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LÖBELL MANIFOLDS REVISED

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Abstract. The first example of a closed orientable hyperbolic 3–manifold was constructed by F. Löbell in 1931. It was an affirmative answer to the Köbe question on the existence of hyperbolic 3–forms. In the present paper we give a short survey of some related results and obtain a simple analytic formula for the volume of the Löbell manifold as well as for volumes of Humbert manifolds.

Introduction

The first example of a closed orientable hyperbolic 3–manifold was constructed by F. Löbell [5] in 1931. It was an affirmative answer to the Köbe question on the existence of hyperbolic 3–forms. Two years later H. Seifert and C. Weber [14] presented an elegant construction of the dodecahedron hyperbolic space, which was much more cited than Löbell's example. As we know, during a long period, there was only one reference made to the Löbell construction: in [13] T. Salenius presented a closed hyperbolic 3–manifold obtained from four copies of Löbell's polyhedron.

We remind that Löbell's example was obtained by gluing eight copies of a right–angled polyhedron $P(6)$ shown in Fig. 1. The construction was described in a purely geometrical form. Later on, it was recognized and widely used in our papers [8, 9, 17, 18, 19, 20] that a Löbell type manifold can be naturally described in terms of 4–coloring of right–angled polyhedra. A similar construction was independently discovered by M. Takahashi [16]. Recently, the Löbell type manifolds as well as right–angled polyhedra became a subject of intensive investigations [1, 2, 7, 12, 15, 4]. In particular, arithmetical properties of these manifolds were investigated in [1]. Upper and lower bounds for complexity of the Löbell type manifolds were

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obtained in [21]. An arrangement of right–angled hyperbolic polyhedra by their volumes was done in [4]. It turns out that the smallest volume is attained by a regular right–angled dodecahedron. Four–dimensional generalizations of the Löbell construction are considered in [12]. Right–angled polyhedra arising as convex cores of quasi–Fuchsian groups are investigated in [7].

1. Construction

Let $P(n)$, $n \geq 5$, be a right–angled polyhedron in $\mathbb{H}^3$ whose boundary consists of two n–gons on the top and bottom and $2n$ pentagons on the lateral surface (see Fig. 1 for $n = 6$). We will call $P(n)$ a *Löbell polyhedron.* Let $\Delta(n)$ be a group generated by reflections in faces of $P(n)$. We recall that every 4–color coloring σ of faces of $P(n)$ induces an epimorphism $\varphi_n^\sigma : \Delta(n) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ such that its kernel $\Gamma_n^\sigma = \mathrm{Ker}\,(\varphi_n^\sigma)$ is torsion free and does not contain orientation reversing isometries. We fix a coloring σ and define a *Löbell manifold* $L(n) = L(n, \sigma)$ as a quotient space $L(n, \sigma) = \mathbb{H}^3/\Gamma_n^\sigma$. Thus, $L(n)$ is obtained by gluing eight copies of $P(n)$. Hence

$$\mathrm{vol}\ L(n) = 8\,\mathrm{vol}\ P(n).$$

We note the volume of the manifold $L(n)$ does not depend on the choice of σ. See [5, 8, 17, 20] for details.

It follows from the result of R. Hidalgo and G. Rosenberger [3] that the commutator subgroup $\Delta(n)'$ of $\Delta(n)$ is torsion free. A quotient space $H(n) = \mathbb{H}^3/\Delta(n)'$ will be referred to as a *Humbert manifold.* Since $\Delta(n)$ is generated by $(2n + 2)$ reflections, we have $\Delta(n)/\Delta(n)' = \mathbb{Z}_2^{2n+2}$. Hence, $|\Delta : \Delta'| = 2^{2n+2}$ and $H(n)$ is obtained from 2^{2n+2} copies of $P(n)$. Therefore,

$$\mathrm{vol}\ H(n) = 2^{2n-1} \cdot \mathrm{vol}\ L(n). \tag{1}$$

Note that $H(n)$ and $L(n)$ are the maximal and the minimal manifold Abelian coverings of orbifold $\mathbb{H}^3/\Delta(n)$, respectively.

2. Volume formulae

In this section we will obtain elementary formulas for volumes of the manifolds $H(n)$ and $L(n)$, which are closed orientable hyperbolic 3–manifolds. A formula expressing volumes of Löbell manifolds in terms of the Lobachevskii function

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin\zeta|d\zeta$$

was obtained by A. Vesnin in [19].

Theorem 1. [19] *Let $L(n)$, $n \geq 5$, be a Löbell manifold. Then*

$$\mathrm{vol}\ L(n) = 4n\left(2\Lambda(\theta) + \Lambda\left(\theta + \frac{\pi}{n}\right) + \Lambda\left(\theta - \frac{\pi}{n}\right) + \Lambda\left(2\theta - \frac{\pi}{2}\right)\right), \tag{2}$$

where $\theta = \dfrac{\pi}{2} - \arccos\dfrac{1}{2\cos\frac{\pi}{n}}$.

A similar formula for a particular case $n = 6$ was established in Ph.D. thesis by D. Surchat [15] advised by P. Buser.

Now we will present a new formula for volume of Löbell manifolds that will be useful for further investigations.

Consider a polyhedron $T(\alpha) = ABCA'B'C'DE$ (see Fig. 1) with dihedral angles as follows: α at AA', $\frac{\pi}{4}$ at DB' and CE, and $\frac{\pi}{2}$ at all other edges. If $0 < \alpha < \frac{\pi}{4}$ then $T(\alpha)$ is a hexahedron in $\mathbb{H}^3$, which can be regarded as a doubly–truncated doubly–rectangular tetrahedron, where hyperbolic triangles ABC and $A'B'C'$ are results of truncations. If $\alpha = \frac{\pi}{n}$, $n \geq 5$, then $T(\frac{\pi}{n})$ is an $\frac{1}{2n}$–piece of the Löbell polyhedron $P(n)$, as presented in Fig. 1. If $\alpha = \frac{\pi}{4}$ then triangles ABC and $A'B'C'$ are Euclidean, and by $T(\frac{\pi}{4})$ we will mean an ideal tetrahedron with two ideal vertices. If $\frac{\pi}{4} < \alpha < \frac{\pi}{3}$ then triangles ABC and $A'B'C'$ are spherical, and by $T(\alpha)$ we will mean a doubly–rectangular tetrahedron. Dihedral angles $\frac{\pi}{4}$, α, $\frac{\pi}{4}$ are *essential* dihedral angles of $T(\alpha)$.

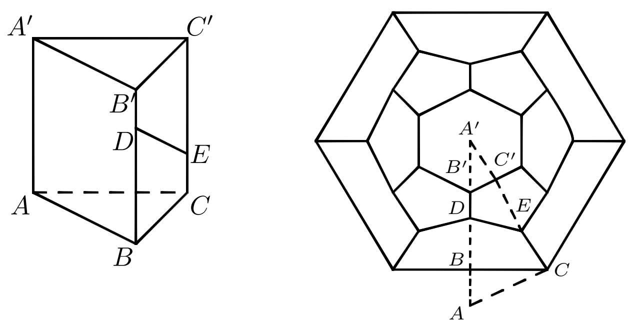

Рис. 1. Truncated tetrahedron $T(\alpha)$ and 14-hedron $P(6)$

Lemma 1. *If* $0 < \alpha < \frac{\pi}{3}$ *then* $T(\alpha)$ *is a hyperbolic polyhedron and*

$$\operatorname{vol} T(\alpha) = \frac{1}{2} \int_{\alpha}^{\frac{\pi}{3}} \operatorname{arccosh} \left| \frac{\cos\theta}{\cos 2\theta} \right| d\theta. \tag{3}$$

Proof. Let ℓ_α be the length of edge of $T(\alpha)$ with prescribed angle α. By the tangent rule from [22, p. 125] we have

$$\frac{\tanh \ell_\alpha}{\tan\alpha} = \frac{\sqrt{\cos^2\alpha - \sin^2\frac{\pi}{4}\sin^2\frac{\pi}{4}}}{\cos\frac{\pi}{4}\cos\frac{\pi}{4}} = \sqrt{4\cos^2\alpha - 1}.$$

Hence, $\tanh \ell_\alpha = \tan\alpha \cdot \sqrt{4\cos^2\alpha - 1}$ and

$$\cosh^2 \ell_\alpha = \frac{1}{1 - \tanh^2 \ell_\alpha} = \left(\frac{\cos\theta}{\cos 2\theta} \right)^2.$$

Obviously, $\dfrac{\cos\theta}{\cos 2\theta} > 0$ for $0 < \alpha < \frac{\pi}{4}$ and $\dfrac{\cos\theta}{\cos 2\theta} < 0$ for $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$. In case $\alpha = \frac{\pi}{4}$ the tetrahedron $T(\frac{\pi}{4})$ has two ideal vertices and hence $\ell_\alpha = \infty$. Moreover, $\ell_\alpha \to 0$ as $\alpha \to \frac{\pi}{3}$. Therefore, vol $T(\alpha) \to 0$ as $\alpha \to \frac{\pi}{3}$. By the Schläfli formula [10] we obtain

$$\text{vol } T(\alpha) = -\int_{\frac{\pi}{3}}^{\alpha} \frac{\ell_\theta}{2}\, d\theta = \frac{1}{2}\int_{\alpha}^{\frac{\pi}{3}} \operatorname{arccosh}\left| \frac{\cos\theta}{\cos 2\theta} \right| d\theta.$$

□

Theorem 2. *Let $L(n)$, $n \geq 5$, be a Löbell manifold. Then*

$$\text{vol } L(n) = 8n \int_{\frac{\pi}{n}}^{\frac{\pi}{3}} \operatorname{arccosh} \left| \frac{\cos\theta}{\cos 2\theta} \right| d\theta. \tag{4}$$

Proof. It can be seen from Fig. 1 that $T(\frac{\pi}{n})$ is an $\frac{1}{2n}$-piece of $P(n)$. Hence, vol $L(n) = 8 \operatorname{vol} P(n) = 8 \cdot 2n \cdot \operatorname{vol} T\left(\frac{\pi}{n}\right)$. The result follows from formula (3). □

As an immediate consequence of the obtained theorem, by (1) we have

Corollary 1. *Let $H(n)$, $n \geq 5$, be a Humbert manifold. Then*

$$\text{vol } H(n) = n \cdot 2^{2n+2} \int_{\frac{\pi}{n}}^{\frac{\pi}{3}} \operatorname{arccosh} \left| \frac{\cos\theta}{\cos 2\theta} \right| d\theta.$$

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