S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 5, стр. 211–214 (2008) УДК 519.172.2
MSC 05C15

LIST 2-ARBORICITY OF PLANAR GRAPHS WITH NO TRIANGLES AT DISTANCE LESS THAN TWO

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Abstract. It is known that not all planar graphs are 4-choosable; neither all of them are vertex 2-arborable. However, planar graphs with no triangles at distance less than two are known to be 4-choosable (Lam, Shiu, Liu, 2001) and 2-arborable (Raspaud, Wang, 2008).

We give a common extension of these two last results in terms of covering the vertices of a graph by induced subgraphs of variable degeneracy. In particular, we prove that every planar graph with no triangles at distance less than two is list 2-arborable.

Keywords: planar graph, 4-choosability, vertex-arboricity.

1. Introduction

It is impossible to strengthen Appel and Haken's Four Color Theorem [1] in the following two ways. First, Chartrand and Kronk [5] constructed planar graphs whose vertex sets cannot be covered by two induced forests. On the other hand, Margit Voigt [9] proved that there exist planar graphs that are not 4-choosable.

However, planar graphs with no triangles at distance less than two are both 4-choosable (as proved by Lam, Shiu, and Liu [7]) and 2-arborable (Raspaud and Wang [8]).

A common extension of the notions of choosability and arboricity, the list point arboricity, was introduced by Borodin, Kostochka and Toft [3]. A graph G is called *list k-arborable* if for any sets $L(v)$ of cardinality at least k at its vertices, one can choose an element (color) for each vertex v from its list $L(v)$ so that the subgraph

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Borodin, O.V., Ivanova, A.O., List 2-arboricity of planar graphs with no triangles at distance less than two.

The authors were supported by grants 08-01-00673 and 06-01-00694 of the Russian Foundation for Basic Research.

Received April, 25, 2008, published May, 5, 2008.

induced by every color class is an acyclic graph (a forest). Note that if the list $L(v)$ does not vary from a vertex to another, we have a problem of usual k-arboricity. Also note that every forest is 2-choosable.

A smaller purpose of this paper is to extend the above mentioned result by Raspaud and Wang [8] as follows:

Theorem 1. *Every planar graph with no triangles at distance less than two is list 2-arborable.*

However, our main purpose here is to give a common extension of the above results in [7] and [8] in terms of covering the vertices of a graph by induced subgraphs of variable degeneracy. These concepts were introduced by Borodin, Kostochka and Toft in [3].

Let f be a function from $V(G)$ to the set of positive integers. We say that G is strictly *f-degenerate* if every subgraph G' of G has a vertex v such that $d_{G'}(v) < f(v)$. (Henceforth, d be the degree function of graph G.) In other words, G can be made empty by a sequence of deletions of vertices such that each vertex v has, at the moment of its deletion, degree less than $f(v)$ in the remaining graph.

Now let f_i, $1 \le i \le s$, be functions from $V(G)$ to the non-negative integers. A graph G is called *$(f_1, \dots, f_s)$-partitionable* if $V(G)$ can be partitioned into subsets $V_1, \dots, V_s$ so that the induced subgraph $G(V_i) = G_i$ is strictly f_i-degenerate for each $1 \le i \le s$.

These subgraphs G_i may be treated as color classes. Observe that if $f_i(v) = 0$, then v cannot be colored with i: otherwise v could never be deleted from G_i.

Thus, the special case of covering by subgraphs of variable degeneracy in which $f_i(v) \in \{0,1\}$ for all $1 \le i \le s$ and $v \in V(G)$ corresponds to list coloring with $L(v) = \{i \mid f_i(v) = 1\}$. If $f_i(v) \in \{0,2\}$ whenever $v \in V(G)$, then we have the problem of list point arboricity [3].

Given functions f_i, $1 \le i \le s$, and a graph G, a *monoblock* H of G is either an end-block of G or G itself if it has only one block, such that there is an index j (depending on H) with the property that

$$f_i = \begin{cases} 0, & \text{for all } i \neq j, \\ d_G(v), & \text{for } i = j, \end{cases}$$

for all v in H, except possibly for the cut-vertex if H is an end-block.

Monoblocks are actually obstacles to $(f_1, \dots, f_s)$-partitionability (see [3] for details). We will deal with a simple special case of monoblocks in the proof of our main Theorem 2.

The main purpose of this paper is the following extension of Theorem 1 and of the results in [7, 8]:

Theorem 2. *Every planar graph with no triangles at distance less than two is $(f_1, \dots, f_s)$-partitionable whenever $s \ge 2$, $f_1(v) + \ldots + f_s(v) \ge 4$ for each vertex v, and $f_i(v) \in \{0, 1, 2\}$ for all i and v.*

Clearly, the case of 4-choosability is $f_i(v) \in \{0, 1\}$, while that of 2-arboricity is $s = 2$, $f_1(v) \equiv f_2(v) \equiv 2$.

2. Proof of main result

Let graph G_0 have the minimum number of vertices among the counterexamples to Theorem 2, and let $(f_1, \dots, f_s)$ be a corresponding vector-function.

First mention some structural properties of planar graphs having no adjacent 3-cycles. Let δ be the minimum degree of such a graph. It was proved in [2] that $\delta \leq 4$; moreover, $\delta \geq 3$ implies that there is an edge $e = xy$ such that $d(x)+d(y) \leq 9$ (both bounds are tight).

As applied to Theorem 2, only the case $\delta = 4$ is of interest, since G_0 has no vertex v with $d(v) \leq 3$. Indeed, any $(f_1, \dots, f_s)$-partition of $G_0 - \{v\}$ can be extended to G_0 by coloring vertex v with such an i that v is adjacent in $G_0 - \{v\}$ to less than $f_i(v)$ vertices colored i.

By f_5^3 denote a 5-face adjacent to a 3-face. It was proved in [6] that every connected planar graph G with $\delta(G) = 4$ and no 4-cycles has an f_5^3 whose all vertices have degree 4. This result was extended in [4] to planar graphs with no 4-cycles adjacent to 3-cycles.

A similar structural fact is established in [8]:

Lemma 1. *If a planar graph G with $\delta(G) = 4$ has no triangles at distance less than two, then G has a 5-cycle $C_5^* = x_1 \dots x_5$ with a chord x_2x_5 such that $d(x_i) = 4$ whenever $1 \leq i \leq 5$.*

So, our G_0 contains a C_5^*. It is easy to see that the subgraph induced by G_0 on the vertex set $\{x_1, \dots, x_5\}$ coincides with C_5^*. In other words, each of x_2 and x_5 is joined to $V(G) - \{x_1, \dots, x_5\}$ by precisely one edge, whereas each x_1, x_3, and x_4, by two edges.

It remains to show that every $(f_1, \dots, f_s)$-partition of $G_0 - \{x_1, \dots, x_5\}$, which exists due to the minimality of G_0, can be extended to the whole G_0.

For each $v \in V(C_5^*)$ and every $1 \leq i \leq s$, we put $f_i^*(v)$ equal $f_i(v)$ minus the number of vertices colored i and adjacent to v in $G_0 - \{x_1, \dots, x_5\}$.

Note that having an $(f_1^*, \dots, f_s^*)$-partition of C_5^*, one can get a desired $(f_1, \dots, f_s)$-partition of G_0. Indeed, for each color class i, we first destroy its vertices in $\{x_1, \dots, x_5\}$ (in the order of its $(f_1^*, \dots, f_s^*)$-destruction), and then, its vertices that belong to $G_0 - \{x_1, \dots, x_5\}$.

So, to complete proving Theorem 2 it suffices to check that C_5^* is $(f_1^*, \dots, f_s)^*$-partitionable. Here, we make use of Theorem 8 in [3], which gives a necessary and sufficient condition for a connected graph G to be $(f_1, \dots, f_s)$-partitionable under the assumption that $f_1(v) + \dots, +f_s(v) \geq d(v)$ for every vertex v.

Since C_5^* is 2-connected, applying this criterion reduces to two observations: (1) C_5^* is neither a complete graph nor an odd cycle, which is obvious, and, (2) it is not a monoblock (see the definition in the introduction above or, for more explanations, [3]).

So suppose C_5^* is a monoblock w.r.t. the vector-function $(f_1^*, \dots, f_s^*)$. Since vertices x_2 and x_5 have degree 3 in C_5^*, we have $f_1^*(x_2) + \dots + f_s^*(x_2) \geq 3$. On the other hand, $f_i(x_2) \leq 2$ for every color i by assumption of Theorem 2, while $f_i^*(x_2) \leq f_i(x_2)$ for all i by definition. This implies that vector $(f_1^*(x_2), \dots, f_s^*(x_2))$ has at least two nonzero components, a contradiction.

This completes the proof of Theorem 2.

Acknowledgement

The authors thank Aleksey Glebov for useful remarks on the proof.

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