S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 5, стр. 193–199 (2008) УДК 512.552
MSC 16S36, 16P40, 16P50, 16U20

IDEAL KRULL-SYMMETRY OF ITERATED EXTENSIONS

V. K. BHAT

Abstract. A ring R is said to be ideal Krull-symmetric if for any ideal I of R, the right Krull dimension of I is equal to the left Krull dimension of I. Let now R be commutative Noetherian ring. In this paper we show that certain Ore extensions of R are ideal Krull-symmetric. The rings that we deal with are:

(1) $S_t(R) = R[x_1;\sigma_1][x_2;\sigma_2]...[x_t,\sigma_t]$, the iterated skew-polynomial ring, where each σ_i is an automorphism of $S_{i-1}(R)$
(2) $L_t(R) = R[x_1, x_1^{-1};\sigma_1][x_2, x_2^{-1};\sigma_2]...[x_t, x_t^{-1};\sigma_t]$, the iterated skew-Laurent polynomial ring, where each σ_i is an automorphism of $L_{i-1}(R)$
(3) $D_t(R) = R[x_1;\delta_1][x_2;\delta_2]...[x_t;\delta_t]$, the iterated differential polynomial ring, where each δ_i is a derivation of $D_{i-1}(R)$ such that each $\delta_i \mid R$ is a derivation of R and,
(4) $A_t(R)$ is any of $S_t(R)$ or $L_t(R)$, where $\sigma_i \mid R$ is an automorphism of R.

With this we prove that $A_t(R)$ and $D_t(R)$ are ideal Krull-symmetric.

Keywords: Automorphism, derivation, Ore extension, annihilator, Krull dimension, Krull-symmetry.

1. Introduction

Throughout this paper all rings are with identity and all modules are unitary. $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{Z}$ denotes the ring of integers unless otherwise stated. Let R be a ring. The prime radical of R is denoted by N(R). The set of associated prime ideals of R (viewed as a right module over itself) is denoted by $Ass(R_R)$. C(0) denotes the set of regular elements of R and C(I) denotes the

Bhat, V.K., Ideal Krull-symmetry of iterated extensions.

The author would like to thank the referee for his valuable suggestions. The author would also like to thank Prof. C. L. Wangneo for his time to time discussions on this area.
Received November, 14, 2007, published April, 22, 2008.

set of elements regular modulo I, where I is an ideal of R. Let I and J be any two subsets of a ring R. Then I $\subset$ J means that I is strictly contained in J.

For any right R-module K, the right Krull dimension of K is denoted by $\mid K \mid_r$ and the annihilator of a subset S of K is denoted by $r(S)$. Similarly if J is a left R-module then left Krull dimension of J is denoted by $\mid J \mid_l$ and the annihilator of a subset L of K is denoted by $l(K)$. Recall that the right Krull dimension of a ring R is defined as the Krull dimension of R, viewed as a right module over itself. Left Krull dimension of a ring R is defined similarly. We recall that a ring R is said to be Krull-symmetric if $|R|_r = |R|_l$. R is said to be right Krull-homogeneous if $|R|_r = |I|_r$, for all right ideals I of R. Left Krull-homogeneity is defined in a similar way. We also recall that a ring R is said to be ideal Krull-symmetric if $\mid I \mid_r = \mid I \mid_l$, where I is any ideal of R. For some more details and results on Krull dimension, the reader is referred to Gordon and Robson [3] and Chapter 13 of Goodearl and Warfield [2].

It is known (proved by Rentschler-Gabriel, and general case by Gordon-Robson, namely Theorem (13.17) of Goodearl and Warfield [2]) that if R is a right Noetherian ring, M a finitely generated right R-module, and x an indeterminate; then $\mid M[x] \mid_r = \mid M \mid_r +1$. In particular, if R is nonzero, then $\mid R[x] \mid_r = \mid R \mid_r +1$.

It is also known (proved by Rentschler-Gabriel, namely Theorem (13.18) of Goodearl and Warfield [2]) that $\mid A_n(K) \mid_r = n$, for each positive integer n, where K is a field of characteristic zero, and $A_n(K)$ is the usual n^{th} Weyl algebra.

Let now R be a commutative Noetherian ring, σ an automorphism of R and δ a derivation of R. In this article, we show that the iterated extensions of the rings $R[x;\sigma]$, $R[x, x^{-1};\sigma]$ and $R[x;\delta]$ are ideal Krull-symmetric (in case of $R[x;\delta]$, R is more over an algebra over $\mathbb{Q}$). We denote these rings by S(R), L(R) and D(R) respectively. If I is an ideal of R such that σ(I) = I, we denote the ideals $I[x;\sigma]$ and $I[x, x^{-1};\sigma]$ as usual by S(I) and L(I) respectively. If J is an ideal of R such that δ(I) $\subseteq$ I, we denote the ideal $I[x;\delta]$ as usual by D(I).
The rings that we deal with are:

(1) $S_t(R) = R[x_1;\sigma_1][x_2;\sigma_2]...[x_t,\sigma_t]$, the iterated skew-polynomial ring, where each σ_i is an automorphism of $S_{i-1}(R)$
(2) $L_t(R) = R[x_1, x_1^{-1};\sigma_1][x_2, x_2^{-1};\sigma_2]...[x_t, x_t^{-1};\sigma_t]$, the iterated skew-Laurent polynomial ring, where each σ_i is an automorphism of $L_{i-1}(R)$
(3) $D_t(R) = R[x_1;\delta_1][x_2;\delta_2]...[x_t;\delta_t]$, the iterated differential polynomial ring, where each δ_i is a derivation of $D_{i-1}(R)$ such that each $\delta_i \mid R$ is a derivation of R and,
(4) $A_t(R)$ is any of $S_t(R)$ or $L_t(R)$, where $\sigma_i \mid R$ is an automorphism of R.

In the main result (Theorem (3.6)) we prove that if A is any of $A_t(R)$ or $D_t(R)$ as above, then:

(1) A is ideal Krull-symmetric.
(2) For any ideal L of A, $\mid A/L \mid_r < \mid A \mid_r$ if and only if $\mid A/L \mid_l < \mid A \mid_l$.

2. Invariance of symbolic powers of certain ideals

We begin this section with the following Proposition:

Proposition 2.1. *Let R be a semiprime Noetherian ring, σ an automorphism of R and δ a σ-derivation of R. Let O(R) = $R[x;\sigma,\delta]$. If $f \in O(R)$ is a regular element, then there exists $g \in O(R)$ such that gf has leading co-efficient regular in R.*

Proof. Let S = $\{a_m \in R$ such that $x^m a_m + ... + a_0 \in$ O(R)f, some m$\} \cup \{0\}$. Then since O(R) is semiprime and Noetherian, O(R)f is an essential left ideal of O(R). Therefore, S is an essential left ideal of R. So S contains a left regular element and since R is semiprime, Proposition (3.2.13) of Rowen [5] implies that S contains a regular element. So there exists $g \in$ O(R) such that gf has leading coefficient regular in R. □

Definition 2.2. Let R be a commutative Noetherian ring and P a semiprime ideal of R. Let $k \geq 1$ be an integer. Then the symbolic power of P is denoted by $P^{(k)}$ and is defined as $P^{(k)} = \{a \in$ R | there exists $d \in$ C(P) such that $ad \in P^k\}$.

Proposition 2.3. *Let R be a commutative Noetherian ring and P be a semiprime ideal of R. Then $P^{(k)}$ is an ideal of R.*

Proof. Let a, $b \in P^{(k)}$. Then there exist d_1, $d_2 \in$ C(P) such that $ad_1 \in P^k$ and $bd_2 \in P^k$. So $ad_1d_2 \in P^k$ and $bd_1d_2 \in P^k$; i.e. $(a - b)d_1d_2 \in P^k$ and since $d_1d_2 \in$ C(P), so $(a - b) \in P^{(k)}$. Now let $a \in P^{(k)}$ and $r \in$ R. Then there exists $d \in$ C(P) such that $ad \in P^k$. Therefore $ard \in P^k$ and since $d \in$ C(P), we have $ar \in P^{(k)}$. Hence $P^{(k)}$ is an ideal of R. □

Proposition 2.4. *Let P be a semiprime ideal of a commutative Noetherian ring R and σ an automorphism of R such that $\sigma(P) = P$. Then $\sigma(P^{(k)}) = P^{(k)}$.*

Proof. Let $a \in P^{(k)}$. Then there exists $d \in$ C(P) such that $ad \in P^k$. So $\sigma(ad) \in \sigma(P^k)$; i.e. $\sigma(a)\sigma(d) \in (\sigma(P))^k = P^k$. Now $\sigma(d) \in$ C(P), therefore $\sigma(a) \in P^{(k)}$. Therefore $\sigma(P^{(k)}) \subseteq P(k)$. Replacing σ by σ^{-1} we get $\sigma^{-1}(P^{(k)}) \subseteq P^{(k)}$; i.e. $P^{(k)} \subseteq \sigma(P^{(k)})$. Hence $\sigma(P^{(k)}) = P^{(k)}$. □

Proposition 2.5. *Let R be commutative Noetherian ring and δ a derivation of R. Let P be a semiprime ideal of R such that $\delta(P) \subseteq P$. Then $\delta(P^{(k)}) \subseteq P^{(k)}$.*

Proof. Let $a \in P^{(k)}$. Then there exists $u \in$ C(P) such that $au \in P^k$. Let $au = p_1.p_2...p_k, p_i \in$ P. Now $\delta(au) = \delta(p_1)p_2...p_k + p_1\delta(p_2)p_3...p_k + ... + p_1p_2...p_{k-1}\delta(p_k) \in P^k$ as $\delta(P) \subseteq$ P; i.e. $\delta(a)u + a\delta(u) \in P^k$. Now $a\delta(u) \in P^{(k)}$, so there exists $u_1 \in$ C(P) such that $a\delta(u)u_1 \in P^k$. Now $\delta(a)uu_1 + a\delta(u)u_1 \in P^k$, therefore $\delta(a)uu_1 \in P^k$ and since $uu_1 \in$ C(P), $\delta(a) \in P^{(k)}$. Hence $\delta(P^{(k)}) \subseteq P^{(k)}$. □

The first important result in the theory of non commutative Noetherian rings was proved in 1958 (Goldie's Theorem) which gives an analog of the field of fractions for factor rings R/P, where R is a Noetherian ring and P is a prime ideal of R. In 1959 the one sided version was proved by Goldie, Lesieur-Croisot (Theorem (5.12) of Goodearl and Warfield [2]) and in 1960 Goldie generalized the result for semiprime rings (Theorem (5.10) of Goodearl and Warfield [2]). For more details on the notion of quotient rings (in particular the existence of artinian quotient rings), the reader is referred to Chapter 10 of Goodearl and Warfield [2]. We now have the following:

Proposition 2.6. *Let R be a commutative Noetherian ring and $A_t(R)$ be the usual skew polynomial ring. Let $Ass(R_R) = \{P_j,\ 1 \leq j \leq n\}$. Let $\sigma_i^{m_i}(P_j) = P_j$, $m_i \geq 1$ for all j, $1 \leq j \leq n$. Let $P_{1j} = \cap\sigma_1^k(P_j)$, $1 \leq k \leq m_1$ and $P_{tj} = \cap\sigma_t^u(P_{t-1j})$, $1 \leq u \leq m_t$, $t \geq 2$ for all j, $1 \leq j \leq n$. Then:*

(1) $\sigma_i(P_{tj}^{(k)}) = P_{tj}^{(k)}$ *for all i, j; $1 \leq j \leq n$, $1 \leq i \leq t$, $k \geq 1$.*

(2) $R/(P_{tj}^{(k)})$ *has an artinian quotient ring for all j, $1 \leq j \leq n$.*

Proof. **(1)** we will use induction on t. For t = 1, clearly $\sigma_1(P_{1j}) = P_{1j}$ Suppose $\sigma_i(P_{t-1j}) = P_{t-1j}$, $1 \leq i \leq t-1$. We now show that $\sigma_i(P_{tj}) = P_{tj}$, $1 \leq i \leq t$. Note that we have an automorphism α_t of R such that $\sigma_t \mid_R$ is same as α_t and α_t can be extended to $S_1(R)$, $S_2(R)$, ... , $S_{t-1}(R)$ such that $\alpha_t(x_i) = x_i$, $1 \leq i \leq t-1$.

Now for all u, $1 \leq u \leq m_t$;

$$\sigma_i(P_{tj}) = \sigma_i(\cap\sigma_t^u(P_{t-1j})) = \sigma_i(\cap\alpha_t^u(P_{t-1j})) = \sigma_i(\cap\alpha_t^u(L_{i-1}(P_{t-1j}) \cap \mathrm{R})).$$

So $\sigma_i(P_{tj}) = \sigma_i(\cap\alpha_t^u(L_{i-1}(P_{t-1j})\cap\mathrm{R})$ by Lemma (10.6.3) of McConnell and Robson [4]. Therefore $\sigma_i(P_{tj}) = (\cap\alpha_t^u(P_{t-1j})) = \cap\sigma_t^u(P_{t-1j})) = P_{tj}$.

(2) $N(R/P_{tj}^{(k)}) = (\cap\sigma_j^i(P_{tj}))/P_{tj}^{(k)}$, $1 \leq i \leq m_j$, where j = t+1 and $\sigma_j^{m_j}(P_{tj}) = P_{tj}$. Let $N_1 = (R/P_{tj}^{(k)})$. Let $a + P_{tj}^{(k)} \in C(N_1)$. Then $a \in \cap\sigma_j^i(P_{tj})$, $1 \leq i \leq m_j$. Therefore $a \in C(\sigma_j^{m_j}(P_{tj})) = C(P_{tj})$. Now let $b \in$ R be such that $ab \in P_{tj}^{(k)}$. Then there exists $d \in C(P_{tj})$ such that $abd \in P_{tj}^k$; i.e. $bad \in P_{tj}^k$, and since $ad \in C(P_{tj})$, therefore $b \in P_{tj}^{(k)}$. Hence by Small's Theorem, namely Theorem (10.9) of Goodearl and Warfield [2] $R/P_{tj}^{(k)}$ has an artinian quotient ring. □

Proposition 2.7. *Let R be a Noetherian $\mathbb{Q}$-algebra and $D_t(R)$ as usual the t^{th} differential polynomial ring. Then:*

(1) $\delta_i(P_j) \subseteq P_j$ *implies* $\delta_i(P_j^{(k)}) \subseteq P_j^{(k)}$ *for all i, j;* $1 \leq i \leq t$*;* $1 \leq j \leq n$*;* $k \geq 1$*, where* P_j *as in Proposition (2.6) above.*

(2) $R/P_j^{(k)}$ *has an artinian quotient ring.*

Proof. **(1)** The proof is obvious by Proposition (2.5)

(2) $N(R/P_j^{(k)}) = \cap\sigma_1^i(P_j))/P_j^{(k)}$, $1 \leq i \leq m_t$, where $\sigma_1^{m_1}(P_j) = P_j$. Let $N_1 = N(R/P_j^{(k)})$ and $a + P_j^{(k)} \in C(N_1)$. Then $a \in C(\cap\sigma_1^i(P_j))$, therefore $a \in \sigma_1^i(P_j)$ for all i, $1 \leq i \leq m_t$. Thus $a \in C(\sigma_1^{m_1}(P_j)) = C(P_j)$. Now let $ab \in P_j^{(k)}$, $b \in$ R. Then there exists $d \in C(P_j)$ such that $abd \in P_j^k$; i.e. $bad \in P_j^k$. But since $ad \in C(P_j)$, therefore $b \in P_j^{(k)}$. Thus by Small's Theorem, namely Theorem (10.9) of Goodearl and Warfield [2] $R/P_j^{(k)}$ has an artinian quotient ring. □

Proposition 2.8. *Let R be a ring and S(R) be as usual and* $S_2(R) = S(R)[x_2, \sigma_2]$*. Let each* $\sigma_i \mid R$ *be an automorphism of R for i = 1, 2. Let I be an ideal of R such that* $\sigma_1(I) = I$ *and* $\sigma_2(I) = I$*. Then* $\sigma_2(S(I)) = S(I)$*.*

Proof. Let

$$f = x^m a_m + ... + a_0 \in S(I),\ a_i \in I.$$

Then

$$\sigma_2(f) = \sigma_2(x^m a_m + ... + a_0) = \sigma_2(x^m a_m) + ... + \sigma_2(a_0) =$$
$$\sigma_2(x^m)\sigma_2(a_m) + ... + \sigma_2(a_0) = g_m b_m + ... + b_0,$$

where $\sigma_2(x^i) = g_i \in S_1(R)$ and $\sigma_2(a_i) = b_i \in I$. Therefore $\sigma_2(f) \in S_1(I)$. Hence $\sigma_2(S(I)) \subseteq S(I)$. Replacing σ_2 by σ_2^{-1} ,we get that $\sigma_2(S(I)) = S(I)$. □

Corollary 2.9. *The above Proposition holds if S(R) is replaced by L(R).*

Theorem 2.10. *Let R be a Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Then* $P \in Ass(D(R)_{D(R)})$ *if and only if* $P = D(P \cap R)$ *and* $P \cap R \in Ass(R_R)$*.*

Proof. See Theorem (3.7) of Bhat [1]. □

Proposition 2.11. *Let R be a Noetherian $\mathbb{Q}$-algebra and $Ass(R_R) = \{P_j, 1 \leq j \leq n\}$. Consider $D_t(R)$. Then $\delta_i(D_{i-1}(P_j^{(k)})) \subseteq D_{i-1}(P_j^{(k)})$ for all i, j; $1 \leq i \leq n$; $1 \leq i \leq t$.*

Proof. $Ass(R_R) = \{P_j, 1 \leq j \leq n\}$. Now $\delta_1(P_j) \subseteq P_j$ by Theorem (1) of Seidenberg [6], and therefore $\delta_1(P_j^{(k)}) \subseteq P_j^{(k)}$ by Proposition (2.5). Now by Theorem (2.10) $D(P_j) = P_j[x_1; \delta_1] \in Ass(D(R)_{D(R)})$. Therefore $\delta_2(D(P_j)) \subseteq D(P_j)$ by Theorem (1) of Seidenberg [6]. We will show that $\delta_2(D(P_j^{(k)})) \subseteq D(P_j^{(k)})$. Let $f_1 = \sum x_1^i a_i \in D(P_j^{(k)})$, $a_i \in P_j^{(k)}$, $0 \leq i \leq s$. Now there exists $d_i \in C(P_j)$ such that $a_i d_i \in P_j^k$. Let $d = d_0.d_1...d_s$. Then $a_i d \in P_j^k$. Therefore $g_1 = \sum x_1^i a_i d \in D(P_j^k)$, which implies that $\delta_2(g_1) \in D(P_j^k)$; i.e. $\delta_2(f_1 d) \in D(P_j^k)$. Thus we have $\delta_2(f_1)d + f_1\delta_2(d) \in D(P_j^k) \subseteq D(P_j^{(k)})$, which implies that $\delta_2(f_1)d \in D(P_j^{(k)})$ as $f_1 \in D(P_j^{(k)})$. Let $\delta_2(f_1) = \sum x^i b_i$, $0 \leq i \leq m$. Then $b_i d \in P_j^{(k)}$. So there exists $v_i \in C(P_j)$ such that $b_i d v_i \in P_j^k$. Let $v = v_0.v_1...v_m$. Now $b_i dv \in P_j^k$, and since $dv \in C(P_j)$, therefore $b_i \in P_j^{(k)}$. Thus we have $\delta_2(D(P_j^{(k)})) \subseteq D(P_j^{(k)})$.

With the same process in a finite number of steps it can be seen that
$\delta_i(D_{i-1}(P_j^{(k)})) \subseteq D_{i-1}(P_j^{(k)})$ for all $i \geq 3$. □

Remark 2.12. Let $0 = \cap I_j$, $1 \leq j \leq n$ be a reduced primary decomposition of a commutative Noetherian ring R. Let $\sqrt{I_j} = P_j$. Then $Ass(R_R) = \{P_j, 1 \leq j \leq n\}$ and there exists an integer $k \geq 1$ such that $P_j^{(k)} \subseteq I_j$. So $\cap P_j^{(k)} = 0$, $1 \leq j \leq n$. Also $\cap P_{tj}^{(k)} = 0$, where $P_{tj}^{(k)}$ as in Proposition (2.6) above. Hence $\cap D_t(P_j^{(k)}) = 0$, $1 \leq j \leq n$ by Proposition (2.7) and by Proposition (2.11). Also $\cap A_t(P_{tj}^{(k)}) = 0$, $1 \leq j \leq n$ by Proposition (2.6) and by Proposition (2.8).

3. Ideal Krull-symmetry of extension rings

In this section we prove that $A_t(R)$ and $D_t(R)$ are ideal Krull-Symmetric, whenever R is a commutative Noetherian ring (in case of $D_t(R)$, R is moreover an algebra over $\mathbb{Q}$). Before that we recall the concept of centralizing elements of a ring, the centralizing extension of a ring and recall some results related to centralizing extensions and Krull dimension.

Definition 3.1. Let S be a ring and R be a subring of S. we say an element $a \in$ S centralizes R if ar = ra for each $r \in$ R. If S has a finite set of generators $\{a_i, 1 \leq i \leq n\}$ each of which centralize R, then S is called a finite centralizing extension of R.

Proposition 3.2. *Let R be a Noetherian ring and let $0 = \cap I_j$, $1 \leq i \leq n$. Let $S = \prod R/I_j$, $1 \leq i \leq n$. Then S is a finite centralizing extension of R.*

Proof. It is easy to see that there exists a monomorphism f : R → S. Let $x_1 = (1 + I_1, 0, ..., 0)$ and $x_j = (0, 0, ..., 0, 1 + I_j, 0...0)$ $1 \leq j \leq n$. For any $s \in$ S, let $s = (r_1 + I_1, r_2 + I_2, ..., r_n + I_n) = \sum x_j r_j$. Now $x_j r = r x_j$ for all $r \in$ R, $1 \leq j \leq n$. Hence the result. □

Proposition 3.3. *If S is a Noetherian centralizing extension of R, then* $| S |_r = | R |_r$ *and* $| S |_l = | R |_l$*. Also for any ideal I of S, S/I is a finite centralizing extension of* $R/I \cap R$.

Proof. The proof is obvious. One may see Corollary (10.1.11) of McConnell and Robson [4]. □

Proposition 3.4. *Let* I_j *be ideals of a Noetherian ring R such that* $0 = \cap I_j$, $1 \leq j \leq n$ *and each* R/I_j *Krull-symmetric and Krull-homogeneous. Then R is ideal Krull-symmetric.*

Proof. Let S = $\prod R/I_j$, $1 \leq i \leq n$. Now by Proposition (3.2) S is a centralizing extension of R. Let I be an ideal of R. Consider the ideal $I_1 = \prod(I + I_j/I_j$ of S. Then it is easy to see that I_1 is a Krull-symmetric ideal of S. Therefore

$$| I_1 |_r = | S/r(I_1) |_r = | S/l(I_1) |_l = | I_1 |_l.$$

Let f : R $\rightarrow$ S be the natural monomorphism. Now notice $r(I) = r(I_1) \cap$ R, where $r(I_1)$ is in S and similarly $l(I) = l(I_1) \cap$ R. Now by Proposition (3.3) $S/r(I_1)$ is a centralizing extension of $R/r(I_1)$. Therefore $| S/r(I_1) |_r = | R/r(I_1) |_r$ by Proposition (3.3) and similarly $| S/l(I_1) |_l = | R/l(I_1) |_l$ and as noted above $| S/r(I_1) |_r = | S/l(I_1) |_l$. Thus $| I |_r = | I |_l$. □

Proposition 3.5. *Let A be any of* $A_t(R)$ *or* $D_t(R)$*. Then A is Krull-symmetric.*

Proof. $S_t(R)$ is Krull-Symmetric is easy. Now let B be any ring and $L(B) = B[x, x^{-1}, s]$. Then there exists an anti-isomorphism $f : L(B) \rightarrow L(B^o)$ where B^o is the opposite ring of B such that $f(x) = x^{-1}$ and $f(x^{-1}) = x$. Then an easy induction gives an anti-isomorphism from $L_t(R)$ onto itself. In $D_t(R)$, the proof is by using the Dixmier map and then an easy induction gives that there exists an anti isomorphism from $D_t(R)$ onto itself. □

Let R be commutative Noetherian ring. Let $S_t(R) = R[x_1; \sigma_1][x_2; \sigma_2]...[x_t; \sigma_t]$, the iterated skew-polynomial ring where each σ_i is an automorphism of $S_{i-1}(R)$; $L_t(R) = R[x_1; x_1^{-1}, \sigma_1][x_2; x_2^{-1}, \sigma_2]...[x_t; x_t^{-1}, \sigma_t]$, the iterated skew-Laurent polynomial ring, where each σ_i is an automorphism of $L_{i-1}(R)$ and the iterated differential polynomial ring $D_t(R) = R[x_1; \delta_1][x_2; \delta_2]...[x_t; \delta_t]$, where each δ_i is a derivation of $D_{i-1}(R)$ such that each $\delta_i \mid R$ is a derivation of R. In case $D_t(R)$, R is moreover an algebra over $\mathbb{Q}$.
$A_t(R)$ is any of $S_t(R)$ or $L_t(R)$ such that each $\sigma_i \mid R$ is an automorphism of R.

With this we are now in a position to state and prove the main result in the form of the following Theorem:

Theorem 3.6. *Let A be any of* $A_t(R)$ *or* $D_t(R)$*. Then :*

(1) *A is ideal Krull-symmetric.*
(2) *For any ideal L of A,* $| A/L |_r < | A |_r$ *if and only if* $| A/L |_l < | A |_l$.

Proof. **(1)** Let $I_j = A_t(P_{tj}^{(k)})$ in case of $A_t(R)$ and $I_j = D_t(P_j^{(k)})$ in case of $D_t(R)$. Then $0 = \cap I_j$, $1 \leq i \leq n$ by Remark (2.12). Let $T = \prod A/I_j$, $1 \leq i \leq n$. Now by Proposition (3.2) T is a centralizing extension of A. Let I be an ideal of A and $I^* = \prod(I+I_j)/I_j$, $1 \leq i \leq n$ of T. Then it is easy to see that I^* is a Krull-symmetric ideal of T. So

$$| I^* |_r = | T/r(I^*) |_r = | T/l(I) |_l = | I^* |_l.$$

Let $f : A \to T$ be the natural monomorphism. Now $r(I) = r(I^*) \cap A$, where $r(I^*)$ is in T and similarly $l(I) = l(I^*) \cap A$. Now by Proposition (3.3) $T/r(I^*)$ is a centralizing extension of $A/r(I)$. Therefore $\mid T/r(I^*) \mid_r = \mid A/r(I) \mid_r$ by Proposition (3.3) and similarly $\mid T/l(I^*) \mid_l = \mid A/l(I) \mid_l$. But $\mid T/r(I^*) \mid_r = \mid T/l(I^*) \mid_l$. Therefore $\mid I \mid_r = \mid I \mid_l$. Hence A is ideal Krull-symmetric.

(2) Let $A = A_t(R)$. Let L be an ideal of A such that $\mid A/L \mid_l < \mid A \mid_l$, and suppose $\mid A/L \mid_r = \mid A \mid_r$. Now $\mid A/L \mid_r = \mid A/P \mid_r = \mid A \mid_r$, where P is a prime deal of A such that $L \subseteq P$. Now $N(A) = \cap A_t(P_{tj})$, $1 \leq j \leq n$ and since $I_j^* = A_t(P_{tj}^k) \subseteq A_t(P_{tj}^{(k)}) = I_j$, $\cap I_j^* = 0$, $1 \leq j \leq n$. Since every $A_t(P_{tj})$, $1 \leq j \leq n$ is associated to A, so P is associated to A and $P = A_t(P_{tj})$ for some j, $1 \leq j \leq n$. Let $A_1 = A/I_j$. Now $L + I_j \subseteq P_{tj}$ and $I_j \subseteq L + I_j \subseteq P_{tj}$, therefore

$$\mid A_1/L + I_j \mid_r = \mid A/L + I_j \mid_r = \mid A/P_{tj} \mid_r .$$

But by Proposition (3.3) in A_1 we have

$$\mid A_1/L + I_j \mid_l = \mid A_1/L + I_j \mid_r = \mid A/L \mid_l = \mid A \mid_l \text{ as } \mid A \mid_r = \mid A \mid_l$$

which is a contradiction. Hence $\mid A/L \mid_r < \mid A \mid_r$.

Converse can be proved on the same lines as above.

For $A = D_t(R)$, replace P_{tj} by P_j. Rest is obvious. □

Remark 3.7. Let $A_t(\mathbb{Z})$ be the Weyl Algebra as in chapter 1 of Goodearl and Warfield [2]. Then every factor ring of $A_t(\mathbb{Z})$ is ideal Krull-symmetric.

Proof. If I is non-zero ideal of $A_t(\mathbb{Z})$, then by (8B) of Goodearl and Warfield [2], $A_t(\mathbb{Z})/I$ is an FBN ring, therefore $A_t(\mathbb{Z})/I$ is ideal Krull-symmetric. Now if I = (0), then $A_t(\mathbb{Z})$ is ideal Krull-symmetric by Theorem (3.6). □

References

[1] V. K. Bhat, *Associated prime ideals of skew polynomial rings*, Beitrдge zur Algebra und Geometrie, **49**: 1 (2008), 277–283.

[2] K. R. Goodearl and R.B. Warfield Jr., *An introduction to non-commutative Noetherian rings*, Cambridge Uni. Press, 1989.

[3] R. Gordon and J. C. Robson, *Krull dimension*, Memoirs of the Amer. Math. Soc., 133, 1973.

[4] J. C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Revised Edition, Amer. Math. Soc., 2000.

[5] L. H. Rowen, *Ring theory*, Academic Press, Inc, 1991.

[6] A. Seidenberg, *Differential ideals in rings of finitely generated Type*, Amer. J Math., 89 (1967), 22–42.

Vijay Kumar Bhat
School of Applied Physics and Mathematics,
SMVD University,
182301, P/o Kakryal, Katra, J and K, India
E-mail address: vijaykumarbhat2000@yahoo.com