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ON CONDITIONS FOR SLLN FOR MARTINGALES WITH IDENTICALLY DISTRIBUTED INCREMENTS

A. I. SAKHANENKO

ABSTRACT. For any random variable X with $\mathbf{E}\big[|X|\log(1+|X|)\big]=\infty$ and $\mathbf{E}X=0$ we construct a sequence $\{X_n : n\geq 1\}$ of martingale differences which are identically distributed with X and such that the strong law of large numbers does not hold.

1. INTRODUCTION

Let $\{X_n : n\geq 1\}$ be a sequence of random variables and let $\{\mathcal{F}_n : n\geq 1\}$ be an increasing sequence of σ-fields with X_n measurable with respect to $\mathcal{F}_n$ for each n. The sequence $\{X_n,\mathcal{F}_n : n\geq 1\}$ is said to be *a sequence of identically distributed martingale differences* if the random variables $\{X_n\}$ are identically distributed and

$$\forall n>1 \qquad \mathbf{E}[X_n|\mathcal{F}_{n-1}]=0=\mathbf{E}X_1 \qquad \text{a.s.} \tag{1}$$

The following assertion is a partial case of Theorem 2.19 in [1].

Theorem 1. *Let $\{X_n,\mathcal{F}_n : n\geq 1\}$ be a sequence of identically distributed martingale differences such that*

$$\mathbf{E}\big[|X_1|\log(1+|X_1|)\big]<\infty. \tag{2}$$

Then

$$n^{-1}\sum_{i=1}^{n}X_i\to 0 \qquad a.s. \qquad as \quad n\to\infty. \tag{3}$$

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On the other hand, if random variables $\{X_n : n \ge 1\}$ are independent and identically distributed, then, as it follows from the classical (Kolmogorov) strong law of large numbers, convergence (3) holds if and only if $\mathbf{E}X_1 = 0$.

Thus, the question arises if condition (2) is necessary in the general case (see also the remark on page 39 in [1]). We give a positive answer for this question in the following theorem which is the main result of the paper.

Theorem 2. *Let a random variable X_1 satisfy the following conditions*

$$\mathbf{E}X_1 = 0 \qquad \text{and} \qquad \mathbf{E}\big[X_1^+ \log(1 + X_1^+)\big] = \infty. \tag{4}$$

Then on some probability space we may define a sequence $\{X_n, \mathcal{F}_n : n \ge 1\}$ of identically distributed martingale differences such that

$$\liminf_{n\to\infty}\ n^{-1}\sum_{i=1}^{n} X_i \le -c < 0 \qquad a.s., \tag{5}$$

where $0 < c = \mathbf{E}X_1^+ = \mathbf{E}|X_1|/2 < \infty$.

The rest of the paper is devoted to the proof of Theorem 2.

2. Construction

First we introduce the following notations

$$a(x) := \mathbf{E}[X_1 : X_1 > x], \qquad b_N = \mathbf{E}[-X_1 : -N \le X_1 < 0]. \tag{6}$$

Lemma 1. *If conditions (4) hold then there exists an integer N such that*

$$b_N \ge c\mathbf{P}(X_1 < 0) + c(c+1)/N \qquad \text{and} \qquad a(N) \le c/2, \tag{7}$$

with $c = \mathbf{E}X_1^+$. In particular,

$$\forall\ m \ge 1 \qquad 0 < p_m := \mathbf{P}(X_1 > N^m) + qa_m \le a_m/c_N < 1/2, \tag{8}$$

where

$$a_n := a(N^m), \qquad q := \mathbf{P}(-N \le X_1 < 0)/b_N, \qquad c_N := c/(1 - 1/N). \tag{9}$$

Now for $m \ge 1$ we define monotone functions

$$G_m(x) := \mathbf{P}(X_1 \le x,\, X_1 > N^m) + a_m\mathbf{P}(X_1 \le x,\, -N \le X_1 < 0)/b_N. \tag{10}$$

Note that

$$p_m := G_m(\infty) \qquad \text{and} \qquad G_m(x) \le \mathbf{P}(X_1 \le x)\ \ \forall\ x.$$

In what follows we are going to introduce a set

$$\{X_1, \eta_m, U_n, V_n :\ m \ge 1, n \ge 2\},$$

of mutually independent random variables with specially chosen distributions. First we suppose that each of the random variables $\{\eta_m : m \ge 1\}$ has only two values with the following probabilities

$$\mathbf{P}(\eta_m = 1) = p_m \qquad \text{and} \qquad \mathbf{P}(\eta_m = 0) = 1 - p_m. \tag{11}$$

After that we introduce random variables U_n and V_n with the following distributions

$$\begin{gathered} \forall\ n \ge 2 \qquad \mathbf{P}(U_n \le x) := G_{k(n)}(x)\big/p_{k(n)}, \\ \mathbf{P}(V_n \le x) := \big(\mathbf{P}(X_n \le x) - G_{k(n)}(x)\big)\big/(1 - p_{k(n)}), \end{gathered} \tag{12}$$

where the integers $k(n) \geq 1$ are defined in the following way:

$$k(m) = m \geq 1 \qquad \text{if and only if} \qquad N^{m-1} < n \leq N^m. \tag{13}$$

Finally, we define the desirable random variables in the following way

$$X_n := \eta_{k(n)} U_n + (1 - \eta_{k(n)}) V_n \qquad \text{for all} \qquad n \geq 2. \tag{14}$$

For all $n \geq 1$ we denote by $\mathcal{F}_n$ the minimal σ-algebra generated by the following random variables:

$$\{X_i, \eta_{k(i+1)} : 1 \leq i \leq n\} = \{X_i, \eta_m : 1 \leq i \leq n, m \geq 1, N^{m-1} \leq n\}. \tag{15}$$

And let $\mathcal{F}_0$ be the trivial σ-algebra.

Lemma 2. *If conditions (4) hold then the sequence $\{X_n, \mathcal{F}_n : n \geq 1\}$, defined above, is a sequence of identically distributed martingale differences. Moreover, in this case for all $n \geq 2$*

$$\mathbf{P}(X_n \leq x | \mathcal{F}_{n-1}) = \eta_{k(n)} \mathbf{P}(U_n \leq x) + (1 - \eta_{k(n)}) \mathbf{P}(V_n \leq x). \tag{16}$$

3. Proof of Theorem 2

For $i \geq 1$ let us introduce random variables

$$Z_i := X_i I(|X_i| > i), \quad z_i := \mathbf{E}[Z_i | \mathcal{F}_{i-1}], \quad y_i := \mathbf{E}[X_i I(|X_i| \leq i) | \mathcal{F}_{i-1}]. \tag{17}$$

It was shown in [1], in the proof of Theorem 2.19 (see formula (2.20) in [1]), that for any sequence of identically distributed martingale differences the following convergence holds

$$\overline{Y}(n) := \frac{1}{n} \sum_{i=1}^{n} (X_i - y_i) \to 0 \qquad \text{a.s.} \qquad \text{as} \quad n \to \infty. \tag{18}$$

It is clear from (17) and (1) that for all $i \geq 1$

$$X_i = X_i I(|X_i| \leq i) + Z_i \quad \text{and} \quad y_i + z_i = \mathbf{E}[X_i | \mathcal{F}_{i-1}] = 0.$$

Hence

$$\overline{X}(n) := \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{Y}(n) - \frac{Z}{n} - \frac{1}{n} \sum_{i=N+1}^{n} z_i \quad \text{with} \quad Z := \sum_{i=1}^{N} z_i. \tag{19}$$

Later on in the paper we suppose that conditions (4) hold and that $\{X_n\}$ and $\{\mathcal{F}_n\}$ are the sequences constructed in Section 2.

Lemma 3. *Under assumptions of Lemma 3 for all $n > N$ we have*

$$-z_n \leq 2w_n - c_{\scriptscriptstyle N} \eta_{k(n)}, \quad \textit{where} \quad w_n := \mathbf{E}[-X_1 : X_1 < -n]. \tag{20}$$

It follows from (20) that

$$w_n \to 0 \quad \text{and} \quad \overline{W}_n := \frac{1}{n} \sum_{i=N+1}^{n} 2w_i \to 0 \qquad \text{as} \quad n \to \infty. \tag{21}$$

On the other hand, we have from (19), (20) and (21) that

$$\overline{X}(n) \leq \overline{Y}(n) - \frac{Z}{n} + \overline{W}(n) - \overline{U}(n) \quad \text{with} \quad \overline{U}(n) := \frac{1}{n} \sum_{i=N+1}^{n} c_{\scriptscriptstyle N} \eta_{k(i)}. \tag{22}$$

So, we obtain from (18), (21) and (22) that

$$\liminf_{n\to\infty} \overline{X}(n) \le \liminf_{n\to\infty} \big(-\overline{U}(n)\big) = -\limsup_{n\to\infty} \overline{U}(n) \le -\limsup_{m\to\infty} \overline{U}(N^m) \quad \text{a.s.} \tag{23}$$

Using (9) and (22) again we obtain for all $m > 1$ that

$$\overline{U}(N^m) = \sum_{i=N+1}^{N^m} \frac{c_N \eta_{k(i)}}{N^m} \ge \sum_{i=N^{m-1}+1}^{N^m} \frac{c_N \eta_m}{N^m} = (1-1/N)c_N \eta_m = c\eta_m.$$

But this fact together with (23) allows us to obtain the inequality

$$\liminf_{n\to\infty} \overline{X}(n) \le -\limsup_{m\to\infty} \overline{U}(N^m) \le -c \limsup_{m\to\infty} \eta_m \quad \text{a.s.} \tag{24}$$

Lemma 4. *Under assumptions of Lemma 3* $\limsup_{m\to\infty} \eta_m = 1$ *a.s.*

Thus, Lemma 4 together with (24) yields (5) with $c = \mathbf{E}X_1^+$.

4. Proofs of Lemmas

Proof of Lemma 1. Using definitions (6) and the first condition in (4) it immediately follows that

$$\begin{gathered} \lim_{N\to\infty} a(N) = 0, \qquad \lim_{N\to\infty} b_N = \mathbf{E}X_1^-, \qquad b_N \le \mathbf{E}X_1^- \quad \forall\, N > 0, \\ 0 = \mathbf{E}X_1 = \mathbf{E}X_1^+ - \mathbf{E}X_1^-, \qquad \mathbf{E}|X_1| = \mathbf{E}X_1^+ + \mathbf{E}X_1^- < \infty. \end{gathered} \tag{25}$$

But $\mathbf{E}X_1^+ > 0$ and $\mathbf{P}(X_1 > 0) > 0$ by the second condition in (4). Hence we have from (25) that

$$\mathbf{E}X_1^- = \mathbf{E}X_1^+ = \mathbf{E}|X_1|/2 = c > 0, \qquad \lim_{N\to\infty} b_N = c > c\mathbf{P}(X_1 < 0). \tag{26}$$

It is evident now from (26) and from the first convergence in (25) that conditions (7) are true for sufficiently large N.

Suppose now that conditions (7) hold. In this case, using definitions (6) and (9), we obtain that

$$\forall\, m \ge 1 \qquad a_m/c_N < a_m/c \le a_1/c = a(N)/c \le 1/2.$$

So, the last inequality in (8) is proved. The first inequality in (8) also holds because $\mathbf{P}(X_1 > x) > 0$ for all $x > 0$ by the second condition in (4). Note also that by Chebyshev's inequality with the first moment

$$\forall\, m \ge 1 \qquad 0 < \mathbf{P}(X_1 > N^m) \le a_m/N^m \le a_m/N. \tag{27}$$

To prove the central inequality in (8) we need to show that $\delta_m \le 0$ for

$$\delta_m := p_m - a_m/c_N = \mathbf{P}(X_1 > N^m) + qa_m - (1-1/N)a_m/c.$$

But we have from (27) that

$$\frac{c\delta_m}{a_m} = \frac{c}{N} + cq - 1 + \frac{1}{N} = \Big(\frac{cb_N}{N} + c\mathbf{P}(-N \le X_1 < 0) - b_N + \frac{b_N}{N}\Big)\Big/ b_N,$$

where we use definition (9) of q. But $b_N \le c$ as it follows from (25) and (26). Hence

$$\frac{cb_N\delta_m}{a_m} \le \frac{c^2}{N} + c\mathbf{P}(X_1 < 0) - b_N + \frac{c}{N} = c\mathbf{P}(X_1 < 0) + \frac{c(c+1)}{N} - b_N \le 0$$

by (7). Thus $\delta_m \le 0$ and we obtain (8) as a partial case of (7).

Proof of Lemma 2. Let $m \geq 1$ and $n \geq 2$. First, note that (16) may be derived immediately from (14) and (15). It follows from (10) that

$$E_m := \int x dG_m(x) = \mathbf{E}[X_1 : X_1 > N^m] + a_m \mathbf{E}[X_1 : -N \leq X_1 < 0]/b_{_N}.$$

Hence, $E_m = a_m + a_m(-b_{_N})/b_{_N} = 0$ by definitions (6) and (9). Now from (12) we have

$$\mathbf{E}U_n = E_{k(n)}/p_{k(n)} = 0, \qquad \mathbf{E}V_n = \big(\mathbf{E}X_n - E_{k(n)}(x)\big)/(1 - p_{k(n)}) = 0.$$

So, we obtain from (16) that

$$\mathbf{E}[X_n|\mathcal{F}_{n-1}] := \eta_{k(n)}\mathbf{E}U_n + (1 - \eta_{k(n)})\mathbf{E}V_n = 0.$$

Thus, $\{X_n, \mathcal{F}_n : n \geq 1\}$ is a sequence of martingale differences.

It follows immediately from (11) and (16) that

$$\mathbf{P}(X_n \leq x) = \mathbf{P}(\eta_{k(n)} = 1)\mathbf{P}(U_n \leq x) + \mathbf{P}(\eta_{k(n)} = 0)\mathbf{P}(V_n \leq x).$$

And now $\mathbf{P}(X_n \leq x) = \mathbf{P}(X_1 \leq x)$ by definitions (11) and (12). Hence, the random variables $\{X_n\}$ are identically distributed.

Proof of Lemma 3. Let $n > N$. From definitions (10) and (12) we have

$$u_n := \mathbf{E}[U_n : |U_n| > n] = \mathbf{E}[X_1 : X_1 > N^{k(n)}]/p_{k(i)} = a_{k(n)}/p_{k(n)},$$
$$v_n := \mathbf{E}[V_n : |V_n| > n] \geq \mathbf{E}[X_n : X_n < -n]/(1 - p_{k(n)}) = -w_n/(1 - p_{k(n)}).$$

It follows from (8) that

$$u_n \geq c_{_N}, \qquad v_n \geq -2w_n, \qquad \eta_{k(n)}w_n \geq 0.$$

Now from definition (17) and representation (16) we obtain

$$z_n = \mathbf{E}[X_n : |X_n| > n|\mathcal{F}_{n-1}] := \eta_{k(n)}u_n + (1 - \eta_{k(n)})v_n \geq \eta_{k(n)}c_{_N} - 2w_n.$$

So, Lemma 3 is proved.

Proof of Lemma 4. First note that

$$\sum_{m\geq 1} a_m = \sum_{m\geq 1}\sum_{j\geq m} A_j = \sum_{j\geq 1} jA_j \quad \text{with} \quad A_j := \mathbf{E}[X_1 : N^j < X_1 \leq N^{j+1}]. \tag{28}$$

On the other hand,

$$\begin{gathered} J_j := \mathbf{E}\big[X_1 \log(X_1/N) : N^j < X_1 \leq N^{j+1}\big] \leq jA_j \log N, \\ J := \mathbf{E}\big[X_1 \log(X_1/N) : X_1 > N\big] = \sum_{j\geq m} J_j, \\ \mathbf{E}\big[X_1^+ \log(1 + X_1^+)\big] \leq \mathbf{E}X_1^+ \log(1 + N) + J, \end{gathered} \tag{29}$$

since $\log(1+x) < \log(1+N) + \log(x/N)$ for $x > N$. So, we have from (28) and (29) that

$$\mathbf{E}\big[X_1^+ \log(1 + X_1^+)\big] \leq \Big(\mathbf{E}X_1^+ + \sum_{m\geq 1} a_m\Big) \log(1 + N).$$

Thus, $\sum_{m\geq 1} a_m = \infty$ if the second condition in (4) holds. But in this case we obtain from (8) that

$$\sum_{m\geq 1} \mathbf{P}(\eta_m = 1) = \sum_{m\geq 1} p_m \geq q \sum_{m\geq 1} a_m = \infty. \tag{30}$$

It follows immediately from (30) that, by the law of zero and one, independent events $\{\eta_m = 1\}$ occur infinitely often. But it means that $\limsup_{m\to\infty} \eta_m = 1$ with probability one.

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Alexander Ivanovich Sakhanenko
Ugra State University,
Chehova st., 16,
Khanty-Mansiysk, RUSSIA
E-mail address: aisakh@mail.ru