. Let $\alpha\in\mathfrak{A}$ and $X_\alpha=[f_\alpha X\cup{}^\alpha\pi^{-1}(Y\setminus([fX]_Y\cup G_\alpha))]_{Y_\alpha}$. The mapping ${}^\alpha\pi$ is perfect due to Theorem 1 of the paper [43]. Moreover, ${}^\alpha\pi X_\alpha=Y$, $f_\alpha X\subseteq$ $\subseteq X_\alpha$ and the mapping ${}^\alpha\pi|_{X_\alpha}$ is irreducible modulo $f_\alpha X$ since the mapping ${}^\alpha\pi$ is one-to-one on the set ${}^\alpha\pi^{-1}(Y\setminus G_\alpha)$ and the set $G_\alpha\setminus[fX]_Y$ is nowhere dense in Y.

On the other hand, if $F\subseteq Y_\alpha$ is a closed subset such that $f_\alpha X\subseteq F$ and ${}^\alpha\pi F=$ $=Y$ then $X_\alpha\subseteq F$ and, hence, the mapping ${}^\alpha\pi$ can be reduced modulo $f_\alpha X$ in a unique way. Therefore the mapping ${}^{\mathfrak{A}}\pi$ can be reduced modulo $f_{\mathfrak{A}}X$ in a unique way by Corollary 2.13. □

3.5. **Corollary.** *Let $f\colon X\xrightarrow{\text{onto}}Y$ be a mapping with the property $\mathfrak{T}^{\mathfrak{E}}\mathfrak{a}$ and $X'\subseteq$ $\subseteq X$ be a subset such that the set $O_0=Y\setminus f[X']_Y$ is open (for example, it is true if f is closed), the mapping $f|_{f^{-1}O_0}\colon f^{-1}O_0\xrightarrow{\text{onto}}O_0$ is perfect and for each $G\in\mathfrak{a}$ the set $G\cap O_0$ is nowhere dense in Y. Then the mapping f can be reduced modulo X' in a unique way.*

§ 4. Algebras of functions on mappings

4.1. Further on we shall fix a mapping $f\colon X\to Y$ such that $[fX]_Y=Y$ until the item 4.16 (except the items 4.12–4.14 where the condition $[fX]_Y=Y$ can be omitted). Let $C^*(X)$ be the algebra of all bounded continuous functions[6] $\tilde{g}\colon X \to \mathbb{R}$ with the usual norm $\|\tilde{g}\| = \sup\{|\tilde{g}x| : x \in X\}$ and let $C(X)$ be the algebra of all continuous functions $\tilde{g}\colon X \to \mathbb{R}$.

A. Algebras of f-bounded functions

4.2. **Definition.** A function $\tilde{g}\colon X \to \mathbb{R}$ will be called *f-bounded* if for each point $y \in Y$ there exists a neighborhood $Uy \subseteq Y$ such that the function $\tilde{g}$ is bounded on the set $f^{-1}Uy$.

Let $B(f)$ be the algebra (over the field $\mathbb{R}$) of all f-bounded functions $\tilde{g}\colon X \to \mathbb{R}$, $C(f) = B(f) \cap C(X)$. Of course, $C^*(X) \subseteq C(f)$ as a subalgebra. The following two statements are very simple (any topology on the set $C(f)$ is not defined).

4.3. **Proposition.** *If the space Y is countably compact then $C(f) = C^*(X)$.*

4.4. **Proposition.** *If the mapping f is closed and $f^{-1}y$ is pseudocompact for every $y \in Y$ then $C(f) = C(X)$.*

B. Semi-norms and topologies on algebras

4.5. For every $y \in Y$ and $\tilde{g} \in B(f)$ let

$$n_y\tilde{g} = \inf\{\sup\{|\tilde{g}x| : x \in f^{-1}Uy\} : Uy \subseteq Y \text{ is a neighborhood of the point } y\}.$$

It is clear that n_y is a seminorm on the algebra $B(f)$ for each point $y \in Y$, and for every $y \in Y$, $\tilde{g} \in B(f)$ and $\varepsilon > 0$ such that $n_y\tilde{g} < \varepsilon$ the set $U_{\tilde{g},\varepsilon}y = \{y' \in Y : n_{y'}\tilde{g} < n_y\tilde{g} + \varepsilon\}$ is an open neighborhood of the point y.

4.6. **Proposition.** *If the mapping f is closed then for each $\tilde{g} \in C(f)$ and $y \in Y$ the equality $n_y\tilde{g} = \sup\{|\tilde{g}x| : x \in f^{-1}y\}$ holds.*

Proof. Let $y \in Y$ and $\tilde{g} \in C(f)$. Obviously, $n_y\tilde{g} \geqslant M = \sup\{|\tilde{g}x| : x \in f^{-1}y\}$. We have to prove the inverse inequality.

Let us take any $\varepsilon > 0$. Since the function $\tilde{g}$ is continuous, for each point $x \in f^{-1}y$ there exists a neighborhood $Ux \subseteq X$ such that $|\tilde{g}x' - \tilde{g}x| < \varepsilon$ for all $x' \in Ux$. Let $U^\varepsilon y = f^{\#}\bigcup\{Ux : x \in f^{-1}y\}$. Then $U^\varepsilon y \subseteq Y$ is open since the mapping f is closed and, hence, "onto" (see 4.1). We have the inequality $\sup\{|\tilde{g}x| : x \in f^{-1}U^\varepsilon y\} \leqslant M + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $n_y\tilde{g} \leqslant \inf\{\sup\{|\tilde{g}x| : x \in f^{-1}U^\varepsilon y\} : \varepsilon > 0\} \leqslant M$.

Thus, the equality $n_y\tilde{g} = M$ is valid. □

4.7. Let us take the family of the sets of the form

$$V_{\varepsilon,M}\tilde{g}_0 = \{\tilde{g} \in B(f) : \max\{n_y(\tilde{g} - \tilde{g}_0) : y \in M\} < \varepsilon\},$$

where $M \subseteq Y$ is a finite subset, $\varepsilon > 0$ and $g_0 \in B(f)$, as a base of a topology of $B(f)$, and let us equip $C(f)$ with the topology of a subspace.

It is easily seen (this is a standard definition) that $B(f)$ and $C(f)$ with these topologies are topological algebras.

C. $C(f)$ and other algebras

4.8. **Theorem.** *The algebra $C(f)$ is closed in $B(f)$.*

Proof. Let $\tilde{g}_0 \in [C(f)]_{B(f)}$. It suffices to prove that the function $\tilde{g}_0$ is continuous.

Let $x_0 \in X$ and $y = fx_0$. We have to prove that $\tilde{g}_0$ is continuous at the point x_0.

Let $\varepsilon > 0$ be an arbitrary number. By the definition of the topology of the space $B(f)$ the set $V_{\frac{\varepsilon}{3},\{y\}}(\tilde{g}_0)$ is an open neighborhood of $\tilde{g}_0$. Therefore there is a continuous function $\tilde{g} \in V_{\frac{\varepsilon}{3},\{y\}}(\tilde{g}_0) \cap C(f)$. By the definition of the seminorm n_y there exists a neighborhood $Uy \subseteq Y$ such that $|\tilde{g}x - \tilde{g}_0x| < \frac{\varepsilon}{3}$ for all $x \in f^{-1}Uy$.

[6] $\mathbb{R}$ is the field of all real numbers with the usual topological structure.

Since the function $\tilde{g}$ is continuous there is a neighborhood $Ux_0 \subseteq f^{-1}Uy$ such that $|\tilde{g}x - \tilde{g}x_0| < \frac{\varepsilon}{3}$ for all $x \in Ux_0$.

Hence, for any $x \in Ux_0$ we have

$$|\tilde{g}_0x - \tilde{g}_0x_0| = |\tilde{g}_0x - \tilde{g}x + \tilde{g}x - \tilde{g}x_0 + \tilde{g}x_0 - \tilde{g}_0x_0| \leqslant \\ \leqslant |\tilde{g}_0x - \tilde{g}x| + |\tilde{g}x - \tilde{g}x_0| + |\tilde{g}x_0 - \tilde{g}_0x_0| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore the function $\tilde{g}_0$ is continuous. □

4.9. **Proposition.** *The identity map $i_X\colon C^*(X) \to C(f)$ is continuous and $i_X C^*(X)$ is dense in $C(f)$.*

Proof. The map i_X is continuous since for every $\tilde{g} \in C^*(X)$ and any $y \in Y$ we have $n_y\tilde{g} \leqslant \|\tilde{g}\|$.

Let $\tilde{g} \in C(f)$. For every number $t \geqslant 0$ let us set

$$\tilde{g}_t x = \begin{cases} t \text{ if } \tilde{g}x \geqslant t, \\ \tilde{g}x \text{ if } |\tilde{g}x| < t, \\ -t \text{ if } \tilde{g}x \leqslant -t. \end{cases}$$

It is obvious that $\tilde{g}_t \in C^*(X)$ for each $t \geqslant 0$, therefore $A = \{\tilde{g}_t : t \geqslant 0\} \subseteq C^*(X)$. If $V_{\varepsilon,M}\tilde{g}$ is any neighborhood of $\tilde{g}$ then for every $t \geqslant \max\{n_y\tilde{g} : y \in M\}$ we have $n_y(\tilde{g}_t - \tilde{g}) = 0$ for all $y \in M$, hence, $\tilde{g}_t \in V_{\varepsilon,M}\tilde{g}$ for such t, and $\tilde{g} \in [A]_{C(f)} \subseteq \subseteq [C^*(X)]_{C(f)}$. □

4.10. **Proposition.** *If there exists a continuous function $g\colon Y \to [0,1]$ such that the set gY is infinite then the identity mapping $i_X\colon C^*(X) \to C(f)$ is not an embedding.*

Proof. Since the segment $[0,1]$ is compact and the set gY is infinite there is a point $t_0 \in [0,1]$ such that $t_0 \in [gY \setminus \{t_0\}]_{\mathbb{R}}$. Let $\{U_nt_0 : n \in \mathbb{N} = \{1,2,3,\dots\}\}$ be a local base of $[0,1]$ at the point t_0.

For every $n \in \mathbb{N}$ let us choose a point $t_n \in gY \cap (U_nt_0 \setminus \{t_0\})$ and a neighborhood $Ut_n \subseteq [0,1]$ such that $[Ut_n]_{\mathbb{R}} \subseteq U_nt_0 \setminus \{t_0\}$, and let $h_n\colon [0,1] \to [0,1]$ be a continuous function such that $h_nt_n = 1$ and $h_nt = 0$ for all $t \in [0,1] \setminus Ut_n$. Let $A = \{h_ngf : n \in \in \mathbb{N}\}$. It is obvious that $A \subseteq C^*(X)$.

Let $\tilde{g}_0 \in C^*(X)$ be a function such that $\tilde{g}_0x = 0$ for all $x \in X$. We have $\|\tilde{g} - \tilde{g}_0\| = \|\tilde{g}\| = 1$ for all $\tilde{g} \in A$, therefore $\tilde{g}_0 \notin [A]_{C^*(X)}$.

On the other hand, for any neighborhood $V_{\varepsilon,M}\tilde{g}_0 \subseteq B(f)$ there exists a number $n \in \mathbb{N}$ such that $M \cap (U_nt_0 \setminus \{t_0\}) = \varnothing$, hence, $n_y(h_ngf - \tilde{g}_0) = n_y(h_ngf) = 0$ for all $y \in M$, that is, $h_ngf \in V_{\varepsilon,M}\tilde{g}_0$ and $V_{\varepsilon,M}\tilde{g}_0 \cap A \neq \varnothing$; therefore $\tilde{g}_0 \in [A]_{C(f)}$. □

4.11. **Proposition.** *If the mapping f is closed and for each $y \in Y$ the set $f^{-1}y$ is finite then the topology of the space $C(f)$ coincides with the topology of pointwise convergence.*

Proof. Let $C_p(X)$ be the algebra of all continuous functions $\tilde{g}\colon X \to \mathbb{R}$ with the topology of pointwise convergence. Due to Proposition 4.4 the sets $C(f)$ and $C_p(X)$ coincide. Let $j_X\colon C(f) \xrightarrow{\text{onto}} C_p(X)$ be the identity map.

It is obvious that the topology of the pointwise convergence can be defined by the family of semi-norms

$$n'_y\tilde{g} = \max\{|\tilde{g}x| : x \in f^{-1}y\} = \sup\{|\tilde{g}x| : x \in f^{-1}y\}$$

for all $y \in fX = Y$ (see 4.1) and $\tilde{g} \in C_p(X)$, because $f^{-1}y$ is finite for each $y \in \in Y$. Due to Proposition 4.6 $n'_yj_X = n_y$ for every $y \in Y$, hence the map j_X is a homeomorphism. □

D. **Globally completely regular mappings**

4.12. **Definition** ([34], §7)**.** We shall say that a mapping $f\colon X \to Y$ *is parallel to a completely regular space* if there exist a completely regular space Z and an embedding $i\colon X \to Y \times Z$ such that the equality $f = p_Y i$ is valid, where $p_Y\colon Y \times \times Z \to Y$ is the projection of the product $Y \times Z$ to its factor Y.

$$\begin{tikzcd} X \arrow[rr, "\subseteq", "i"'] \arrow[dr, "f"'] & & Y \times Z \arrow[dl, "p_Y"] \\ & Y & \end{tikzcd}$$

Obviously, if a mapping $f\colon X \to Y$ is parallel to a completely regular space then f is Tychonoff. On the other hand, if the space X is completely regular then the mapping f is parallel to a completely regular space (we can take $Z = X$).

4.13. **Definition.** A mapping $f\colon X \to Y$ will be called *globally completely regular* if for an arbitrary point $x \in X$ in each of the following two cases

a) for every point $x' \in f^{-1}fx \setminus \{x\}$ and

b) for every neighborhood $Ux \subseteq X$

there exist a continuous function $\tilde{g}\colon X \to \mathbb{R}$ and a neighborhood $Ofx \subseteq Y$ such that, respectively,

a) $\tilde{g}x' \neq \tilde{g}x$ or

b) $\tilde{g}x \notin [\tilde{g}(f^{-1}Ofx \setminus Ux)]_{\mathbb{R}}$.

Note that we can use the segment $[0,1]$ instead of $\mathbb{R}$ with the same result.

It is possible to generalize this definition analogously to Definition 1.6, but we shall consider the simplest case. For the general case we can prove all results of this paper about mappings with the properties $\mathfrak{T}^{\mathfrak{C}}\mathfrak{a}$ and $\mathfrak{T}\mathfrak{a}$ with trivial modifications.

4.14. **Proposition.** *A mapping $f\colon X \to Y$ is parallel to a completely regular space iff it is globally completely regular.*

Proof. Necessity. Let the mapping f be parallel to a completely regular space Z, that is, there exists an embedding $i\colon X \to Y \times Z$ such that $f = p_Y i$, where $p_Y\colon Y \times Z \to Y$ is the projection. We have to consider two cases.

a) Let $x \in X$ and $x' \in f^{-1}fx \setminus \{x\}$. Let us denote $z = p_Z ix$ and $z' = p_Z ix'$, where $p_Z\colon Y \times Z \to Z$ is the projection. Then $z' \neq z$, therefore there is a continuous function $g\colon Z \to [0,1]$ such that $gz = 0$ and $gz' = 1$, because the space Z is completely regular. Then the function $\tilde{g} = gp_Z i$ has all necessary properties.

b) Let $x \in X$ be a point and $Ux \subseteq X$ be its neighborhood. By the definition of the topological product there are neighborhoods $Ofx \subseteq Y$ and $Op_Z ix \subseteq Z$ such that $x \in f^{-1}Ofx \cap i^{-1}p_Z^{-1}Op_Z ix = i^{-1}(Ofx \times Op_Z ix) \subseteq Ux$, since the mapping i is embedding. Analogously there is a continuous function $g\colon Z \to [0,1]$ such that $gp_Z ix = 0$ and $gz = 1$ for all $z \in Z \setminus Op_Z ix$. Then the function $\tilde{g} = gp_Z i$ has all necessary properties.

Sufficiency. Let the mapping f be globally completely regular. Let $\mathfrak{A}_1 = = \{(x, x') : x \in X, x' \in f^{-1}fx \setminus \{x\}\}$, $\mathfrak{A}_2 = \{(x, Ux) : x \in X, Ux \subseteq X$ is a neighborhood of the point $x\}$, $\mathfrak{A} = \mathfrak{A}_1 \dot\cup \mathfrak{A}_2$.

Let $\alpha = (x, x') \in \mathfrak{A}_1$. By Definition 4.12 there is a continuous function $\tilde{g}_\alpha\colon X \to \to Z_\alpha = \mathbb{R}$ (or $[0,1]$) such that $\tilde{g}_\alpha x' \neq \tilde{g}_\alpha x$.

Let $\alpha = (x, Ux) \in \mathfrak{A}_2$. By Definition 4.12 there are a continuous function $\tilde{g}_\alpha\colon X \to Z_\alpha = \mathbb{R}$ (or $[0,1]$) and a neighborhood $Ofx \subseteq Y$ such that $\tilde{g}_\alpha x \notin \notin [\tilde{g}_\alpha(f^{-1}Ofx \setminus Ux)]_{\mathbb{R}}$.

Let $Z = \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$ and let $i = f \,\Delta(\Delta\{\tilde{g}_\alpha : \alpha \in \mathfrak{A}\})\colon X \to Y \times \prod\{Z_\alpha : \alpha \in \in \mathfrak{A}\} = Y \times Z$ be the diagonal mapping (see [51], §2.3). It is easily seen that the mapping i is an embedding, the space Z is completely regular ([51], Theorem 2.3.11) and $f = p_Y i$. □

4.15. **Theorem.** *There exist a space X' and two mappings $h_1\colon X \to X'$ and $h_2\colon X' \to Y$ such that*

1) $h_2 h_1 = f$;
2) $[h_1 X]_{X'} = X'$;
3) *h_2 is perfect (hence, $h_2 X' = Y$; see 4.1) and globally completely regular (hence, h_2 is separable);*
4) *the map $\varphi\colon C(h_2) \xrightarrow{\text{onto}} C(f)$, defined by the equality $\varphi\bar g = \bar g h_1$ for all $\bar g \in C(h_2)$, is an isomorphism of topological algebras preserving all semi-norms n_y, $y \in Y$.*

Moreover, if the mapping f is globally completely regular (and perfect) then the mapping h_1 is an embedding (a homeomorphism onto X').

$$\begin{tikzcd}
X \arrow[dr, "h_1"] \arrow[drr, "\varphi\bar g"] \arrow[dd, "f"'] & & \\
& X' \arrow[r, "\bar g"'] \arrow[dl, "h_2"] & \mathbb{R} \\
Y & & C(h_2) \xrightarrow{\varphi} C(f)
\end{tikzcd}$$

Proof. Let $C(f) = \{\tilde g_\alpha : \alpha \in \mathfrak{A}\}$. Let us set $Z_\alpha = \mathbb{R}$, $O_\alpha = Y$, $G_\alpha = Y$ for every $\alpha \in \mathfrak{A}$, and let us define a function $g_\alpha\colon Y \to Z_\alpha$ by the equality

$$g_\alpha y = \inf\{\sup\{\tilde g_\alpha x : x \in f^{-1}Uy\} : Uy \subseteq Y \text{ is a neighborhood of the point } y\}$$

for each $y \in Y$.

Using Construction 3.2 we get the mappings $f_{\mathfrak{A}}\colon X \to X_{\mathfrak{A}} \subseteq Y_{\mathfrak{A}} = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha|_{O_\alpha \setminus G_\alpha}\}, \alpha \in \mathfrak{A}) = Y \times \prod\{Z_\alpha : \alpha \in \mathfrak{A}\}$, ${}^{\mathfrak{A}}\pi = p_Y\colon Y_{\mathfrak{A}} \xrightarrow{\text{onto}} Y$ and ${}^{\mathfrak{A}}_{\alpha}\psi = p_{Z_\alpha}\colon Y_{\mathfrak{A}} \to Z_\alpha = \mathbb{R}$, $\alpha \in \mathfrak{A}$, such that ${}^{\mathfrak{A}}\pi f_{\mathfrak{A}} = f$ and ${}^{\mathfrak{A}}_{\alpha}\psi f_{\mathfrak{A}} = \tilde g_\alpha$ for all $\alpha \in \mathfrak{A}$.

Since all functions $\tilde g_\alpha$, $\alpha \in \mathfrak{A}$, are f-bounded, the mapping ${}^{\mathfrak{A}}\pi|_{X_{\mathfrak{A}}}$ is perfect by Proposition 3.3. Let $X' = [f_{\mathfrak{A}} X]_{X_{\mathfrak{A}}}$, $h_2 = {}^{\mathfrak{A}}\pi|_{X'}$, and let $h_1\colon X \to X'$ be the mapping which coincides with $f_{\mathfrak{A}}$.

It is easily seen that the map $\psi\colon C(f) \to C(h_2)$, defined by the equality $\psi\tilde g_\alpha = {}^{\mathfrak{A}}_{\alpha}\psi|_{X'}$ for $\alpha \in \mathfrak{A}$, is inverse to the mapping φ, and that $n_y \tilde g_\alpha = n_y(\psi\tilde g_\alpha)$ for all $\alpha \in \mathfrak{A}$ and $y \in Y$. Therefore the map φ is a topological isomorphism preserving all seminorms n_y, $y \in Y$.

$$\begin{tikzcd}
X \arrow[rrr, "\tilde g_\alpha"] \arrow[dr, "h_1"] \arrow[drr, "f_{\mathfrak{A}}"] \arrow[dd, "f"'] & & & Z_\alpha = \mathbb{R} \\
& X' \arrow[r, "\subseteq"'] \arrow[urr, "\psi\tilde g_\alpha"] \arrow[dl, "h_2"] & X_{\mathfrak{A}} \arrow[r, "\subseteq"] & Y_{\mathfrak{A}} \arrow[u, "{}^{\mathfrak{A}}_{\alpha}\psi"'] \arrow[dlll, "{}^{\mathfrak{A}}\pi"] \\
Y & & C(f) \xrightarrow{\psi} C(h_2) &
\end{tikzcd}$$

The mapping h_2 is globally completely regular by Proposition 4.14.

It is easily seen that if the mapping f is globally completely regular then the mapping h_1 is embedding in consequence of Definition 4.13. Moreover, if the mapping f is perfect then the mapping h_1 is perfect by Lemma 8 of [43] and, hence, h_1 is a homeomorphism onto X'. □

4.16. For the statements 4.17 and 4.18 let $f_1: X_1 \to Y_1$, $f_2: X_2 \to Y_2$, $\phi_1: X_1 \to X_2$ and $\phi_2: Y_1 \to Y_2$ be mappings such that $[f_1 X_1]_{Y_1} = Y_1$, $[f_2 X_2]_{Y_2} = Y_2$ and $\phi_2 f_1 = f_2 \phi_1$.

4.17. **Lemma.** *For $i = 1, 2$ let the space X_i' and the mappings h_{i1} and h_{i2} be such as they were constructed in Theorem 4.15 for the mapping f_i. Then there exists and is unique a mapping $\phi_0: X_1' \to X_2'$ such that $\phi_0 h_{11} = h_{21}\phi_1$ and $h_{22}\phi_0 = \phi_2 h_{12}$. If the mapping ϕ_2 is perfect (or separable) then the mapping ϕ_0 is perfect (or separable) too.*

$$\begin{tikzcd}
X_1 \arrow[rrrr, "\phi_1"] \arrow[drr, "\tilde{g}_{1h\alpha}"] \arrow[ddr, "h_{11}"] \arrow[dddr, "f_{1\mathfrak{A}_1}"'] \arrow[dddd, "f_1"'] & & & & X_2 \arrow[dll, "\tilde{g}_{2\alpha}"'] \arrow[ddl, "h_{21}"'] \arrow[dddl, "f_{2\mathfrak{A}_2}"] \arrow[dddd, "f_2"] \\
& & \mathbb{R} & & \\
& X_1' \arrow[ur, "\bar{g}_{1h\alpha}"] \arrow[rr, "\phi_0"] \arrow[d, "\cap|"'] \arrow[ddl, "h_{12}"'] & & X_2' \arrow[ul, "\bar{g}_{2\alpha}"'] \arrow[d, "\cap|"] \arrow[ddr, "h_{22}"] & \\
& Y_{1\mathfrak{A}_1} \arrow[uur, "{}^{\mathfrak{A}_1}_{h\alpha}\psi_1"'] \arrow[rr, "\bar{\phi}"] \arrow[dl, "{}^{\mathfrak{A}_1}\pi_1"] & & Y_{2\mathfrak{A}_2} \arrow[uul, "{}^{\mathfrak{A}_2}_{\alpha}\psi_2"] \arrow[dr, "{}^{\mathfrak{A}_2}\pi_2"'] & \\
Y_1 \arrow[rrrr, "\phi_2"] & & & & Y_2
\end{tikzcd}$$

Proof. Let $C(f_i) = \{\tilde{g}_{i\alpha} : \alpha \in \mathfrak{A}_i\}$, $i = 1, 2$. For each $\alpha \in \mathfrak{A}_2$ there is a unique element $\alpha' \in \mathfrak{A}_1$ such that $\tilde{g}_{1\alpha'} = \tilde{g}_{2\alpha}\phi_1$; therefore we can define a map $h: \mathfrak{A}_2 \to \mathfrak{A}_1$ by setting $h\alpha = \alpha'$ for all $\alpha \in \mathfrak{A}_2$. Due to Theorem 2 of [43] there exists and is unique a mapping $\bar{\phi}: Y_{1\mathfrak{A}_1} \to Y_{2\mathfrak{A}_2}$ such that $\phi_2{}^{\mathfrak{A}_1}\pi_1 = {}^{\mathfrak{A}_2}\pi_2\bar{\phi}$ and ${}^{\mathfrak{A}_2}_{\alpha}\psi_2\bar{\phi} = {}^{\mathfrak{A}_1}_{h\alpha}\psi_1$ for all $\alpha \in \mathfrak{A}_2$ (see 4.15). By the definition of the map h we have $\bar{\phi} f_{1\mathfrak{A}_1} = f_{2\mathfrak{A}_2}\phi_1$, hence, $\bar{\phi} f_{1\mathfrak{A}_1} X_1 \subseteq f_{a\mathfrak{A}_2} X_2$. Therefore $\bar{\phi} X_1' \subseteq X_2'$, and we can define the mapping ϕ_0 as the restriction of the mapping $\bar{\phi}$.

The mapping ϕ_0 satisfying the conditions $\phi_0 h_{11} = h_{21}\phi_1$ and $h_{22}\phi_0 = \phi_2 h_{12}$ is unique since the first condition defines it on the dense subset $h_{11}X_1 \subseteq X_1'$ and the mapping h_{22} is separable.

If the mapping ϕ_2 is perfect then the mapping $h_{22}\phi_0 = \phi_2 h_{12}$ also is perfect, and hence ϕ_0 is perfect by Lemma 8 of [43]. Analogously we can prove that ϕ_0 is separable if ϕ_2 is separable. $\square$

E. **Homomorphisms of algebras**

4.18. **Theorem.** *The map $\varphi: C(f_2) \to C(f_1)$ defined by the formula $\varphi\tilde{g}_2 = \tilde{g}_2\phi_1$ for all $\tilde{g}_2 \in C(f_2)$ is a continuous homomorphism of the topological algebras and $n_y(\varphi\tilde{g}_2) \leqslant n_{\phi_2 y}\tilde{g}_2$ for all $y \in Y_1$ and $\tilde{g} \in C(f_2)$. Moreover,*

1) *if $[\phi_1 X_1]_{X_2} = X_2$ then φ is a continuous isomorphism onto a subalgebra of $C(f_1)$;*
2) *if, in addition to 1), the mapping ϕ_2 is perfect then φ is a continuous isomorphism onto a closed subalgebra of $C(f_1)$;*
3) *if, in addition to 1) and 2), the set $\phi_2^{-1}y$ is finite for each $y \in Y_2$ then φ is a topological isomorphism onto a closed subalgebra of $C(f_1)$.*

Proof is very simple except for the closedness of $\varphi C(f_2)$ in $C(f_1)$ in the statement 2).

By Lemma 4.17 we can assume that the mappings ϕ_1 and f_2 are perfect and $\phi_1 X_1 = X_2$; otherwise we can replace X_1, X_2, f_1, f_2 and ϕ_1 by X_1', X_2', h_{12}, h_{22} and ϕ_0.

Consider a function $\tilde{g}_0 \in [\varphi C(f_2)]_{C(f_1)}$. We shall show that for any points $x_1, x_2 \in X_1$ such that $\phi_1 x_1 = \phi_1 x_2$ we have $\tilde{g}_0 x_1 = \tilde{g}_0 x_2$. Let $M = \{f_1 x_1, f_1 x_2\}$. For each $\varepsilon > 0$ there is $\tilde{g}_2 \in C(f_2)$ such that $\tilde{g}_1 = \varphi \tilde{g}_2 \in V_{\frac{\varepsilon}{2}, M} \tilde{g}_0$ (see 4.7); therefore

$$|\tilde{g}_0 x_1 - \tilde{g}_0 x_2| = |\tilde{g}_0 x_1 - \tilde{g}_1 x_1 + \tilde{g}_1 x_1 - \tilde{g}_1 x_2 + \tilde{g}_1 x_2 - \tilde{g}_0 x_2| \leqslant$$
$$\leqslant |\tilde{g}_0 x_1 - \tilde{g}_1 x_1| + |\tilde{g}_1 x_1 - \tilde{g}_1 x_2| + |\tilde{g}_1 x_2 - \tilde{g}_0 x_2| < \frac{\varepsilon}{2} + |\tilde{g}_2 \phi_1 x_1 - \tilde{g}_2 \phi_1 x_2| + \frac{\varepsilon}{2} = \varepsilon,$$

since $\phi_1 x_1 = \phi_1 x_2$. Thus, $|\tilde{g}_0 x_1 - \tilde{g}_0 x_2| < \varepsilon$ for any $\varepsilon > 0$, so that $\tilde{g}_0 x_1 = \tilde{g}_0 x_2$ for any $x_1, x_2 \in X_1$ such that $\phi_1 x_1 = \phi_1 x_2$.

Therefore we can define a function $\tilde{g} \colon X_2 \to \mathbb{R}$ by the equality $\tilde{g} x = \tilde{g}_0 x'$ for any $x \in X_2$ and $x' \in \phi_1^{-1} x$. For every closed set $F \subseteq \mathbb{R}$ the set $\tilde{g}^{-1} F = \phi_1 \tilde{g}_0^{-1} F$ is closed since ϕ_1 is closed, hence $\tilde{g}$ is continuous; by Proposition 4.4 $\tilde{g} \in C(f_2)$. Obviously, $\varphi \tilde{g} = \tilde{g}_0$, therefore $\varphi C(f_2)$ is closed subalgebra of $C(f_1)$. □

4.19. **Corollary.** *Let mappings $f_1 \colon X_1 \to Y$, $f_2 \colon X_2 \to Y$ and $\phi_0 \colon X_1 \to X_2$ such that $f_1 = f_2 \phi_0$, $[\phi_0 X_1]_{X_2} = X_2$ and $[f_1 X_1]_Y = Y$ be given. Then the map $\varphi \colon C(f_2) \to C(f_1)$, defined by the equality $\varphi \tilde{g}_2 = \tilde{g}_2 \phi_0$ for all $\tilde{g}_2 \in C(f_2)$, is a topological isomorphism onto a closed subalgebra of $C(f_1)$, preserving all seminorms n_y, $y \in Y$.*

§ 5. Sheaves

5.1. From now on we shall fix a mapping $f \colon X \to Y$ and a closed family $\mathfrak{a}$ of locally closed subsets of the space Y.

Let T be the family of all open subsets of the space Y.

Let us denote by $T_{\mathfrak{a}}$ the family of all ordered couples (O, G), where $O \in T$ and $G \in \mathfrak{a}$ are sets such that $[G]_Y \cap O = G$. For $(O_1, G_1), (O_2, G_2) \in T_{\mathfrak{a}}$ we shall write $(O_1, G_1) \subseteq (O_2, G_2)$ if $O_1 \subseteq O_2$ and $O_1 \setminus G_1 \subseteq O_2 \setminus G_2$. This relation defines a partial order on the set $T_{\mathfrak{a}}$.

5.2. **Lemma.** *The partially ordered set $T_{\mathfrak{a}}$ has the following properties:*

a) *the couple $(Y, \varnothing)$ is the largest element of $T_{\mathfrak{a}}$;*

b) *for each $(O_1, G_1), (O_2, G_2) \in T_{\mathfrak{a}}$ there is their minimum*

$$(O_1, G_1) \cap (O_2, G_2) = (O_1 \cap O_2, (G_1 \cup G_2) \cap (O_1 \cap O_2));$$

c) *for each point $y \in Y$ and each set $O \in T$ the families $T_y = \{(O, G) \in T_{\mathfrak{a}} : O \ni y\}$ and $T_O = \{(O', G) \in T_{\mathfrak{a}} : O' = O\}$ are directed by the relation "$\supseteq$" inverse to "$\subseteq$".*

Proof. The statements a) and b) are evident, the statement c) is a consequence of the fact that for any $(O_1, G_1), (O_2, G_2) \in T_y$ (or T_O) their minimum belongs to T_y (or T_O). □

5.3. Further on we shall fix a couple $(O, G) \in T_{\mathfrak{a}}$ until the item 5.14.

A. Sets of couples of functions

5.4. Let us denote by $\mathcal{C}_{\mathfrak{a}}(O, G)$ the set of all ordered couples $(g, \tilde{g})$ of functions $g \colon O \to \mathbb{R}$ and $\tilde{g} \colon f^{-1} O \to \mathbb{R}$ which satisfy the following conditions:

1) $g|_{O \setminus G}$ and $\tilde{g}$ are continuous;
2) $\tilde{g}|_{f^{-1}(O \setminus G)} = g f|_{f^{-1}(O \setminus G)}$;
3) for every point $y \in G$ there is a neighborhood $Uy \subseteq O$ such that the functions $g|_{Uy}$ and $\tilde{g}|_{f^{-1} Uy}$ are bounded;

4) for every point $y \in G$ the following equality holds:

$$(2) \quad \begin{cases} gy = \inf\{\max\{\sup\{gy' : y' \in Uy \setminus ([fX]_Y \cup E)\}, \sup\{\tilde{g}x : x \in f^{-1}Uy\}\} : \\ \qquad Uy \subseteq O \text{ is a neighborhood of the point } y, \\ \qquad E \subseteq Uy \text{ is a nowhere dense set}\}. \end{cases}$$

We shall assume that $\mathcal{C}_{\mathfrak{a}}(\varnothing, \varnothing) = \{(0,0)\}$ is the zero-algebra.

Let $C^*_{\mathfrak{a}}(O,G) = \{(g,\tilde{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G) : g \text{ and } \tilde{g} \text{ are bounded}\}$.

The following statement is obvious.

5.5. **Proposition.** *If $(g,\tilde{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ then for every point $y_0 \in O$ we have*

$$gy_0 = \inf\{\sup\{gy : y \in Uy_0\} : Uy_0 \subseteq O \textit{ is a neighborhood of the point } y_0\},$$

and for every $t \in \mathbb{R}$ the set $\{y \in O : gy < t\}$ is open.

5.6. **Definition.** A couple $(g,\tilde{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ will be called *factorizable* if the function g is continuous on the set O and $\tilde{g} = gf|_{f^{-1}O}$.

The set of all factorizable couples $(g,\tilde{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ will be denoted by $\mathcal{C}_f(O,G)$. Let us observe that $\mathcal{C}_f(O,G)$ is independent of the set G since $\mathcal{C}_f(O,G)$ "coincides" with the set $C(O)$ of all continuous functions on the space O.

5.7. **Definition.** A couple $(g,\tilde{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ will be called *constant* if there exists a number $c \in \mathbb{R}$ such that $gy = c$ for all $y \in O$ and $\tilde{g}x = c$ for all $x \in f^{-1}O$.

B. **Algebras of couples of functions**

5.8. Let $\mathcal{C}_{\mathfrak{a}}(O,G) = \{(g_\alpha, \tilde{g}_\alpha) : \alpha \in \mathfrak{A}\}$. For each $\alpha \in \mathfrak{A}$ let us set $Z_\alpha = \mathbb{R}$, $G_\alpha = G$, $O_\alpha = O$.

Using Construction 3.1 let us construct the spaces $Y_\alpha = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha|_{O_\alpha \setminus G_\alpha}\}, \alpha \in \{\alpha\})$ and the mappings ${}^\alpha\pi \colon Y_\alpha \xrightarrow{\text{onto}} Y$, $\alpha \in \mathfrak{A}$, and, after that, the space $Y_{\mathfrak{A}} = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha|_{O_\alpha \setminus G_\alpha}\}, \alpha \in \mathfrak{A})$ and the mapping ${}^{\mathfrak{A}}\pi \colon Y_{\mathfrak{A}} \xrightarrow{\text{onto}} Y$.

Using Construction 3.2 let us construct the mappings $f_\alpha \colon X \to Y_\alpha$, the maps $\varphi_\alpha \colon Y \to Y_\alpha$ and the closed subsets $X_\alpha = [f_\alpha X \cup \varphi_\alpha Y]_{Y_\alpha} \subseteq Y_\alpha$, $\alpha \in \mathfrak{A}$, and, after that, the closed subset $X_{\mathfrak{A}} = \prod_Y(\{X_\alpha\}, \{{}^\alpha\pi|_{X_\alpha}\}, \alpha \in \mathfrak{A}) \subseteq Y_{\mathfrak{A}}$ and the mapping $f_{\mathfrak{A}} \colon X \to X_{\mathfrak{A}}$. Due to the condition 3) of the item 5.4 and Proposition 3.3, the mappings ${}^{\mathfrak{A}}\pi|_{X_{\mathfrak{A}}}$ and $p_\alpha = {}^\alpha\pi|_{X_\alpha}$, $\alpha \in \mathfrak{A}$, are perfect.

5.9. **Lemma.** *For each $\alpha \in \mathfrak{A}$ the mapping $p_\alpha \colon X_\alpha \xrightarrow{\text{onto}} Y$ is irreducible modulo $f_\alpha X$.*

Proof. Let $\alpha \in \mathfrak{A}$; by the conditions 2) and 4) of the item 5.4 we have $\varphi_\alpha y \in [f_\alpha X]_{Y_\alpha}$ for all $y \in \operatorname{Int}_Y [fX]_Y$.[7] Therefore in order to prove that the mapping p_α is irreducible modulo f_α it is suffices to prove that $p_\alpha^{\#} U = \{y \in Y : p_\alpha^{-1} y \subseteq U\} \neq \varnothing$ for an arbitrary non-empty open set $U \subseteq p_\alpha^{-1}(Y \setminus [fX]_Y)$. Moreover, we can assume that $U \subseteq p_\alpha^{-1} G_\alpha$ since the mappings ${}^\alpha\pi$ and p_α are one-to-one on the set $p_\alpha^{-1}(Y \setminus G_\alpha)$.

Since the set $\varphi_\alpha(Y \setminus [fX]_Y)$ is dense in the set $p_\alpha^{-1}(Y \setminus [fX]_Y)$, there exists $y_0 \in Y \setminus [fX]_Y$ such that $\varphi_\alpha y_0 \in U$. Of course, $y_0 \in G_\alpha \setminus [fX]_Y$. By the definition of the topology of the space X_α there are a neighborhood $Wy_0 \subseteq O_\alpha \setminus [fX]_Y$ and a number $\varepsilon > 0$ such that $\varphi_\alpha y_0 \in p_\alpha^{-1} Wy_0 \cap {}^\alpha\psi^{-1} U_{2\varepsilon} g_\alpha y \subseteq U$, where $U_{2\varepsilon} g_\alpha y_0 = \{t \in Z_\alpha = \mathbb{R} : g_\alpha y_0 - 2\varepsilon < t < g_\alpha y_0 + 2\varepsilon\}$ is a neighborhood of the point $g_\alpha y_0$ in the space Z_α.

For every $t \in \mathbb{R}$ let $U_t = \{y \in Wy_0 : g_\alpha y < t\}$; by Proposition 5.5 the sets $V_1 = U_{g_\alpha y_0 + \varepsilon}$ and $V_0 = U_{g_\alpha y_0 - \varepsilon}$ are open. The set $E = V_1 \setminus V_0$ is locally closed and $y_0 \in E \subseteq V_1 \subseteq Wy_0$. It is impossible for the set E to be nowhere dense, since in such

[7] If $A \subseteq Y$ then $\operatorname{Int}_Y A = Y \setminus [Y \setminus A]_Y$.

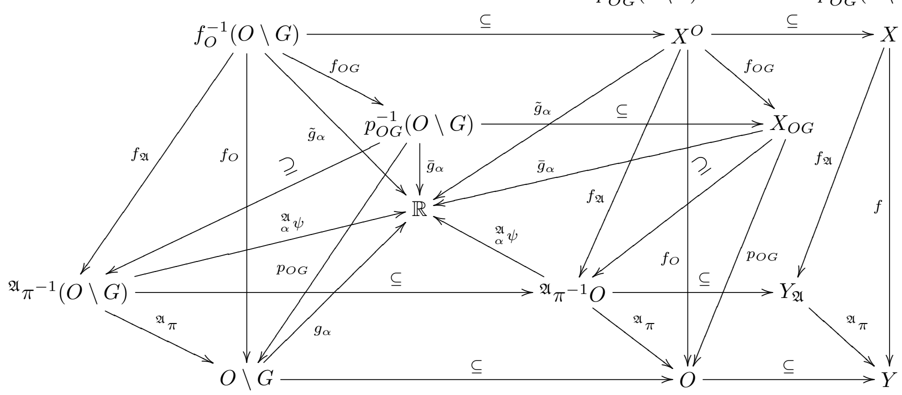
a case the condition (2) (see 5.4) would not be valid for the point y_0 (we can take $Uy_0 = V_1$ and our $E \subseteq Uy_0$ since $g_\alpha y < g_\alpha y_0 - \varepsilon$ for all $y \in Uy_0 \setminus E = V_0$). Therefore the set $V = \operatorname{Int}_Y E$ is non-empty. We have $\varnothing \neq p_\alpha^{-1}V \subseteq p_\alpha^{-1}Wy_0 \cap [\varphi_\alpha V]_{Y_\alpha} \subseteq$ $\subseteq p_\alpha^{-1}Wy_0 \cap {}^\alpha\psi^{-1}[g_\alpha V]_{Z_\alpha} \subseteq p_\alpha^{-1}Wy_0 \cap {}^\alpha\psi^{-1}U_{2\varepsilon}g_\alpha y_0 \subseteq U$, hence $p_\alpha^{\#}U \subseteq V \neq \varnothing$ and the mapping p_α is irreducible modulo $f_\alpha X$. □

5.10. Thus we have the fan product $X_{\mathfrak{A}}$ of the spaces X_α relative to the perfect irreducible modulo $f_\alpha X$ mappings p_α, $\alpha \in \mathfrak{A}$ (see 5.8). Due to Corollary 2.13 there exists a unique closed subset $X_{\mathfrak{A}}^r \subseteq X_{\mathfrak{A}}$ such that $f_{\mathfrak{A}}X \subseteq X_{\mathfrak{A}}^r$, ${}^{\mathfrak{A}}\pi X_{\mathfrak{A}}^r = Y$, and the mapping $p_{\mathfrak{A}} = {}^{\mathfrak{A}}\pi|_{X_{\mathfrak{A}}^r}$ is irreducible modulo $f_{\mathfrak{A}}X$.

Let $X_{OG} = p_{\mathfrak{A}}^{-1}O$, $X^O = f^{-1}O$, $f_{OG} = f_{\mathfrak{A}}|_{X^O} \colon X^O \to X_{OG}$, $f_O = f|_{X^O} \colon X^O \to$ $\to O$, $p_{OG} = p_{\mathfrak{A}}|_{X_{OG}} \colon X_{OG} \to O$.

$$\begin{tikzcd}
X^O \arrow[rrrrr, "\subseteq"] \arrow[dr, "f_{OG}"] \arrow[dd, "f_O"'] & & & & & X \arrow[dlll, "f_{\mathfrak{A}}"'] \arrow[dd, "f"] \\
& X_{OG} \arrow[r, "\subseteq"] \arrow[dl, "p_{OG}"] & X_{\mathfrak{A}}^r \arrow[r, "\subseteq"] \arrow[drrr, "p_{\mathfrak{A}}"'] & X_{\mathfrak{A}} \arrow[r, "\subseteq"] \arrow[drr, "{}^{\mathfrak{A}}\pi"] & Y_{\mathfrak{A}} \arrow[dr, "{}^{\mathfrak{A}}\pi"] & \\
O \arrow[rrrrr, "\subseteq"] & & & & & Y
\end{tikzcd}$$

5.11. For each $\alpha \in \mathfrak{A}$ let us define the continuous function $\bar{g}_\alpha \colon X_{OG} \to \mathbb{R}$ by the equality $\bar{g}_\alpha = {}^{\mathfrak{A}}_{\alpha}\psi|_{X_{OG}}$. Obviously, $\tilde{g}_\alpha = \bar{g}_\alpha f_{OG}$ and $\bar{g}_\alpha|_{p_{OG}^{-1}(O\setminus G)} = g_\alpha p_{OG}|_{p_{OG}^{-1}(O\setminus G)}$.



Thus we have got the map $\varphi_{OG} \colon \mathcal{C}_{\mathfrak{a}}(O,G) \to C(p_{OG})$ defined by the formula $\varphi_{OG}(g_\alpha, \tilde{g}_\alpha) = \bar{g}_\alpha$ for all $\alpha \in \mathfrak{A}$ (see 4.4).

5.12. **Theorem.** *The map φ_{OG} is "onto" and one-to-one. Moreover, for each $y_0 \in O$ and $\alpha \in \mathfrak{A}$ we have* (see 4.5)

$$\text{(3)} \qquad \begin{cases} n_{y_0}\bar{g}_\alpha = \inf\{\max\{\sup\{|g_\alpha y| : y \in Uy_0\}, \sup\{|\tilde{g}_\alpha x| : x \in f^{-1}Uy_0\}\} : \\ \qquad Uy_0 \subseteq O \textit{ is a neighborhood of the point } y_0\}. \end{cases}$$

Proof. Let us define a map $\varphi^{OG} \colon C(p_{OG}) \to \mathcal{C}_{\mathfrak{a}}(O,G)$ in the following way. For a function $\bar{g} \in C(p_{OG})$ let $\tilde{g} = \bar{g}f_{OG}$, and for all $y \in O$ let $gy = \sup\{\bar{g}z : z \in p_{OG}^{-1}y\}$. It is clear that the functions $\tilde{g} \colon f^{-1}O = X^O \to \mathbb{R}$ and $g|_{O\setminus G} \colon O \setminus G \to \mathbb{R}$ are continuous; moreover, $\tilde{g}|_{f^{-1}(O\setminus G)} = gf|_{f^{-1}(O\setminus G)}$ since the mapping $p_{OG}|_{p_{OG}^{-1}(O\setminus G)}$ is a homeomorphism of the set $p_{OG}^{-1}(O \setminus G)$ onto $O \setminus G$.

Since the mapping p_{OG} is closed, it follows for every $y \in O$ that

$$gy = \inf\{\sup\{\bar{g}z : z \in p_{OG}^{-1}Uy\} : Uy \subseteq O \text{ is a neighborhood of the point } y\}$$

(see 4.6). Using the irreducibility of the mapping p_{OG} modulo $f_{OG}X^O$, one obtains easily the equality (2) (in analogous way we can prove the equality (3)). Thus

$(g,\tilde g) \in \mathcal{C}_{\mathfrak{a}}(O,G)$, and we shall get the map φ^{OG} letting $\varphi^{OG}\bar g = (g,\tilde g)$ for all $\bar g \in C(p_{OG})$.

For $(g,\tilde g) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ let $M_{g\tilde g} = f_{OG}X^O \cup (X_{OG} \cap {}^{\mathfrak{A}}_{\alpha}\pi^{-1}\varphi_\alpha O)$,[8] and for $\bar g \in \in C(p_{OG})$ let $M_{\bar g} = f_{OG}X^O \cup \{z \in X_{OG} : \bar g z = \sup\{\bar g z' : z' \in p_{OG}^{-1}p_{OG}z\}\}$.

It is easily seen that both $\bar g = \varphi_{OG}(g,\tilde g)$, $(g,\tilde g) \in \mathcal{C}_{\mathfrak{a}}(O,G)$, and $(g,\bar g) = \varphi^{OG}\bar g$, $\bar g \in C(p_{OG})$, imply the equality $M_{g\tilde g} = M_{\bar g}$, and this set is dense in X_{OG}. Using the definitions of maps φ_{OG} and φ^{OG} and the last equality, it is easily proved that these maps are inverse to each other. Hence the maps φ_{OG} and φ^{OG} are "onto" and one-to-one. □

5.13. By Theorem 5.12 we can transfer the structure of the topological algebra $C(p_{OG})$ onto $\mathcal{C}_{\mathfrak{a}}(O,G)$. Moreover, we can write $n_y(g,\tilde g) = n_y\varphi_{OG}(g,\tilde g)$ for all $y \in O$ and $(g,\tilde g) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ (see (3)).

C. **Presheaves of algebras**

5.14. If $(O_1,G_1),(O_2,G_2) \in T_{\mathfrak{a}}$, $(O_1,G_1) \subseteq (O_2,G_2)$, then there is the restriction homomorphism $h\colon \mathcal{C}_{\mathfrak{a}}(O_2,G_2) \to \mathcal{C}_{\mathfrak{a}}(O_1,G_1)$, defined by the formula $h(g,\tilde g) = = (g|_{O_1}, \tilde g|_{f^{-1}O_1})$ for $(g,\tilde g) \in \mathcal{C}_{\mathfrak{a}}(O_2,G_2)$. Let $h_f = \varphi_{O_1G_1}h\varphi^{O_2G_2}$, where the maps $\varphi_{O_1G_1}$ and $\varphi^{O_2G_2}$ were defined in the items 5.11 and 5.12.

$$\begin{tikzcd}
C(p_{O_2G_2}) \arrow[r, shift left, "\varphi^{O_2G_2}"] \arrow[d, "h_f"'] & \mathcal{C}_{\mathfrak{a}}(O_2,G_2) \arrow[l, shift left, "\varphi_{O_2G_2}"] \arrow[d, "h"] \\
C(p_{O_1G_1}) \arrow[r, shift left, "\varphi^{O_1G_1}"] & \mathcal{C}_{\mathfrak{a}}(O_1,G_1) \arrow[l, shift left, "\varphi_{O_1G_1}"]
\end{tikzcd}$$

It is easily seen that the maps C and $\mathcal{C}_{\mathfrak{a}}$ which attribute to each couple $(O,G) \in \in T_{\mathfrak{a}}$ the algebras $C(p_{OG})$ and $\mathcal{C}_{\mathfrak{a}}(O,G)$ with the corresponding restriction homomorphisms h_f and h are presheaves (see Definition 1.44). These presheaves are isomorphic in a natural sense. We shall prove that the restriction homomorphisms are continuous, and that $\mathcal{C}_{\mathfrak{a}}$ (and, of course, C) is a sheaf.

5.15. **Lemma.** *If $(O_1,G_1),(O_2,G_2) \in T_{\mathfrak{a}}$, $(O_1,G_1) \subseteq (O_2,G_2)$, then there exists and is unique a mapping $\phi\colon X_{O_1G_1} \to X_{O_2G_2}$ such that $\phi f_{O1,G_1} = f_{O_2G_2}|_{X^{O_1}}$ and $p_{O_2G_2}\phi = p_{O_1G_1}$. Moreover,*

1) *$h_f\bar g = \bar g\phi$ for all $\bar g \in C(p_{O_2G_2})$;*
2) *if $[O_1]_Y \cap O_2 = O_1$ then the mapping ϕ is perfect;*
3) *if $O_1 = O_2$ then the mapping ϕ is "onto" and irreducible modulo $f_{O_1G_1}X^{O_1}$;*
4) *if $[O_1]_Y \cap O_2 = O_1$ and $G_2 \cap O_1 = G_1$ then the mapping ϕ is a homeomorphism onto the clopen set $p_{O_2G_2}^{-1}O_1$.*

$$\begin{tikzcd}
X^{O_1} \arrow[rrrr, "\subseteq"] \arrow[drr, "\tilde g"] \arrow[ddr, "f_{O_1G_1}"] \arrow[ddd, "f_{O_1}"'] & & & & X^{O_2} \arrow[dll, "\tilde g"'] \arrow[ddl, "f_{O_2G_2}"'] \arrow[ddd, "f_{O_2}"] \\
& & \mathbb{R} & & \\
& X_{O_1G_1} \arrow[ur, "h_f\bar g"] \arrow[rr, "\phi"] \arrow[dl, "p_{O_1G_1}"] & & X_{O_2G_2} \arrow[ul, "\bar g"'] \arrow[dr, "p_{O_2G_2}"'] & \\
O_1 \arrow[rrrr, "\subseteq"] & & & & O_2
\end{tikzcd}$$

[8]The map φ_α, where $\alpha \in \mathfrak{A}$ satisfies the condition $(g_\alpha,\tilde g_\alpha) = (g,\tilde g)$, is defined in 3.2 (see 5.8).

Proof. Let $\mathcal{C}_{\mathfrak{a}}(O_2, G_2) = \{(g_\alpha, \tilde{g}_\alpha) : \alpha \in \mathfrak{A}\}$ and $\mathcal{C}_{\mathfrak{a}}(O_1, G_1) = \{g_\beta, \tilde{g}_\beta) : \beta \in \mathfrak{B}\}$. Let us define the map $k\colon \mathfrak{A} \to \mathfrak{B}$ by the formula $k\alpha = \beta$, where $\beta \in \mathfrak{B}$ is the unique element such that $(g_\beta, \tilde{g}_\beta) = h(g_\alpha, \tilde{g}_\alpha)$, $\alpha \in \mathfrak{A}$.

Let all objects with the index $\alpha \in \mathfrak{A}$ (respectively, $\beta \in \mathfrak{B}$) correspond to the constructions 5.8 and 5.10 of $X_{O_2G_2}$ (respectively, $X_{O_1G_1}$).

Let $\alpha \in \mathfrak{A}$ and $\beta = k\alpha \in \mathfrak{B}$. We can define the map ${}^{\beta}_{\alpha}i\colon p_\beta^{-1}O_1 \to X_\alpha$ as follows (see Construction 3.2):

$$ {}^{\beta}_{\alpha}iz = \begin{cases} p_\beta z \text{ for } z \in p_\beta^{-1}(O_1 \setminus G_2), \\ (p_\beta z, {}^{\beta}\psi z) \text{ for } z \in p_\beta^{-1}(G_2 \cap O_1). \end{cases} $$

The map ${}^{\beta}_{\alpha}i$ is an embedding onto the set $p_\alpha^{-1}O_1 \subseteq X_\alpha$, because $g_\beta = g_\alpha|_{O_1}$, $\tilde{g}_\beta = \tilde{g}_\alpha|_{f^{-1}O_1}$ and ${}^{\beta}\psi|_{p_\beta^{-1}}(O_1 \setminus G_2) = g_\alpha p_\beta|_{p_\beta^{-1}}(O_1 \setminus G_2)$. Of course, ${}^{\beta}\psi = {}^{\alpha}\psi\, {}^{\beta}_{\alpha}i$.

Thus for each $\alpha \in \mathfrak{A}$ and $\beta = k\alpha$ we have the mapping $\bar{f}_\alpha\colon X_{O_1G_1} \to X_\alpha$ defined by the equality $\bar{f}_\alpha = {}^{\beta}_{\alpha}i\, {}^{\mathfrak{B}}_{\beta}\pi|_{X_{O_1G_1}}$. The existence of the mapping $\phi\colon X_{O_1G_1} \to \to X_{O_2G_2}$ follows now from Proposition 2.7. Let us prove its uniqueness.

Let $\phi_i\colon X_{O_1G_1} \to X_{O_2G_2}$, $i = 1, 2$, be mappings such that $\phi_i f_{O_1G_1} = f_{O_2G_2}|_{X^{O_1}}$ and $p_{O_2G_2}\phi_i = p_{O1G1}$ for $i = 1, 2$. We have to prove that $\phi_1 = \phi_2$.

If $x \in [f_{O_1G_1}X^{O_1}]_{X_{O_1G_1}}$ then the equality $\phi_1 x = \phi_2 x$ follows from pointed out conditions and the separability of the mapping $p_{O_2G_2}$. Therefore let $x \in X_{O_1G_1} \setminus \setminus [f_{O_1G_1}X^{O_1}]_{X_{O_1G_1}}$ be a point such that $\phi_1 x \neq \phi_2 x$. Since $p_{O_2G_2}\phi_1 x = p_{O_1G_1}x = = p_{O_2G_2}\phi_2 x$ and the mapping $p_{O_2G_2}$ is separable, there are disjoint neighborhoods $U\phi_1 x, U\phi_2 x \subseteq X_{O_2G_2} \setminus [f_{O_2G_2}X^{O_2}]_{X_{O_2G_2}}$. Let us set $Ux = \phi_1^{-1}U\phi_1 x \cap \phi_2^{-1}U\phi_2 x$. Then $Ux \subseteq X_{O_1G_1} \setminus [f_{O_1G_1}X^{O_1}]_{X_{O_1G_1}}$ is a neighborhood of the point x. The mapping $p_{O_1G_1}$ is irreducible modulo $f_{O_1G_1}X^{O_1}$, hence, $p^{\#}_{O_1G_1}Ux \neq \varnothing$. On the other hand, we have $p^{\#}_{O_1G_1}Ux = p^{\#}_{O_1G_1}Ux \cap p^{\#}_{O_1G_1}Ux = p^{\#}_{O_2G_2}\phi_1^{\#}Ux \cap p^{\#}_{O_2G_2}\phi_2^{\#}Ux \subseteq \subseteq p^{\#}_{O_2G_2}U\phi_1 x \cap p^{\#}_{O_2G_2}U\phi_2 x = \varnothing$. This contradiction shows that $\phi_1 x = \phi_2 x$, that is, the mapping ϕ is unique.

Obviously, the mapping ϕ is perfect as the mapping onto $\phi X_{O_1G_1}$ by Lemma 8 of [43] (but not into $X_{O_2G_2}$ in general case), and it is irreducible modulo $f_{O_1G_1}X^{O_1}$ under the same condition.

Finally, if $[O_1]_Y \cap O_2 = O_1$ and $G_1 = G_2 \cap O_1$ then for each couple $(g_\beta, \tilde{g}_\beta) \in \in \mathcal{C}_{\mathfrak{a}}(O_1, G_1)$ there exists a couple $(g_\alpha, \tilde{g}_\alpha) \in \mathcal{C}_{\mathfrak{a}}(O_2, G_2)$ such that $h(g_\alpha, \tilde{g}_\alpha) = = (g_\beta, \tilde{g}_\beta)$, that is, $k\alpha = \beta$ and, hence, $k\mathfrak{A} = \mathfrak{B}$ (for example, we can define

$$ g_\alpha y = \begin{cases} g_\beta y \text{ for } y \in O_1, \\ 0 \text{ for } y \in O_2 \setminus O_1, \end{cases} \qquad \tilde{g}_\alpha x = \begin{cases} \tilde{g}_\beta x \text{ for } x \in f^{-1}O_1, \\ 0 \text{ for } x \in f^{-1}(O_2 \setminus O_1); \end{cases} $$

all conditions of the definition 5.4 are obviously satisfied). Therefore the map h is “onto” and, hence, the mapping ϕ separates points of $X_{O_1G_1}$; since ϕ is perfect, it is a homeomorphism onto the clopen set $p^{-1}_{O_2G_2}O_1$.

The remaining statements follow from the proved statements and the construction of the mapping ϕ. □

5.16. **Corollary.** *If $(O_1, G_1), (O_2, G_2) \in T_{\mathfrak{a}}$, $(O_1, G_1) \subseteq (O_2, G_2)$, then the map $h\colon \mathcal{C}_{\mathfrak{a}}(O_2, G_2) \to \mathcal{C}_{\mathfrak{a}}(O_1, G_1)$ is a continuous homomorphism of topological algebras. Moreover,*

1) *if $[O_1]_Y \cap O_2 = O_1$ then h is a homomorphism onto a closed subalgebra of $\mathcal{C}_{\mathfrak{a}}(O_1, G_1)$;*
2) *if $O_1 = O_2$ then h is a topological isomorphism onto a closed subalgebra of $\mathcal{C}_{\mathfrak{a}}(O_1, G_1)$.*

D. **Sheaves of algebras**

5.17. **Definition.** A family $\{(O_\alpha, G_\alpha) : \alpha \in A\} \subseteq T_\mathfrak{a}$ will be called *a covering of the couple* $(O, G) \in T_\mathfrak{a}$ if $O = \bigcup\{O_\alpha : \alpha \in A\}$ and $O \setminus G = \bigcup\{O_\alpha \setminus G_\alpha : \alpha \in A\}$.

All conditions of Definition 1.43 are obviously valid.

5.18. **Theorem.** *The maps $\mathcal{C}_\mathfrak{a}$ and $\mathcal{C}_f$ which assign to each couple $(O, G) \in T_\mathfrak{a}$ the topological algebras $\mathcal{C}_\mathfrak{a}(O, G)$ and $\mathcal{C}_f(O, G)$* (see 5.4 and 5.6) *are sheaves.*

Proof follows from Proposition 2.1.11 of the book [51]. □

5.19. *Remark.* The analogous map $C^*_\mathfrak{a}$ (see 5.4) is a presheaf but it is not a sheaf.

5.20. We shall meet sheaves on the set T (see 5.4) which are usual sheaves over a topological space Y (see Example 1.47), but we shall formulate all definitions for sheaves on the set $T_\mathfrak{a}$ and use (if it is possible) some of them for sheaves on the set T too.

We shall consider sheaves of topological algebras only; therefore, the phrases "sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$" or "subsheaf $\mathcal{C}$ of the sheaf $\mathcal{C}_\mathfrak{a}$" and so on will mean, particularly, that $\mathcal{C}(O, G)$ contains all constant couples and is a subalgebra of the topological algebra $\mathcal{C}_\mathfrak{a}(O, G)$ for every $(O, G) \in T_\mathfrak{a}$.

E. **Properties of sheaves**

5.21. **Definition.** A sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ will be called *complete* if the algebra $\mathcal{C}(O, G)$ is closed in $\mathcal{C}_\mathfrak{a}(O, G)$ for any $(O, G) \in T_\mathfrak{a}$.

5.22. Let $T_O = \{(O, G_\alpha) : \alpha \in \mathfrak{A}\}$ be the family of all couples $(O, G) \in T_\mathfrak{a}$ with a fixed set $O \in T$ (see 5.2 c)). For each $\alpha \in \mathfrak{A}$ we have the mappings $f_O \colon X^O \to O$, $f_{OG_\alpha} \colon X^O \to X_{OG_\alpha}$ and $p_{OG_\alpha} \colon X_{OG_\alpha} \xrightarrow{\text{onto}} O$ constructed in the item 5.10. Let $X_\mathfrak{A} = \prod_O(\{X_{OG_\alpha}\}, \{p_{OG_\alpha}\}, \alpha \in \mathfrak{A})$ and ${}^\mathfrak{A}p = \prod_O\{p_{OG_\alpha} : \alpha \in \mathfrak{A}\}$ be the fan products. By Theorem 2.11 the mapping ${}^\mathfrak{A}p$ is perfect.

By Proposition 2.7 there is a mapping $f_\mathfrak{A} \colon X^O \to X_\mathfrak{A}$ such that ${}^\mathfrak{A}pf_\mathfrak{A} = f_O$ and ${}^\mathfrak{A}_\alpha pf_\mathfrak{A} = f_{OG_\alpha}$ for all $\alpha \in \mathfrak{A}$, where ${}^\mathfrak{A}_\alpha p \colon X_\mathfrak{A} \to X_{OG_\alpha}$, $\alpha \in \mathfrak{A}$, is the projection of the fan product to its factor (see 2.1). By Corollary 2.13 there exists a unique closed subset $X_O \subseteq X_\mathfrak{A}$ such that $f_\mathfrak{A} X^O \subseteq X_O$, ${}^\mathfrak{A}pX_O = O$ and the mapping $p_O = {}^\mathfrak{A}p|_{X_O} \colon X_O \xrightarrow{\text{onto}} O$ is irreducible modulo $f_\mathfrak{A} X^O$. Let us denote by $q_O \colon X^O \to X_O$ the mapping coinciding with $f_\mathfrak{A}$.

$$
\begin{tikzcd}
X^O \arrow[rrrr, "f_{OG_\alpha}"] \arrow[rrr, "f_{\mathfrak{A}}", to=2-4] \arrow[rd, "q_O"] \arrow[rrdd, "f_O"'] & & & & X_{OG_\alpha} \arrow[lldd, "p_{OG_\alpha}"] \\
& X_O \arrow[rr, "\subseteq"] \arrow[rd, "p_O"'] & & X_{\mathfrak{A}} \arrow[ru, "{}^{\mathfrak{A}}_{\alpha}p"] \arrow[ld, "{}^{\mathfrak{A}}p"] & \\
& & O & &
\end{tikzcd}
$$

5.23. By Theorem 4.18 we can assume that for each $\alpha \in \mathfrak{A}$ the algebra $\mathcal{C}_\mathfrak{a}(O, G_\alpha)$ is a closed subalgebra of $C(p_O)$ (the embedding $\mathcal{C}_\mathfrak{a}(O, G_\alpha) \to C(p_O)$ is defined by the formula $(g, \tilde{g}) \to (\varphi_{OG_\alpha}(g, \tilde{g}))^\mathfrak{A}_\alpha p$ for each $(g, \tilde{g}) \in \mathcal{C}_\mathfrak{a}(O, G_\alpha)$ and $\alpha \in \mathfrak{A}$; see 5.11). This embedding is coordinated with the restriction homomorphisms (see 5.14), which are closed embeddings by Corollary 5.16.

By Lemma 5.2 c) for every sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ the set $\hat{\mathcal{C}}(O) = \bigcup\{\mathcal{C}(O, G_\alpha) : \alpha \in \mathfrak{A}\}$ is a subalgebra of the algebra $C(p_O)$. In general case this algebra is not closed in $C(p_O)$ and in $\hat{\mathcal{C}}_\mathfrak{a}(O)$ even if the sheaf $\mathcal{C}$ is complete.

5.24. **Definition.** We shall say that the family $\mathfrak{a}$ *has the largest representative in a set* $O \in T$ if there exists an element $G_O \in \mathfrak{a}$ such that $(O, G_O) \in T_\mathfrak{a}$ and $(O, G_O) \subseteq (O, G)$ for every couple $(O, G) \in T_O$.

5.25. **Proposition.** *Let a sheaf* $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ *be complete, and let the family* $\mathfrak{a}$ *have the largest representative* $G_O \in \mathfrak{a}$ *in the set* $O \in T$. *Then the algebra* $\hat{\mathcal{C}}(O)$ *is closed in* $C(p_O)$.

Proof. Obviously, in this case we have $\mathcal{C}(O, G) \subseteq \mathcal{C}(O, G_O)$ for all $(O, G) \in \in T_O$, therefore the algebra $\hat{\mathcal{C}}(O) = \bigcup\{\mathcal{C}(O, G) : (O, G) \in T_O\} = \mathcal{C}(O, G_O)$ is closed in $\mathcal{C}_\mathfrak{a}(O, G_O)$; on the other hand, using Lemma 5.15, the construction 5.22 and Proposition 2.7, we can prove that the projection of X_O onto X_{OG_O} is a homeomorphism, therefore $\mathcal{C}_\mathfrak{a}(O, G_O) = C(p_O)$. □

5.26. **Definition.** A sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ will be called *hereditary*, if for every (O, G_1), $(O, G_2) \in T_O$, $O \in T$, such that $(O, G_1) \subseteq (O, G_2)$ the condition $\mathcal{C}(O, G_2) = = \mathcal{C}(O, G_1) \cap \mathcal{C}_\mathfrak{a}(O, G_2)$ is satisfied.[9]

5.27. **Lemma.** *If a sheaf* $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ *is hereditary then the map* $\hat{\mathcal{C}}$ *which assigns to each* $O \in T$ *the topological algebra* $\hat{\mathcal{C}}(O)$ *is a sheaf over the topological space* Y. *In particular, the map* $\hat{\mathcal{C}}_\mathfrak{a}$ *is a sheaf.*

Proof. It is clear that the map $\hat{\mathcal{C}}$ is a presheaf. It is necessary to prove the following property only (Definition 1.46): if $\{O_\alpha : \alpha \in \mathfrak{A}\}$ is a family of open sets of the space Y, $O = \bigcup\{O_\alpha : \alpha \in \mathfrak{A}\}$, for each $\alpha \in \mathfrak{A}$ a couple $(g_\alpha, \tilde{g}_\alpha) \in \mathcal{C}(O_\alpha)$ is given, and for all $\alpha, \beta \in \mathfrak{A}$ the equalities $g_\alpha|_{O_\alpha \cap O_\beta} = g_\beta|_{O_\alpha \cap O_\beta}$ and $\tilde{g}_\alpha|_{f^{-1}(O_\alpha \cap O_\beta)} = = \tilde{g}_\beta|_{f^{-1}(O_\alpha \cap O_\beta)}$ hold, then there exists a unique couple $(g, \tilde{g}) \in \hat{\mathcal{C}}(O)$ such that $g|_{O_\alpha} = g_\alpha$ and $\tilde{g}|_{f^{-1}O_\alpha} = \tilde{g}_\alpha$ for all $\alpha \in \mathfrak{A}$.

Such a couple $(g, \tilde{g})$ exists by Proposition 2.1.11 of [51]. We have to prove that $(g, \tilde{g}) \in \hat{\mathcal{C}}(O)$. For every $\alpha \in \mathfrak{A}$ let $G_\alpha \in \mathfrak{a}$ be a set such that $(O_\alpha, G_\alpha) \in T_\mathfrak{a}$ and $(g_\alpha, \tilde{g}_\alpha) \in \mathcal{C}(O_\alpha, G_\alpha)$. Let

$$U = \{y \in O : \text{ there exists a neighborhood } Uy \subseteq O \text{ such that the function } g|_{Uy} \text{ is continuous and } \tilde{g}|_{f^{-1}Uy} = gf|_{f^{-1}Uy}\}.$$

It is clear that the set U is open and the set $G = O \setminus U$ is closed in O, hence, G is a locally closed subset of Y. Let us show that $G \in \mathfrak{a}$. For each $y \in G$ there exists $\alpha \in \mathfrak{A}$ such that $y \in O_\alpha$. Obviously, $G \cap O_\alpha \subseteq G_\alpha$ by the definition of the set G. Since the family $\mathfrak{a}$ is closed, we have $G \in \mathfrak{a}$ (see Definition 1.5, the condition 3)).

For every $\alpha \in \mathfrak{A}$ let $G'_\alpha = G \cap O_\alpha$; since $(g_\alpha, \tilde{g}_\alpha) \in \mathcal{C}_\mathfrak{a}(O_\alpha, G_\alpha)$, $(O_\alpha, G_\alpha) \subseteq \subseteq (O_\alpha, G'_\alpha)$, the sheaf $\mathcal{C}$ is hereditary and $(g_\alpha, \tilde{g}_\alpha) \in \mathcal{C}(O_\alpha, G_\alpha)$, we have $(g_\alpha, \tilde{g}_\alpha) \in \in \mathcal{C}(O_\alpha, G'_\alpha)$ for all $\alpha \in \mathfrak{A}$. The family $\{(O_\alpha, G'_\alpha) : \alpha \in \mathfrak{A}\}$ is a covering of the couple (O, G) and $\mathcal{C}$ is a sheaf. Therefore $(g, \tilde{g}) \in \mathcal{C}(O, G)$ and, hence, $(g, \tilde{g}) \in \hat{\mathcal{C}}(O)$. Thus, the map $\hat{\mathcal{C}}$ is a sheaf over the topological space Y. □

5.28. **Proposition.** *There exists a one-to-one correspondence between the set of all hereditary subsheaves of the sheaf* $\mathcal{C}_\mathfrak{a}$ *and the set of all subsheaves of the sheaf* $\hat{\mathcal{C}}_\mathfrak{a}$ *which preserves the relation "$\subseteq$".*

Proof. It suffices to note that if $\hat{\mathcal{C}} \subseteq \hat{\mathcal{C}}_\mathfrak{a}$ is a subsheaf then one can define the sheaf $\mathcal{C}$ by the equality $\mathcal{C}(O, G) = \hat{\mathcal{C}}(O) \cap \mathcal{C}_\mathfrak{a}(O, G)$ for all $(O, G) \in T_\mathfrak{a}$, and this construction is inverse to the construction 5.23. □

5.29. It is known ([61], Theorem 4.5.3, or [8], Theorem 0.24) that a sheaf over a space Y has a representation by a local homeomorphism $p\colon E \to Y$. Therefore

[9] Analogously to the item 5.23 we shall assume that if $(O, G_1), (O, G_2) \in T_\mathfrak{a}$ and $(O, G_1) \subseteq \subseteq (O, G_2)$, then $\mathcal{C}_\mathfrak{a}(O, G_2)$ is a closed subalgebra of the algebra $\mathcal{C}_\mathfrak{a}(O, G_1)$ (see Corollary 5.16).

we can consider that hereditary subsheaves of $\mathcal{C}_\mathfrak{a}$ have similar representations. We shall not use these representations.

5.30. **Definition.** A sheaf $\hat{\mathcal{C}} \subseteq \hat{\mathcal{C}}_\mathfrak{a}$ will be called *closed* if for every set $O \in T$ the algebra $\hat{\mathcal{C}}(O)$ is closed in $\hat{\mathcal{C}}_\mathfrak{a}(O)$.

5.31. **Definition.** A sheaf $\hat{\mathcal{C}} \subseteq \hat{\mathcal{C}}_\mathfrak{a}$ will be called *complete* if for every set $O \in T$ the algebra $\hat{\mathcal{C}}(O)$ is closed in $C(p_O)$ (see 5.23).

5.32. *Remark.* If a sheaf $\hat{\mathcal{C}} \subseteq \hat{\mathcal{C}}_\mathfrak{a}$ is complete then it is closed too, but the inverse statement is not valid (for example, the sheaf $\hat{\mathcal{C}}_\mathfrak{a}$ is closed but, in general case, is not complete). It is also quite probable that a sheaf $\hat{\mathcal{C}}$ need not be closed, even if the sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ is hereditary and complete (of course, if $\hat{\mathcal{C}}$ is closed and $\mathcal{C}$ is hereditary then $\mathcal{C}$ is complete; see 5.21).

5.33. **Definition.** Let $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ be a subsheaf, $(O,G) \in T_\mathfrak{a}$. A couple $(g_0,\tilde{g}_0) \in \mathcal{C}_\mathfrak{a}(O,G)$ will be called $\mathcal{C}$*-separated* if for any numbers $a,b \in \mathbb{R}$, $a<b$, and a point $y \in O$ there exist couples $(O',G') \in T_\mathfrak{a}$, $(g,\tilde{g}) \in \mathcal{C}(O',G')$ and numbers $a',b' \in \mathbb{R}$, $a'<b'$, such that $y \in O'$,

$$O' \cap g_0^{-1}H_a \subseteq g^{-1}H_{a'} \subseteq g^{-1}H_{b'} \subseteq g_0^{-1}H_b \cap O' \text{ and}$$
$$f^{-1}O' \cap \tilde{g}_0^{-1}H_a \subseteq \tilde{g}^{-1}H_{a'} \subseteq \tilde{g}^{-1}H_{b'} \subseteq \tilde{g}_0^{-1}H_b \cap f^{-1}O',$$

where $H_c = \{t \in \mathbb{R} : t < c\}$ for $c \in \mathbb{R}$.

5.34. **Definition.** A sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ will be called *saturated* if for each $(O,G) \in T_\mathfrak{a}$ the algebra $\mathcal{C}(O,G)$ contains all $\mathcal{C}$-separated couples $(g,\tilde{g}) \in \mathcal{C}_\mathfrak{a}(O,G)$.

5.35. **Theorem.** *If a sheaf $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ is saturated then it is also hereditary and complete, $\mathcal{C}_f \subseteq \mathcal{C}$, and the sheaf $\hat{\mathcal{C}}$ is closed.*

Proof. The sheaf $\mathcal{C}$ is hereditary, because if $(O,G_1),(O,G_2) \in T_\mathfrak{a}$, $(O,G_1) \subseteq (O,G_2)$ and $(g_0,\tilde{g}_0) \in \mathcal{C}(O,G_1) \cap \mathcal{C}_\mathfrak{a}(O,G_2)$, then the couple $(g_0,\tilde{g}_0)$ is $\mathcal{C}$-separated and, hence, $(g_0,\tilde{g}_0) \in \mathcal{C}(O,G_2)$ (we can take $(O',G')=(O,G_1)$, $(g,\tilde{g})=(g_0,\tilde{g}_0)$ and $(a',b')=(a,b)$ in Definition 5.33).

Let $(g_0,\tilde{g}_0) \in \mathcal{C}_f(O,G)$ be an arbitrary couple, where $(O,G) \in T_\mathfrak{a}$. By Definition 5.6 the function g_0 is continuous and $\tilde{g}_0 = g_0 f|_{f^{-1}O}$. Let $a,b \in \mathbb{R}$, $a<b$. The sets $F_0 = [g_0^{-1}H_a]_Y \cap O$ and $F_1 = O \setminus g^{-1}H_b$ are disjoint and closed in O. For any $y \in O$ let

$$O' = \begin{cases} O \setminus F_1 \text{ if } y \in F_0, \\ O \setminus F_0 \text{ if } y \in O \setminus F_0, \end{cases} \qquad c = \begin{cases} a \text{ if } y \in F_0, \\ b \text{ if } y \in O \setminus F_0, \end{cases}$$

$G' = \varnothing$, $gy' = c$ and $\tilde{g}x = c$ for all $y' \in O'$ and $x \in f^{-1}O'$, $a'=a$, $b'=b$. Then all conditions of Definition 5.33 are satisfied; therefore $(g_0,\tilde{g}_0) \in \mathcal{C}(O,G)$ and, hence, $\mathcal{C}_f(O,G) \subseteq \mathcal{C}(O,G)$.

We shall show that the sheaf $\hat{\mathcal{C}}$ is closed. The completeness of the hereditary sheaf $\mathcal{C}$ is a consequence of the closedness of $\hat{\mathcal{C}}$.

Let $(g_0,\tilde{g}_0) \in [\hat{\mathcal{C}}(O)]_{\hat{\mathcal{C}}_\mathfrak{a}(O)}$ for some $O \in T$. Let $a,b \in \mathbb{R}$, $a<b$, $y \in O$. Let us set $\varepsilon = \frac{b-a}{3}$, $a' = a+\varepsilon$, $b' = b-\varepsilon$. By the definition of the topology of the algebra $\hat{\mathcal{C}}_\mathfrak{a}(O)$ there exist a couple $(g,\tilde{g}) \in \hat{\mathcal{C}}(O)$ and a set $O' \in T$ such that $y \in O' \subseteq O$ and $n_{y'}((g,\tilde{g})-(g_0,\tilde{g}_0)) < \varepsilon$ for all $y' \in O'$ (see 5.13). Let $G \in \mathfrak{a}$ be a set such that $(O,G) \in T_\mathfrak{a}$, $(g,\tilde{g}) \in \mathcal{C}(O,G)$ and $(g_0,\tilde{g}_0) \in \mathcal{C}_\mathfrak{a}(O,G)$. Let $G' = G \cap O'$; it is easily seen that the couple $(g|_{O'},\tilde{g}|_{f^{-1}O'}) \in \mathcal{C}(O',G')$ satisfy all conditions of Definition 5.33. Therefore the couple $(g_0,\tilde{g}_0)$ is $\mathcal{C}$-separated and, hence, $(g_0,\tilde{g}_0) \in \mathcal{C}(O,G) \subseteq \hat{\mathcal{C}}(O)$. □

5.36. Let $\mathcal{C} \subseteq \mathcal{C}_\mathfrak{a}$ be a subsheaf. For each $(O,G) \in T_\mathfrak{a}$ let us set

$$\bar{\mathcal{C}}(O,G) = \{(g,\tilde{g}) \in \mathcal{C}_\mathfrak{a}(O,G) : (g,\tilde{g}) \text{ is } \mathcal{C}\text{-separated}\}.$$

5.37. **Theorem.** *For every subsheaf $\mathcal{C} \subseteq \mathcal{C}_{\mathfrak{a}}$ the map $\bar{\mathcal{C}}$ is a saturated sheaf.*

Proof. It suffices to note that each $\bar{\mathcal{C}}$-separated couple is also $\mathcal{C}$-separated (we can prove, that $\bar{\mathcal{C}}(O,G)$ is a subalgebra of $\mathcal{C}_{\mathfrak{a}}(O,G)$ for every $(O,G) \in T_{\mathfrak{a}}$, using the reasonings 6.6, 6.11, 6.3). $\square$

Let us note that this theorem is also true if $\mathcal{C} \subseteq \mathcal{C}_{\mathfrak{a}}$ is a presheaf.

§ 6. $\mathfrak{Ta}$-BICOMPACTIFICATIONS

6.1. From now on we shall fix a mapping $f\colon X \to Y$ with the property $\mathfrak{Ta}$.

A. From a bicompactification to a sheaf

6.2. Let $f_v\colon v_f X \xrightarrow{\text{onto}} Y$ be a $\mathfrak{Ta}$-bicompactification of the mapping f, $O \in T$. Let us denote $X_O = f_v^{-1}O$, $X^O = f^{-1}O$, $f_O = f|_{X^O}$, $p_O = f_v|_{X_O}$.

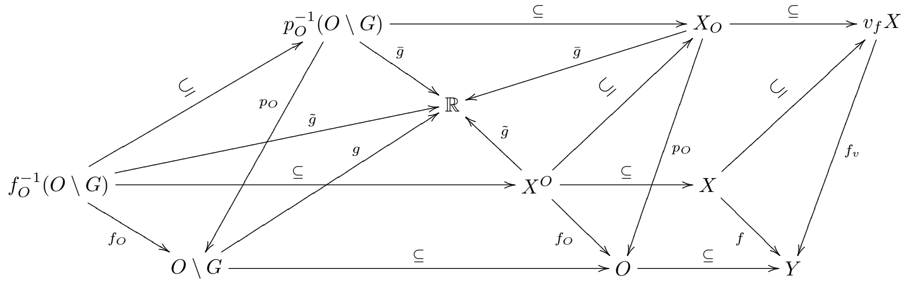
For each $\bar{g} \in C(p_O)$ let us define functions $\tilde{g}\colon X^O \to \mathbb{R}$ and $g\colon O \to \mathbb{R}$ as follows:

$$\tilde{g} = \bar{g}|_{X^O}, \qquad gy = \sup\{\bar{g}z : z \in p_O^{-1}y\} \text{ for all } y \in O. \tag{4}$$

Since the mapping p_O is irreducible modulo X^O, we have $(g_1, \tilde{g}_1) \neq (g_2, \tilde{g}_2)$ for any different $\bar{g}_1, \bar{g}_2 \in C(p_O)$.

For every couple $(O,G) \in T_{\mathfrak{a}}$ let

$$\mathcal{C}_v(O,G) = \{(g,\tilde{g}) : (g,\tilde{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G) \text{ and there is } \bar{g} \in C(p_O) \text{ such that the equalities (4) hold}\}.$$



6.3. **Lemma.** *The map $\mathcal{C}_v$ defined above is a saturated sheaf.*

Proof. It is obvious that $\mathcal{C}_v$ is a sheaf. We shall prove that it is saturated. Let $(O,G) \in T_{\mathfrak{a}}$.

Let a couple $(g_0, \tilde{g}_0) \in \mathcal{C}_{\mathfrak{a}}(O,G)$ be $\mathcal{C}_v$-separated. We have to show that there exists a function $\bar{g}_0 \in C(p_O)$ such that the equalities (4) hold.

For every $b \in \mathbb{R}$ let $U_b = \{y \in O : g_0 y < b\}$, $V_b = \{x \in X^O : \tilde{g}_0 x < b\}$ and $W_b = \bigcup\{\tilde{W}_a : a < b\}$, where for all $a \in \mathbb{R}$

$$\tilde{W}_a = \{z \in X_O : \text{ there is a neighborhood } Wz \subseteq X_O \text{ such that } X^O \cap Wz \subseteq V_a \text{ and } p_O^{\#} Wz \subseteq U_a\}.$$

Let $b \in \mathbb{R}$; the sets U_b, V_b, $\tilde{W}_b$ and W_b are open in Y, X, $v_f X$ and $v_f X$ respectively. Let us prove the equality $W_b \cap X^O = \tilde{W}_b \cap X^O = V_b$. Let us take any point $x \in V_b$ and a number $a \in \mathbb{R}$ such that $\tilde{g}_0 x < a < b$ and denote $y = fx$. Since the couple $(g_0, \tilde{g}_0)$ is $\mathcal{C}_v$-separated, there exist couples $(O', G') \in T_{\mathfrak{a}}$, $(g, \tilde{g}) \in \mathcal{C}_v(O', G')$ and numbers $a', b' \in \mathbb{R}$, $a' < b'$, such that $y \in O'$,

$$O' \cap U_a \subseteq g^{-1} H_{a'} \subseteq g^{-1} H_{b'} \subseteq U_b \cap O' \text{ and}$$

$$x \in f_O^{-1} O' \cap V_a \subseteq \tilde{g}^{-1} H_{a'} \subseteq \tilde{g}^{-1} H_{b'} \subseteq V_b \cap f^{-1} O'.$$

For a function $\bar{g} \in C(p_O)$ satisfying the condition (4), the inverse image $\bar{g}^{-1}H_{a'}$ is an open subset of X_O,

$$x \in f_O^{-1}O' \cap V_a \subseteq \bar{g}^{-1}H_{a'} \subseteq \bar{g}^{-1}H_{b'},$$
$$\bar{g}^{-1}H_{a'} \cap X^O \subseteq \bar{g}^{-1}H_{b'} \cap X^O \subseteq f_O^{-1}O' \cap V_b \subseteq V_b \text{ and}$$
$$p_O^{\#}\bar{g}^{-1}H_{a'} = g^{-1}H_{a'} \subseteq g^{-1}H_{b'} \subseteq O' \cap U_b \subseteq U_b.$$

Thus, $x \in \tilde{W}_a \subseteq W_b \subseteq \tilde{W}_b$ and, hence, $V_b \subseteq W_b \subseteq \tilde{W}_b$. Since the inclusion $\tilde{W}_b \cap X^O \subseteq V_b$ is obvious, we have the required equality.

Let us prove that $p_O^{\#}W_b = U_b$. The inclusion $U_b \subseteq p_O^{\#}W_b$ is obvious. Consider a point $y \in p_O^{\#}W_b$; since $p_O^{-1}y$ is compact, by the definition of the set W_b there is $a < b$ such that $p_O^{-1}y \subseteq \tilde{W}_a$. From the definition of the set $\tilde{W}_a$ and the irreducibility of the mapping p_O modulo X^O it follows that for every point $z \in p_O^{-1}y$ and its arbitrary neighborhood $Wz \subseteq X_O$ at least one of the sets $Wz \cap V_a$ and $Wz \cap p_O^{-1}U_a$ is non-empty, therefore $p_O^{-1}y \subseteq [V_a \cup p_O^{-1}U_a]_{X_O} = [\tilde{W}_a]_{X_O}$ (the last equality is true since the same property holds for all points $z \in [\tilde{W}_a]_{X_O}$). Since the couple $(g_0, \tilde{g}_0)$ is $\mathcal{C}_v$-separated, there exist couples $(O', G') \in T_{\mathfrak{a}}$ and $(g, \tilde{g}) \in \mathcal{C}_v(O', G')$ and numbers $a', b' \in \mathbb{R}$, $a' < b'$, such that $y \in O'$,

$$O' \cap U_a \subseteq g^{-1}H_{a'} \subseteq g^{-1}H_{b'} \subseteq U_b \cap O' \text{ and}$$
$$f_O^{-1}O' \cap V_a \subseteq \tilde{g}^{-1}H_{a'} \subseteq \tilde{g}^{-1}H_{b'} \subseteq V_b \cap f_O^{-1}O'.$$

For a function $\bar{g} \in C(p_{O'})$ satisfying the condition (4) we have

$$\begin{aligned} p_O^{-1}y \subseteq [\tilde{W}_a \cap X_{O'}]_{X_{O'}} &\subseteq [(V_a \cup p_O^{-1}U_a) \cap X_{O'}]_{X_{O'}} \subseteq \\ &\subseteq [\tilde{g}^{-1}H_{a'} \cup p_{O'}^{-1}H_{a'}]_{X_{O'}} = [\bar{g}^{-1}H_{a'}]_{X_{O'}} \subseteq \bar{g}^{-1}H_{b'}, \end{aligned}$$

thus, $y \in p_O^{\#}\bar{g}^{-1}H_{b'} = g^{-1}H_{b'} \subseteq U_b \cap O' \subseteq U_b$ and, hence, $p_O^{\#}W_b = U_b$.

Analogously we can prove the inclusion $[W_a]_{X_O} \subseteq W_b$ for all $a, b \in \mathbb{R}$, $a < b$. From that it follows that a function $\bar{g}_0 \colon X_O \to \mathbb{R}$, defined by the equality $\bar{g}_0 z = \inf\{t \in \mathbb{R} : z \in W_t\}$ for all $z \in X_O$, is continuous (see [51], proof of Theorem 1.5.10).

Thus we have constructed the function $\bar{g}_0 \in C(P_O)$. The equalities (4) follow from the equalities $\bar{g}_0^{-1}H_t \cap X^O = W_t \cap X^O = V_t$ and $p_O^{\#}\bar{g}_0^{-1}H_t = p_O^{\#}W_t = U_t$ for all $t \in \mathbb{R}$. Hence, $(g_0, \tilde{g}_0) \in \mathcal{C}_v(O, G)$ and the sheaf $\mathcal{C}_v$ is saturated. □

6.4. **Definition.** A sheaf $\mathcal{C} \subseteq \mathcal{C}_{\mathfrak{a}}$ will be called *dismembering* if for an arbitrary point $x \in X$ in each of the following two cases

a) for every point $x' \in f^{-1}fx \setminus \{x\}$ and
b) for every neighborhood $Ux \subseteq X$

there exist couples $(O, G) \in T_{\mathfrak{a}}$ and $(g, \tilde{g}) \in \mathcal{C}(O, G)$ such that $fx \in O$ and, respectively,

a) $\tilde{g}x \neq \tilde{g}x'$ or
b) $\tilde{g}x \notin [\tilde{g}(f^{-1}O \setminus Ux)]_{\mathbb{R}}$.

6.5. **Lemma.** *The sheaf $\mathcal{C}_v$ is dismembering; moreover, for an arbitrary point $z \in v_f X$ in each of the following two cases*

a) *for every point $z' \in f_v^{-1}z \setminus \{z\}$ and*
b) *for every neighborhood $Uz \subseteq v_f X$*

there are couples $(O, G) \in T_{\mathfrak{a}}$, $(g, \tilde{g}) \in \mathcal{C}_v(O, G)$ and a continuous function $\bar{g} \colon f_v^{-1}O \to \mathbb{R}$ satisfying the condition (4) such that $f_v z \in O$ and, respectively,

a) *$\bar{g}z' \neq \bar{g}z$ or*
b) *$\bar{g}z \notin [\bar{g}(f_v^{-1}O \setminus Uz)]_{\mathbb{R}}$.*

Proof follows from the definition of $\mathfrak{Ta}$-bicompactification. □

B. **From a sheaf to a bicompactification**

6.6. Let a dismembering sheaf $\mathcal{C} \subseteq C_{\mathfrak{a}}$ be given. Let us denote by B the set of all quadruplets $(O, G, g, \tilde{g})$ where $(O, G) \in T_{\mathfrak{a}}$ and $(g, \tilde{g}) \in \mathcal{C}(O, G)$. Let $B = \{(O_\alpha, G_\alpha, g_\alpha, \tilde{g}_\alpha) : \alpha \in \mathfrak{A}\}$. For all $\alpha \in \mathfrak{A}$ let us set $Z_\alpha = \mathbb{R}$.

By analogy with the item 5.8, using Constructions 3.1 and 3.2, Proposition 3.3, Lemma 5.9 and Corollary 2.13, we obtain the space

$$v_{\mathcal{C}}X = X^r_{\mathfrak{A}} \subseteq X_{\mathfrak{A}} \subseteq Y_{\mathfrak{A}} = \mathfrak{P}(Y, \{Z_\alpha\}, \{G_\alpha\}, \{O_\alpha\}, \{g_\alpha|_{O_\alpha \setminus G_\alpha}\}, \alpha \in \mathfrak{A})$$

and the mappings $i_{\mathcal{C}} = f_{\mathfrak{A}} \colon X \to v_{\mathcal{C}}X$, $f_{\mathcal{C}} = {}^{\mathfrak{A}}\pi|_{v_{\mathcal{C}}X} \colon v_{\mathcal{C}}X \xrightarrow{\text{onto}} Y$ and $\bar{g}_\alpha = {}^{\mathfrak{A}}_{\alpha}\psi|_{f_{\mathcal{C}}^{-1}O_\alpha} \colon f_{\mathcal{C}}^{-1}O_\alpha \to Z_\alpha = \mathbb{R}$, $\alpha \in \mathfrak{A}$, satisfying the conditions $f_{\mathcal{C}} i_{\mathcal{C}} = f$, $\bar{g}_\alpha|_{f_{\mathcal{C}}^{-1}(O_\alpha \setminus G_\alpha)} = g_\alpha f_{\mathcal{C}}|_{f_{\mathcal{C}}^{-1}(O_\alpha \setminus G_\alpha)}$ and $\tilde{g}_\alpha = \bar{g}_\alpha i_{\mathcal{C}}|_{f^{-1}O_\alpha}$ for all $\alpha \in \mathfrak{A}$. Moreover, the mapping $f_{\mathcal{C}}$ is perfect and irreducible modulo $i_{\mathcal{C}}X$.

Let $C(O, G) = \{\bar{g}_\alpha : \alpha \in \mathfrak{A}, O_\alpha = O, G_\alpha = G\}$ for all $(O, G) \in T_{\mathfrak{a}}$. The map C is a sheaf which is naturally isomorphic to the sheaf $\mathcal{C}$.

6.7. **Lemma.** *The mapping $f_{\mathcal{C}} \colon v_{\mathcal{C}}X \xrightarrow{\text{onto}} Y$ is a $\mathfrak{T}\mathfrak{a}$-bicompactification of the mapping f.*

Proof. Since the sheaf $\mathcal{C}$ is dismembering, it is easily seen that the mapping $i_{\mathcal{C}}$ is an embedding. Let us identify X and $i_{\mathcal{C}}X$, that is, we shall assume that $X \subseteq v_{\mathcal{C}}X$. Then we have $f_{\mathcal{C}}|_X = f$. Moreover, the mapping $f_{\mathcal{C}}$ has the property $\mathfrak{T}\mathfrak{a}$ by Assertion 1.8. Therefore $f_{\mathcal{C}}$ is a $\mathfrak{T}\mathfrak{a}$-bicompactification of the mapping f. □

C. **Bicompactifications and sheaves**

6.8. **Lemma.** *Let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{R}^n = \{\vec{x} = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}$ for all $i = 1, 2, \dots, n\}$ with the usual topology, $\Phi \subseteq \mathbb{R}^n$ be a compact set, $F \subseteq \mathbb{R}^n$ be a closed set and $\Phi \cap F = \varnothing$. Then there exists a polynomial $h \colon \mathbb{R}^n \to \mathbb{R}$ such that*

1) *$h\vec{x} \geqslant -\frac{1}{2}$ for all $\vec{x} \in \mathbb{R}^n$,*
2) *$-\frac{1}{2} \leqslant h\vec{x} < 0$ for all $\vec{x} \in \Phi$,*
3) *$h\vec{x} \geqslant 1$ for all $\vec{x} \in F$.*

Proof. Let $c = \inf\{\sum\limits_{i=1}^{n} (x_i - x_{0i})^2 : (x_1, x_2, \dots, x_n) \in F, (x_{01}, x_{02}, \dots, x_{0n}) \in \Phi\}$. It is clear that $c > 0$ since the set Φ is compact, the set F is closed and $\Phi \cap F = \varnothing$ (of course, we assume that $\Phi \neq \varnothing$ and $F \neq \varnothing$).

For each point $\vec{x}_0 = (x_{01}, x_{02}, \dots, x_{0n}) \in \Phi$ let us define a polynomial $h_{\vec{x}_0}$ by the formula $h_{\vec{x}_0}\vec{x} = \frac{3}{2c} \cdot \sum\limits_{i=1}^{n} (x_i - x_{0i})^2 - \frac{1}{2}$ for $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, and let $U\vec{x}_0 = h_{\vec{x}_0}^{-1}[-\frac{1}{2}, 0) \subseteq \mathbb{R}^n$. These neighborhoods form an open covering of the compactum Φ. Let us choose a finite subcovering. Let it be formed by sets $U_1, U_2, \dots, U_m$, and let $h_1, h_2, \dots, h_m$ be the corresponding polynomials. It is clear that for all $i = 1, 2, \dots, m$ the conditions $h_i\vec{x} \geqslant -\frac{1}{2}$ for $\vec{x} \in \mathbb{R}^n$, $-\frac{1}{2} \leqslant h_i\vec{x} < 0$ for $\vec{x} \in \Phi \cap U_i$ and $h_i\vec{x} \geqslant 1$ for $\vec{x} \in F$ are satisfied.

If $m = 1$ then our Lemma is proved. Let us suppose that $m > 1$ and show that the number m can be made smaller.

Let us denote $U_{1,2} = U_1 \cup U_2$, $M = \sup\{\max\{h_1\vec{x}, h_2\vec{x}, 0\} : \vec{x} \in \Phi \cap U_{1,2}\}$ ($0 \leqslant M \in \mathbb{R}$ since Φ is compact and the polynomials h_1 and h_2 are continuous), and let $h'_{1,2}$ be a polynomial defined by the equality $h'_{1,2}\vec{x} = 2 \cdot h_1\vec{x} \cdot h_2\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. This polynomial has the following properties: $-M \leqslant h'_{1,2}\vec{x} \leqslant \frac{1}{2}$ if $\vec{x} \in \Phi \cap U_{1,2}$ and $h'_{1,2}\vec{x} \geqslant 2$ if $\vec{x} \in F$.

Let us choose a number $k \in \mathbb{N}$ such that $\left(\frac{2M+1}{2M+2}\right)^{2k} < \frac{1}{2}$. Since $\frac{2M+1}{2M+2} \cdot \frac{M+2}{M+1} \geqslant 1$, we have $\left(\frac{M+2}{M+1}\right)^{2k} > 2 > \frac{3}{2}$. Therefore the polynomial $h_{1,2}$, defined by the equality $h_{1,2}\vec{x} = \left(\frac{h'_{1,2}\vec{x}+M}{M+1}\right)^{2k} - \frac{1}{2}$ for all $\vec{x} \in \mathbb{R}^n$, satisfies the conditions $h_{1,2}\vec{x} \geqslant -\frac{1}{2}$ for $\vec{x} \in \mathbb{R}^n$, $-\frac{1}{2} \leqslant h_{1,2}\vec{x} < 0$ for $\vec{x} \in \Phi \cap U_{1,2}$ and $h_{1,2}\vec{x} \geqslant 1$ for $\vec{x} \in F$.

In consequence, we have the covering $\{U_{1,2}, U_3, \dots, U_m\}$ consisting of $m-1$ elements and the corresponding polynomials $h_{1,2}$, h_3, $\dots$, h_m. Repeating these reasonings we shall get the required polynomial. $\square$

6.9. **Lemma.** *For each point $z_0 \in v_{\mathcal{C}}X$ and each neighborhood $Uz_0 \subseteq v_{\mathcal{C}}X$ there exists $\alpha \in \mathfrak{A}$* (see 6.6) *such that $f_{\mathcal{C}}z_0 \in O_\alpha$ and*

1) $\bar{g}_\alpha z \geqslant -\frac{1}{2}$ *for all $z \in f_{\mathcal{C}}^{-1}O_\alpha$,*
2) $\bar{g}_\alpha z_0 = -\frac{1}{2}$,
3) $\bar{g}_\alpha z \geqslant 1$ *for all $z \in f_{\mathcal{C}}^{-1}O_\alpha \setminus Uz_0$.*

Proof. By the definition of the topology of the space $Y_{\mathfrak{A}}$, there are a finite set $\mathfrak{B} \subseteq \mathfrak{A}$ and neighborhoods $Uf_{\mathcal{C}}z_0 \subseteq Y$ and $U\bar{g}_\alpha z_0 \subseteq Z_\alpha = \mathbb{R}$, $\alpha \in \mathfrak{B}$, such that $z_0 \in f_{\mathcal{C}}^{-1}Uf_{\mathcal{C}}z_0 \cap \bigcap\{\bar{g}_\alpha^{-1}U\bar{g}_\alpha z_0 : \alpha \in \mathfrak{B}\} \subseteq Uz_0$. Let $O = Uf_{\mathcal{C}}z_0 \cap \bigcap\{O_\alpha : \alpha \in \mathfrak{B}\}$, $G = \bigcup\{G_\alpha \cap O : \alpha \in \mathfrak{B}\}$. Since $\mathcal{C}$ is a sheaf we can suppose that $O_\alpha = O$ and $G_\alpha = G$ for all $\alpha \in \mathfrak{B}$ (for a simplification of notations).

Let $\mathfrak{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, and let $\bar{g} = \Delta\{\bar{g}_\alpha : \alpha \in \mathfrak{B}\}\colon f_{\mathcal{C}}^{-1}O \to \mathbb{R}^n$ be the diagonal mapping (see [51], §2.3). Then $\bar{g}z_0 \notin F = [\bar{g}(f_{\mathcal{C}}^{-1}O \setminus Uz_0)]_{\mathbb{R}^n}$. Let us denote $c = \inf\{\sum\limits_{i=1}^{n}(x_i - \bar{g}_{\alpha_i}z_0)^2 : (x_1, x_2, \dots, x_n) \in F\}$. Obviously, $c > 0$. Since $C(O,G)$ is an algebra containing all constants (see the items 5.20 and 6.6), it contains the function $\bar{g}_0 : f_{\mathcal{C}}^{-1}O \to \mathbb{R}$ defined by the equality $\bar{g}_0 z = \frac{3}{2c} \cdot \sum\limits_{i=1}^{n}(\bar{g}_{\alpha_i}z - \bar{g}_{\alpha_i}z_0)^2 - \frac{1}{2}$ for $z \in f_{\mathcal{C}}^{-1}O$. Therefore there exists $\alpha \in \mathfrak{A}$ such that $O_\alpha = O$, $G_\alpha = G$ and $\bar{g}_\alpha = \bar{g}_0$. This α is the one we were looking for. $\square$

6.10. **Lemma.** *Let $y \in Y$, $\Phi \subseteq f_{\mathcal{C}}^{-1}y$ be a compact subset, $U\Phi \subseteq v_{\mathcal{C}}X$ be a neighborhood of the set Φ. Then there exists $\alpha \in \mathfrak{A}$* (see 6.6) *such that $y \in O_\alpha$ and*

1) $\bar{g}_\alpha z \geqslant -\frac{1}{2}$ *for all $z \in f_{\mathcal{C}}^{-1}O_\alpha$,*
2) $-\frac{1}{2} \leqslant \bar{g}_\alpha z < 0$ *for all $z \in \Phi$,*
3) $\bar{g}_\alpha z \geqslant 1$ *for all $z \in f_{\mathcal{C}}^{-1}O_\alpha \setminus U\Phi$.*

Proof. Using Lemma 6.9 we can find for each $z \in \Phi$ a function $\bar{g}_z \in C(O_z, G_z)$ for some $(O_z, G_z) \in T_{\mathfrak{a}}$ (see 6.6) such that $f_{\mathcal{C}}z \in O_z$, $\bar{g}_z z = -\frac{1}{2}$, $\bar{g}_z z' \geqslant 1$ for all $z' \in f_{\mathcal{C}}^{-1}O_z \setminus U\Phi$ and $\bar{g}'_z \geqslant -\frac{1}{2}$ for all $z' \in f_{\mathcal{C}}^{-1}O_z$. Let $Uz = \bar{g}_z^{-1}[-\frac{1}{2}, 0)$. Then $\{Uz : z \in \Phi\}$ is an open covering of the compact set Φ. Let $\{Uz_1, Uz_2, \dots, Uz_n\}$ be its finite subcovering, $O = \bigcap\{O_{z_i} : i = 1, 2, \dots, n\}$, $G = \bigcup\{G_{z_i} \cap O : i = 1, 2, \dots, n\}$. Then $g_i = g_{z_i}|_{f_{\mathcal{C}}^{-1}O} \in C(O,G)$ for $1 \leqslant i \leqslant n$.

Let $\bar{g} = \Delta\{\bar{g}_i : i = 1, 2, \dots, n\}\colon f_{\mathcal{C}}^{-1}O \to \mathbb{R}^n$ be the diagonal mapping. Then $\bar{g}\Phi$ is a compact subset of $\mathbb{R}^n$, $F = [\bar{g}(f_{\mathcal{C}}^{-1}O \setminus U\Phi)]_{\mathbb{R}^n}$ is a closed subset and $F \cap \bar{g}\Phi = \varnothing$. By Lemma 6.8 there exists a polynomial $h\colon \mathbb{R}^n \to \mathbb{R}$ such that $-\frac{1}{2} \leqslant h\vec{x} < 0$ for all $\vec{x} \in \bar{g}\Phi$, $h\vec{x} \geqslant 1$ for all $\vec{x} \in F$ and $h\vec{x} \geqslant -\frac{1}{2}$ for all $\vec{x} \in \mathbb{R}^n$. Then the function $h\bar{g}$ belongs to $C(O,G)$ since $C(O,G)$ is an algebra containing all constants (see 5.20), that is, there is $\alpha \in \mathfrak{A}$ such that $O_\alpha = O$, $G_\alpha = G$ and $\bar{g}_\alpha = h\bar{g}$. $\square$

6.11. **Lemma.** *Let $\mathcal{C}_v$ be a sheaf constructed using the $\mathfrak{Ta}$-bicompactification $f_{\mathcal{C}}\colon v_{\mathcal{C}} \xrightarrow{\text{onto}} Y$ as in the item 6.2. Then for every $(O,G) \in T_{\mathfrak{a}}$ each couple $(g, \tilde{g}) \in \mathcal{C}_v(O,G)$ is $\mathcal{C}$-separated.*

Proof. Let $(O,G) \in T_{\mathfrak{a}}$, $(g_0, \tilde{g}_0) \in \mathcal{C}_v(O,G)$, $\bar{g} \in C(p_O)$ be a function satisfying the conditions (4), $a, b \in \mathbb{R}$, $a < b$, $y \in O$.

Let $F_0 = \{z \in f_{\mathcal{C}}^{-1}O : \bar{g}_0 z \leqslant a\}$ and $F_1 = \{z \in f_{\mathcal{C}}^{-1}O : \bar{g}_0 z \geqslant b\}$. The sets F_0 and F_1 are closed in $f_{\mathcal{C}}^{-1}O$ and $F_0 \cap F_1 = \varnothing$. Since the mapping $f_{\mathcal{C}}$ is perfect, the sets $\Phi_0 = F_0 \cap f_{\mathcal{C}}^{-1}y$ and $\Phi_1 = F_1 \cap f_{\mathcal{C}}^{-1}y$ are compact.

By Lemma 6.10 there exists $\alpha_0 \in \mathfrak{A}$ such that $y \in O_{\alpha_0} \subseteq O$, $-\frac{1}{2} \leqslant \bar{g}_{\alpha_0} z < < 0$ for $z \in \Phi_0$, $\bar{g}_{\alpha_0} z \geqslant -\frac{1}{2}$ for $z \in f_{\mathcal{C}}^{-1}O_{\alpha_0}$ and $\bar{g}_{\alpha_0} z \geqslant 1$ for $z \in F_1 \cap f_{\mathcal{C}}^{-1}O_{\alpha_0}$. Analogously there exists $\alpha_1 \in \mathfrak{A}$ such that $y \in O_{\alpha_1} \subseteq O$, $-\frac{1}{2} \leqslant \bar{g}_{\alpha_1} z < 0$ for $z \in \Phi_1$, $\bar{g}_{\alpha_1} z \geqslant -\frac{1}{2}$ for $z \in f_{\mathcal{C}}^{-1}O_{\alpha_1}$ and $\bar{g}_{\alpha_1} z \geqslant 1$ for $z \in F_0 \cap f_{\mathcal{C}}^{-1}O_{\alpha_1}$. Since C is a sheaf of algebras (see 6.6), there exists an element $\alpha \in \mathfrak{A}$ such that $O_\alpha = O_{\alpha_0} \cap O_{\alpha_1}$, $G_\alpha = (G_{\alpha_0} \cup G_{\alpha_1}) \cap O_\alpha$ and $\bar{g}_\alpha = \frac{1}{2}(\bar{g}_{\alpha_0}|_{O_\alpha} + 1 - \bar{g}_{\alpha_1}|_{O_\alpha})$.

Let $U_0 = \{z \in f_{\mathcal{C}}^{-1}O_\alpha : \bar{g}_{\alpha_0} z < 0\}$, $U_1 = \{z \in f_{\mathcal{C}}^{-1}O_\alpha : \bar{g}_{\alpha_1} z < 0\}$, $U_2 = = f_{\mathcal{C}}^{-1}O_\alpha \setminus (F_0 \cup F_1)$, $O' = f_{\mathcal{C}}^{\#}(U_0 \cup U_1 \cup U_2)$, $G' = O' \cap G_\alpha$, $g = g_\alpha|_{O'}$, $\tilde{g} = \tilde{g}_\alpha|_{f^{-1}O'}$, $a' = 0$, $b' = 1$. It is easily seen that all conditions of Definition 5.33 are satisfied, therefore the couple $(g_0, \tilde{g}_0)$ is $\mathcal{C}$-separated. □

6.12. **Corollary.** *Under the assumptions of Lemma 6.11 we have* $\mathcal{C}_v = \bar{\mathcal{C}}$ (see 5.36). *Particularly, if the sheaf* $\mathcal{C}$ *is saturated then* $\mathcal{C}_v = \mathcal{C}$.

6.13. **Lemma.** *If* $f_v : v_f X \xrightarrow{\text{onto}} Y$ *is a* $\mathfrak{T}\mathfrak{a}$*-bicompactification of the mapping* f, $\mathcal{C}_v$ *is a sheaf constructed using* f_v *as in the item 6.2, and* $f_{\mathcal{C}} : v_{\mathcal{C}} X \xrightarrow{\text{onto}} Y$ *is a* $\mathfrak{T}\mathfrak{a}$*-bicompactification constructed using* $\mathcal{C}_v$ *as in the item 6.6, then the* $\mathfrak{T}\mathfrak{a}$*-bicompactifications* f_v *and* $f_{\mathcal{C}}$ *are equivalent.*

Proof. Let us repeat the construction 6.6 using the space $v_f X$ instead of X and the sheaf $\mathcal{C}'_v$ instead of $\mathcal{C}_v$, where

$$\mathcal{C}'_v(O,G) = \{(g, \bar{g}) : (g, \tilde{g}) \in \mathcal{C}_v(O,G) \text{ and } \bar{g} \in C(p_O) \text{ is a function satisfying the condition (4)}\}$$

for all $(O,G) \in T_{\mathfrak{a}}$ (the sheaf $\mathcal{C}'_v$ is naturally isomorphic to the sheaf $\mathcal{C}_v$). Due to Lemma 6.5 we obtain an embedding $i_v : v_f X \to v_{\mathcal{C}} X$ which is a homeomorphism onto $v_{\mathcal{C}} X$, because the mapping f_v is perfect, and the mapping $f_{\mathcal{C}}$ is separable and irreducible modulo X (see Lemma 8 of the paper [43]). □

6.14. **Theorem.** *There exists a one-to-one correspondence between the set of all* $\mathfrak{T}\mathfrak{a}$*-bicompactifications of the mapping* f *and the set of all dismembering saturated subsheaves of the sheaf* $\mathcal{C}_{\mathfrak{a}}$ *which preserves the partial order.*

Proof. The existence of a one-to-one correspondence follows from Corollary 6.12 and Lemma 6.13. If $f_v : v_f X \xrightarrow{\text{onto}} Y$ and $f_w : w_f X \xrightarrow{\text{onto}} Y$ are $\mathfrak{T}\mathfrak{a}$-bicompactifications of the mapping f, $f_v \geqslant f_w$, and $\mathcal{C}_v, \mathcal{C}_w \subseteq \mathcal{C}_{\mathfrak{a}}$ are the corresponding subsheaves, then $\mathcal{C}_v \supseteq \mathcal{C}_w$ by Corollary 5.16 and the construction 6.2. If $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}_{\mathfrak{a}}$ are dismembering subsheaves, $\mathcal{C}_1 \supseteq \mathcal{C}_2$, and $f_{\mathcal{C}_1} : v_{\mathcal{C}_1} X \xrightarrow{\text{onto}} Y$, $f_{\mathcal{C}_2} : v_{\mathcal{C}_2} X \xrightarrow{\text{onto}} Y$ are the corresponding $\mathfrak{T}\mathfrak{a}$-bicompactifications, then the inequality $f_{\mathcal{C}_1} \geqslant f_{\mathcal{C}_2}$ can be proved as Lemma 6.13. □

6.15. **Definition.** Subsheaves $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}_{\mathfrak{a}}$ will be called *equivalent* if $\bar{\mathcal{C}}_1 = \bar{\mathcal{C}}_2$ (see 5.36).

6.16. **Proposition.** *Let* $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}_{\mathfrak{a}}$ *be dismembering subsheaves and let* $f_{\mathcal{C}_1} : v_{\mathcal{C}_1} X \xrightarrow{\text{onto}} Y$, $f_{\mathcal{C}_2} : v_{\mathcal{C}_2} X \xrightarrow{\text{onto}} Y$ *be the corresponding* $\mathfrak{T}\mathfrak{a}$*-bicompactifications. The sheaves* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are equivalent iff the* $\mathfrak{T}\mathfrak{a}$*-bicompactifications* $f_{\mathcal{C}_1}$ *and* $f_{\mathcal{C}_2}$ *are equivalent.*

Proof follows from Corollary 6.12 and Lemma 6.13. □

6.17. *Remark.* Of course, we can use subsheaves of the sheaf $\hat{\mathcal{C}}_{\mathfrak{a}}$ (see the items 5.23, 5.28 and 5.35) instead of subsheaves of the sheaf $\mathcal{C}_{\mathfrak{a}}$, but the sheaf $\hat{\mathcal{C}}_{\mathfrak{a}}$ is not complete (see 5.32). This defect can make some difficulties.

This defect does not occur if the family $\mathfrak{a}$ has the largest representative in each set $O \in T$ (see Definition 5.24 and Proposition 5.25). For example, such situation holds for Tychonoff mappings in the sense of paper [34] (see the item 1.10), when $\mathfrak{a}$ is the family of all locally closed subsets of the space Y.

§ 7. Maximal ideals of sheaves

7.1. We shall assume that the notions of an ideal and a maximal ideal of an algebra are known (see, for example, [51], the item 3.12.21, or [28], Chapter II, §7 (4)).

We shall consider a fixed perfect mapping $f\colon X \xrightarrow{\text{onto}} Y$ with the property $\mathfrak{T}\mathfrak{a}$. For each couple $(O,G) \in T_{\mathfrak{a}}$ let $f_O = f|_{f^{-1}O}\colon f^{-1}O \xrightarrow{\text{onto}} O$ and

$$C_{\mathfrak{a}}(O,G) = \{\bar{g} \in C(f_O) : \text{ there is a function } g\colon O \to \mathbb{R} \text{ such that } (g,\bar{g}) \in \mathcal{C}_{\mathfrak{a}}(O,G)\}.$$

The map $C_{\mathfrak{a}}$ is a sheaf which is naturally isomorphic to the sheaf $\mathcal{C}_{\mathfrak{a}}$ (see 5.13).

7.2. Let $C \subseteq C_{\mathfrak{a}}$ be a dismembering subsheaf. For each point $y \in Y$ and each couple $(O,G) \in T_y$ (see 5.2 c)) let $I_y(O,G) = \{\bar{g} \in C(O,G) : n_y\bar{g} = 0\}$.

It is easily seen that $I_y(O,G)$ is a closed ideal of the algebra $C(O,G)$ for each couple $(O,G) \in T_y$. Moreover, if $(O_1,G_1),(O_2,G_2) \in T_y$, $(O_1,G_1) \subseteq (O_2,G_2)$ and $h\colon C(O_2,G_2) \to C(O_1,G_1)$ is the restriction homomorphism (see 5.14), then $I_y(O_2,G_2) = h^{-1}I_y(O_1,G_1)$.

7.3. **Definition.** Let $y \in Y$ be any point. A map M, which assign to each couple $(O,G) \in T_y$ a closed maximal ideal $M(O,G) \subseteq C(O,G)$ of the algebra $CO,G)$, will be called *a closed maximal y-ideal of the sheaf C*, if the following conditions are fulfilled:

1) if $(O,G) \in T_y$ then $I_y(O,G) \subseteq M(O,G)$;
2) if $(O_1,G_1),(O_2,G_2) \in T_y$ and $(O_1,G_1) \subseteq (O_2,G_2)$ then $M(O_2,G_2) = h^{-1}M(O_1,G_1)$, where h is the restriction homomorphism (see 7.2).

Let us denote by $\mathfrak{M}_y$ the set of all closed maximal y-ideals of the sheaf C for $y \in Y$.

7.4. **Lemma.** *Let $y \in Y$, $x \in f^{-1}y$, $M_x(O,G) = \{\bar{g} \in C(O,G) : \bar{g}x = 0\}$ for all $(O,G) \in T_y$. Then the map M_x is a closed maximal y-ideal of the sheaf C.*

Proof. For each couple $(O,G) \in T_y$ let us define a homomorphism $\varphi_{OG}\colon C(O,G) \to \mathbb{R}$ by the formula $\varphi_{OG}\bar{g} = \bar{g}x$ for all $\bar{g} \in C(O,G)$. This homomorphism is continuous and "onto", since the algebra $C(O,G)$ contains all constants (see 5.20), and $|\bar{g}x| \leqslant n_y\bar{g}$ for all $\bar{g} \in C(O,G)$. It is easily seen that $M_x(O,G) = \varphi_{OG}^{-1}0$, therefore $M_x(O,G)$ is a closed ideal of the algebra $C(O,G)$. This ideal is maximal since the algebra $\mathbb{R}$ has no ideals except $\{0\}$. The conditions 1) and 2) of Definition 7.3 are satisfied obviously. $\square$

7.5. **Lemma.** *For each closed maximal y-ideal $M \in \mathfrak{M}_y$, $y \in Y$, there exists a point $x \in f^{-1}y$ such that $M = M_x$.*

Proof. Let us suppose that for every point $x \in f^{-1}y$ there are a couple $(O_x,G_x) \in T_y$ and a function $\bar{g}_x \in M(O_x,G_x)$ such that $\bar{g}_x x \neq 0$; let us denote $Ux = \{x' \in f^{-1}O_x : |\bar{g}_x x'| > \frac{1}{2}|\bar{g}_x x|\}$. Then the set $\{Ux : x \in f^{-1}y\}$ is an open covering of the compact set $f^{-1}y$. Let $\{Ux_i : i = 1,2,\dots,n\}$ be its finite subcovering; the set

$$O = \Big(f^{\#}\bigcup\{Ux_i : i = 1,2,\dots,n\}\Big) \cap \Big(\bigcap\{O_{x_i} : i = 1,2,\dots,n\}\Big)$$

is open and $y \in O \subseteq \bigcap\{O_{x_i} : i = 1, 2, \dots, n\}$ since the mapping f is closed. Let $G = \bigcup\{G_{x_i} \cap O : i = 1, 2, \dots, n\}$, $\varepsilon_0 = \min\{\frac{1}{4}(\bar{g}_{x_i} x_i)^2 : i = 1, 2, \dots, n\} > > 0$ and $\bar{g}_i = \bar{g}_{x_i}|_{f^{-1}O}$ for $i = 1, 2, \dots, n$; then $(O, G) \in T_y$ and $\bar{g}_i \in M(O, G)$ for all $i = 1, 2, \dots, n$ by the condition 2) of Definition 7.3. Therefore the function $\bar{g}\colon f^{-1}O \to \mathbb{R}$, defined by the formula $\bar{g}x = \sum_{i=1}^{n} (\bar{g}_i x)^2$ for all $x \in f^{-1}O$, belongs to the ideal $M(O, G)$ and satisfies the condition $\bar{g}x > \varepsilon_0 > 0$ for all $x \in f^{-1}O$.

Let us denote by $\bar{g}_e$ the function such that $\bar{g}_e x = 1$ for all $x \in f^{-1}O$. Of course, $\bar{g}_e \in C(O, G)$ (see 5.20). The function $\bar{g}_e$ is the unit of the algebra $C(O, G)$. We shall prove that $\bar{g}_e \in M(O, G)$.

Let $V^C_{\varepsilon,A}\bar{g}_e = V_{\varepsilon,A}\bar{g}_e \cap C(O, G) \subseteq C(O, G)$, where $A \subseteq O$ is a finite set and $\varepsilon > 0$, be an arbitrary neighborhood of the function $\bar{g}_e$ (see 4.7), $B = \varepsilon_0 + \max\{n_{y'}\bar{g} : y' \in \in A\}$. For each $n \in \mathbb{R}$ the function[10] $\bar{g}_n = \frac{2}{B} \cdot \sum_{i=0}^{n-1} \left(\bar{g}_e - \frac{2\bar{g}}{B}\right)^i$ belongs to $C(O, G)$ and $\bar{g}_e - \bar{g} \cdot \bar{g}_n = \left(\bar{g}_e - \frac{2\bar{g}}{B}\right)^n$.

Since $\varepsilon_0 \leqslant \bar{g}x \leqslant B - \varepsilon_0$ for all $x \in f^{-1}A$, we have $\left|1 - \frac{2\bar{g}x}{B}\right| \leqslant 1 - \frac{2\varepsilon_0}{B} < 1$ for $x \in f^{-1}A$. There is a number $n_0 \in \mathbb{R}$ such that $\left(1 - \frac{2\varepsilon_0}{B}\right)^{n_0} < \varepsilon$. Then for every $y' \in A$ we have

$$n_{y'}(\bar{g}_e - \bar{g} \cdot \bar{g}_{n_0}) = n_{y'}\left(\left(\bar{g}_e - \frac{2\bar{g}}{B}\right)^{n_0}\right) \leqslant \left(1 - \frac{2\varepsilon_0}{B}\right)^{n_0} < \varepsilon,$$

that is, $\bar{g} \cdot \bar{g}_{n_0} \in V^C_{\varepsilon,A}\bar{g}_e$. Hence, $V^C_{\varepsilon,A}\bar{g}_e \cap M(O, G) \neq \varnothing$ because $\bar{g} \cdot \bar{g}_{n_0} \in M(O, G)$ by the definition of an ideal. Therefore $\bar{g}_e \in [M(O, G)]_{C(O,G)} = M(O, G)$.

The latter inclusion is impossible because $M(O, G)$ is an ideal. Hence, there exists a point $x \in f^{-1}y$ such that $\bar{g}x = 0$ for all $\bar{g} \in M(O, G)$. Obviously, $M(O, G) \subseteq M_x(O, G)$, but really $M(O, G) = M_x(O, G)$ since $M(O, G)$ is a maximal ideal. □

7.6. **Theorem.** *For each point $y \in Y$ there exists a one-to-one map $\phi_y\colon f^{-1}y \xrightarrow{\text{onto}} \mathfrak{M}_y$.*

Proof. Due to Lemma 7.4 we can define the map ϕ_y by the formula $\phi_y x = M_x$ for all $x \in f^{-1}y$. By Lemma 7.5 we have $\phi_y f^{-1}y = \mathfrak{M}_y$. Let us note that if $x_1, x_2 \in f^{-1}y$ and $x_1 \neq x_2$ then $M_{x_1} \neq M_{x_2}$ because the sheaf C is dismembering and, hence, there exist a couple $(O, G) \in T_y$ and a function $\bar{g} \in C(O.G)$ such that $\bar{g}x_1 = 0$ and $\bar{g}x_2 \neq 0$ (see 5.20); then $\bar{g} \in M_{x_1}(O, G)$ and $\bar{g} \notin M_{x_2}(O, G)$. □

7.7. Thanks to Theorem 7.6 for each $y \in Y$ we can define a Hausdorff compact topology on the set $\mathfrak{M}_y$ such that the map ϕ_y is a homeomorphism.

Let $X' = \dot{\bigcup}\{\mathfrak{M}_y : y \in Y\}$, where the symbol "$\dot{\bigcup}$" denotes the disjunctive union, and let $f'\colon X' \xrightarrow{\text{onto}} Y$ be the map defined by the formula $f'M = y$ for all $M \in \mathfrak{M}_y$ and $y \in Y$. For each couple $(O, G) \in T_{\mathfrak{a}}$ and each function $\bar{g} \in C(O, G)$ we can define a function $\hat{g}\colon f'^{-1}O \to \mathbb{R}$ by the equality $\hat{g}M = \bar{g}\phi^{-1}_{f'M}M$ for $M \in f'^{-1}O$. Let us equip X' with the smallest topology in which the map f' and all functions $\hat{g}$, where $\bar{g} \in \bigcup\{C(O, G) : (O, G) \in T_{\mathfrak{a}}\}$, are continuous.

7.8. **Theorem.** *The map $\phi\colon X \xrightarrow{\text{onto}} X'$, defined by the equality $\phi x = \phi_{fx}x$ for all $x \in X$, is a homeomorphism satisfying the condition $f'\phi = f$.*

Proof. The map ϕ is one-to-one by the construction. It is continuous since the map f and all functions $\bar{g} \in \bigcup\{C(O, G) : (O, G) \in T\mathfrak{a}\}$ are continuous. It is easily seen that the mapping f' is separable because for each $M, M' \in X'$ such that

[10] We assume that $\left(\bar{g}_e - \frac{2\bar{g}}{B}\right)^0 = \bar{g}_e$.

$M \neq M'$ and $f'M = f'M' = y$ there are a couple $(O, G) \in T_y$ and a function $\bar{g} \in \in C(O, G)$ such that $\hat{g}M \neq \hat{g}M'$, and the space $\mathbb{R}$ is Hausdorff. Therefore by Lemma 8 of the paper [43] the mapping ϕ is perfect, that is, ϕ is a homeomorphism. $\square$

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[12]The translation of this article into English contains significant errors which do not exist in the Russian text. For the correction it is necessary

1) to omit the word "open" in the second line of §1;
2) to replace the word "continuous" by "irreducible" in Proposition 2;
3) to replace the word "compact" by "Hausdorff compact" in Corollaries 4 and 9 (the Russian term "бикомпакт" means "Hausdorff compact space").

Vladimir M. Ulyanov
Independent University of Russian Academy of Education,
Novomoskovsk's branch
Berezhnogo st., 12,
301650, Novomoskovsk, Russia
E-mail address: umax@newmsk.tula.net
URL: http://www.someoneltd.boom.ru/