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SOME CHARACTERIZATIONS OF COSMIC-SPACES

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Abstract. In this paper, we prove that a space X is a cosmic space if and only if it has a σ-strong network consisting of countable covers of X, which give a new characterization of cosmic-spaces.

Keywords: cosmic space, separable metric space.

To give some characterizations of certain images of a separable metric spaces is one of interesting questions on generalized metric spaces. In the past years, many interesting results were obtained (see [2, 3, 5, 7], for example). Recall a space X is a cosmic-space if X has a countable network. In [5], E. Michael gave the following characterization of cosmic-spaces.

Theorem 1. *A space X is a cosmic-space if and only if X is an images of a separable metric space.*

In this paper, we give some new characterizations of cosmic-spaces. Assume that all spaces are regular and all mappings are continuous and onto. N denotes the set of all natural numbers. Let $\mathcal{P}$ be a family of subsets of a space X and let $x \in X$. $st(x, \mathcal{P})$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P \in \mathcal{P} : x \in P\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$, respectively. If $f : X \longrightarrow Y$ is a mapping, then $f(\mathcal{P})$ denotes $\{f(P) : P \in \mathcal{P}\}$.

Definition 2. *(1) Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x$. $\mathcal{P}$ is called a network for X* [5], *if for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.*

(2) Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$. $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is called a σ-strong network for X [4, 7], *if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$. Moreover, if every*

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$\mathcal{P}_n$ is also a countable cover of X, then $\mathcal{P}$ is called a σ-strong network consisting of countable covers.

(3) A space is called a cosmic-space [1] *if it has a countable network.*

Remark 3. *It is clear that for a space X, every σ-strong network consisting of countable covers of X is a countable network, so every space X with a σ-strong network consisting of countable covers of X is a cosmic-space.*

Definition 4. *Let $f : M \longrightarrow X$ be a mapping, where M is a metric space with a metric d. f is called a π-mapping* [6]*, if for every $x \in X$ and for every neighborhood U of x in X, $d(f^{-1}(x), X - f^{-1}(U)) > 0$.*

Theorem 5. *For a space X, the following are equivalent.*

(1) X is a π-image of a separable metric space.

(2) X has a σ-strong network consisting of countable covers.

Proof. (1)$\Longrightarrow$(2). Let M be a separable metric space with a metric d and $f : M \longrightarrow X$ be a π-mapping. Write $B_n(a) = \{b \in M : d(a,b) < 1/n\}$ for every $a \in M$ and every $n \in \mathbb{N}$. Put D is a countable dense subset of M, and put $\mathcal{U}_n = \{B_n(a) : a \in D\}$ for every $n \in \mathbb{N}$. It is easy to see that every $\mathcal{U}_n$ is a countable cover of M. Put $\mathcal{P}_n = f(\mathcal{U}_n)$ for every $n \in \mathbb{N}$, and put $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. Then for every $n \in \mathbb{N}$, $\mathcal{P}_n$ is a countable cover of X and $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. We only need to prove that for every $x \in X$, $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X. Let $x \in U$ with U open in X. Since f is a π-mapping, there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) > 1/n$. Pick $m \in \mathbb{N}$ such that $m > 2n$. It suffices to prove that $st(x, \mathcal{P}_m) \subset U$. Pick $a \in D$ such that $x \in f(B(a, 1/m)) \in \mathcal{P}_m$. We claim that $B(a, 1/m) \subset f^{-1}(U)$. If fact, if $B(a, 1/m) \not\subset f^{-1}(U)$, then there exists $b \in B(a, 1/m) - f^{-1}(U)$. Note that $f^{-1}(x) \bigcap B(a, 1/m) \neq \emptyset$, pick $c \in f^{-1}(x) \bigcap B(a, 1/m)$. Then $d(f^{-1}(x), M - f^{-1}(U)) \leq d(c, b) \leq d(c, a) + d(a, b) < 2/m < 1/n$. This is a contradiction. So $B(a, 1/m) \subset f^{-1}(U)$, thus $f(B(a, 1/m)) \subset ff^{-1}(U) = U$. This proves that $st(x, \mathcal{P}_m) \subset U$.

(2)$\Longrightarrow$(1). Let X have a σ-strong network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ consisting of countable covers. For every $n \in \mathbb{N}$, we denote $\mathcal{P}_n$ by $\mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\}$, and Λ_n is endowed with discrete topology, then Λ_n is a separable metric space.

Put

$$M = \{b = (\beta_n) \in \Pi_{n \in \mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \textit{ is a network at some } x_a \textit{ in } X\}.$$

Then M, which is a subspace of the product space $\Pi_{n \in N} \Lambda_n$ of countable many separable metric spaces $\{\Lambda_n : n \in \mathbb{N}\}$, is a separable metric space with metric d described as follows:

Let $b = (\beta_n), c = (\gamma_n) \in M$. If $b = c$, then $d(b, c) = 0$. If $b \neq c$, then $d(b, c) = 1/min\{n \in \mathbb{N} : \beta_n \neq \gamma_n\}$.

Claim 1. Let $b = (\beta_n) \in M$. Then there exists unique x_b such that $\{P_{\beta_n}\}$ is a network at x_b in X.

The existence comes from the construction of M, we only need to prove the uniqueness. Let $\{P_{\beta_n}\}$ be a network at both x_b and x'_b in X, then $\{x_b, x'_b\} \subset P_{\beta_n}$ for every $n \in \mathbb{N}$. If $x_b \neq x'_b$, then there exists an open neighborhood U of x_b such that $x'_b \notin U$. Because $\{P_{\beta_n}\}$ is a network at x_b in X, there exists $n \in \mathbb{N}$ such that $x_b \in P_{\beta_n} \subset U$, thus $x'_b \notin P_{\beta_n}$, a contradiction. This proves the uniqueness.

By Claim 1, for every $b = (\beta_n) \in M$, there exists unique x_b such that $\{P_{\beta_n}\}$ is a network at x_b in X. Define a correspondence $f : M \longrightarrow X$ by $f(b) = x_b$.

Claim 2. f is continuous and onto, so f is a mapping.

(a) f is continuous: let $b = (\beta_n) \in M$ and let $f(b) = x_b$. If U is an open neighborhood of x_b, then there exists $k \in \mathbb{N}$ such that $x_b \in P_{\beta_k} \subset U$ because $\{P_{\beta_n}\}$ is a network at x_b in X. Put $V = \{c = (\gamma_n) \in M : \gamma_k = \beta_k\}$, then V is an open neighborhood of b. We claim that $f(V) \subset P_{\beta_k}$. In fact, let $c = (\gamma_n) \in V$, then $\{P_{\gamma_n}\}$ forms a network at $f(c)$ in X, thus $f(c) \in P_{\gamma_k}$. Note that $\gamma_k = \beta_k$, so $f(c) \in P_{\beta_k}$ This proves that $f(V) \subset P_{\beta_k}$. Thus $f(V) \subset U$, hence f is continuous.

(b) f is onto: Let $x \in X$. For every $n \in \mathbb{N}$, $\mathcal{P}_n$ is a cover of X, so there exists $\beta_n \in \Lambda_n$ such that $x \in P_{\beta_n}$. Thus $\{P_{\beta_n}\}$ is a network at x in X. In fact, Let $x \in U$ with U open in X. Since $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for X, there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. Note that $P_{\beta_n} \subset st(x, \mathcal{P}_n)$, so $P_{\beta_n} \subset U$, hence $\{P_{\beta_n}\}$ is a network at x in X. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$. So f is onto.

Claim 3. f is a π-mapping.

Let $x \in U$ with U open in X. Since $\mathcal{P}$ is a σ-strong network for X, there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. Then $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. In fact, let $a = (\alpha_n) \in M$ such that $d(f^{-1}(x), a) < 1/2n$. Then there is $b = (\beta_n) \in f^{-1}(x)$ such that $d(a, b) < 1/n$, so $\alpha_k = \beta_k$ if $k \leq n$. Notice that $x \in P_{\beta_n} \in \mathcal{P}_n$, $P_{\alpha_n} = P_{\beta_n}$, so $f(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_n) \subset U$, hence $a \in f^{-1}(U)$. Thus $d(f^{-1}(x), a) \geq 1/2n$ if $a \in M - f^{-1}(U)$, so $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. This proves that f is a π-mapping. □

Theorem 6. *For a space X, the following are equivalent.*

(1) X is a cosmic-space.

(2) X is an image of a separable metric space.

(3) X is a π-image of a separable metric space.

(4) X has a σ-strong network consisting of countable covers of X.

Proof. (1)$\Longleftrightarrow$(2) and (3)$\Longleftrightarrow$(4) from Theorem 1 and Theorem 5, respectively. (4)$\Longrightarrow$(1) from Remark 3. It suffices to prove that (1)$\Longrightarrow$(4).

(1)$\Longrightarrow$(4). Let X be a cosmic-space, and let $\mathcal{A} = \{P_n : n \in \mathbb{N}\}$ be a countable network for X. For every $n \in \mathbb{N}$, put $\mathcal{F}_n = \{P_n\} \cup \{P_k - P_n : k \in \mathbb{N}\}$. Then $\{st(x, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$. In fact, for every $x \in U$ with U open in X, since $\mathcal{A}$ is a network for X, there exists $n \in \mathbb{N}$ such that $x \in P_n \subset U$. Note that $st(x, \mathcal{F}_n) = P_n$, so $st(x, \mathcal{F}_n) \subset U$, thus $\{st(x, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a network at x in X. Let $\mathcal{U}$ and $\mathcal{V}$ be two families of subsets of X. We write $\mathcal{U} \bigwedge \mathcal{V} = \{U \bigcap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$. Put $\mathcal{P}_1 = \mathcal{F}_1$, and for every $n > 1$, put $\mathcal{P}_n = \mathcal{F}_n \bigwedge \mathcal{P}_{n-1}$. Then, for every $n \in \mathbb{N}$, $\mathcal{P}_n$ is a countable cover of X and $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. Put $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, note that $\mathcal{P}_n$ refines $\mathcal{F}_n$, so $st(x, \mathcal{P}_n) \subset st(x, \mathcal{F}_n)$ for every $x \in X$. Thus $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$. This proves that $\mathcal{P}$ is a σ-strong network consisting of countable covers of X. □

References

[1] A.V.Arhangel'skii, *On a class of spaces containing all metric spaces and all bicompact spaces*, Dokl. Akad. Nauk. SSSR., 151 (1963), 751–754.

[2] Y.Ge, *Spaces with countable sn-networks*, Comment Math. Univ. Carolinae, 45 (2004), 169–176.

[3] Y.Ge, $\aleph_0$-*spaces and images of separable metric spaces*, Siberian Electronic Mathematical Reports, 2 (2005), 62–67.
[4] Y.Ikeda, C.Liu and Y.Tanaka, *Quotient compact images of metric spaces, and related matters*, Topology Appl., 122 (2002), 237–252.
[5] E.Michael, $\aleph_0$-*spaces*, J. Math. Mech., 15 (1966), 983–1002.
[6] V.I.Ponomarev, *Axiom of countability and continuous mappings*, Bull Pol. Acad Math., 8 (1960), 127–133.
[7] Y.Tanaka and Y.Ge, *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math., 32 (2006), 99–117.

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