S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 5, стр. 255–278 (2008) УДК 519.17
MSC 05C15

TWO SERIES OF EDGE–4–CRITICAL GRÖTZSCH–SACHS GRAPHS GENERATED BY FOUR CURVES IN THE PLANE

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Abstract. Let G be a 4–regular planar graph and suppose that G has a cycle decomposition S (i.e., each edge of G is in exactly one cycle of the decomposition) with every pair of adjacent edges on a face always in different cycles of S. Such a graph G arises as a superposition of simple closed curves in the plane with tangencies disallowed. Graphs of this class are called Grötzsch–Sachs graphs. Two infinite families of edge–4–critical Grötzsch–Sachs graphs generated by four curves in the plane have been announced in [4]. In this paper, we present a complete proof of this result.

Keywords: planar graphs, vertex coloring, chromatic number, 4–critical graphs, Grötzsch–Sachs graphs.

1. Introduction

A simple graph is called *k–chromatic* if its chromatic number is equal to k. A graph is *edge (vertex)–4–critical* if it is vertex 4–chromatic and the removal of any edge (vertex) decreases its chromatic number. Numerous results and problems related to critical graphs can be found in the book [9]. In this paper, we deal with 4–regular planar graphs. Consider a graph $G = G(S)$ formed by the superposition of a set S of simple closed curves in the plane, no two of which are tangent and no three of which meet at a point. Vertices and edges of G correspond to crossing points and arcs of S, respectively (see, for example, Figure 1). The first results concerning coloring of graphs generated by curves in the plane are due to Grötzsch. At several international meetings on graph theory [15, 16], Sachs discussed problems related

Dobrynin, A. A., Mel'nikov, L. S., Two series of edge–4–critical Grötzsch–Sachs graphs generated by four curves in the plane.

The work is supported by Russian Foundation for Basic Research (project codes: 08–01–00516 and 08–01–00673).

Received March, 12, 2008, published June, 10, 2008.

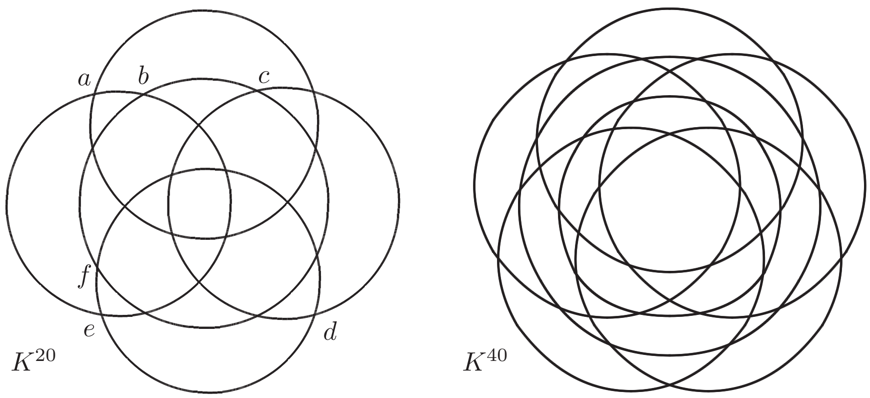


FIGURE 1. Koester's 4–regular 4–chromatic graphs.

to vertex coloring of these graphs. Such 4–regular planar graphs will be called *Grötzsch–Sachs graphs*.

Since every two closed curves have an even number of crossing points, each Grötzsch–Sachs graph has an even number of vertices. The closed curves in S can be partitioned into several *parallel* classes, where the curves in each class are pairwise disjoint. Call the minimum number of parallel classes in S the *class number* s of S or of the corresponding graph $G(S)$. Jaeger proved [6, 7] that if $s \leq 3$, then $\chi(G) \leq 3$. In 1984, Koester constructs two 4–regular 4–chromatic graphs generated by sets of 5 and 7 circles in the plane (see Figure 1) [10, 11, 12]. The first graph K^{20} has order 20 and it is neither vertex critical, nor edge critical. Its edge–4–critical subgraphs are $K^{20} - a$, $K^{20} - \{(b,c),(e,d)\}$, and $K^{20} - \{(b,c),(e,f)\}$. The second graph K^{40} of order 40 is the first example of a 4–regular edge–4–critical planar graph. It has 6 parallel classes, only one of which contains two curves. For $s = 4$, two 4–chromatic Grötzsch–Sachs graphs of order 18 have been recently presented in [1, 3] (see Figure 2).

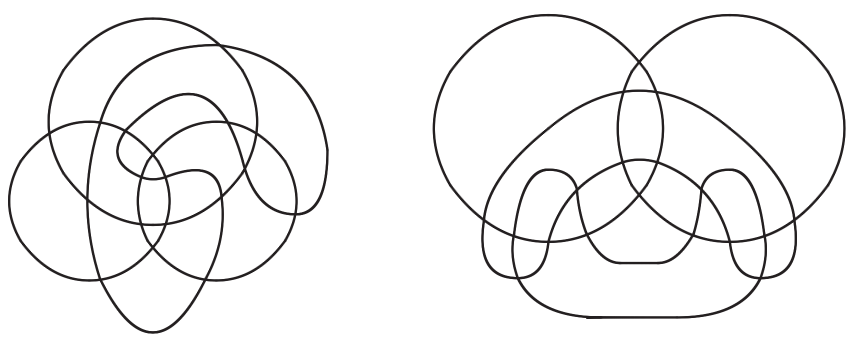
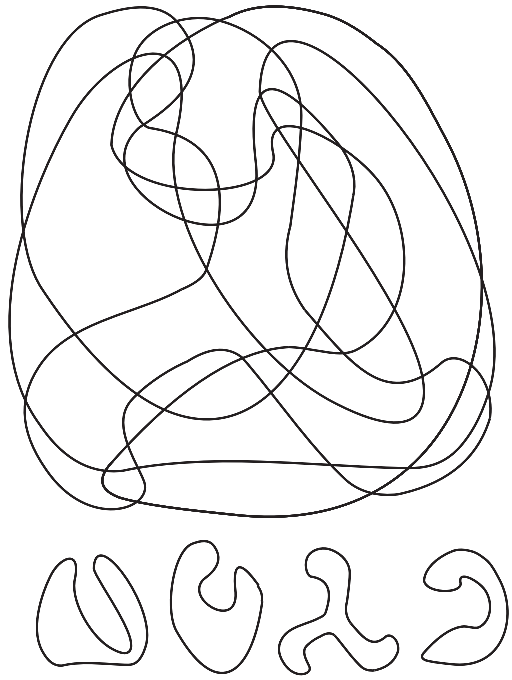


FIGURE 2. 4–chromatic Grötzsch–Sachs graphs generated by four curves.

G



H

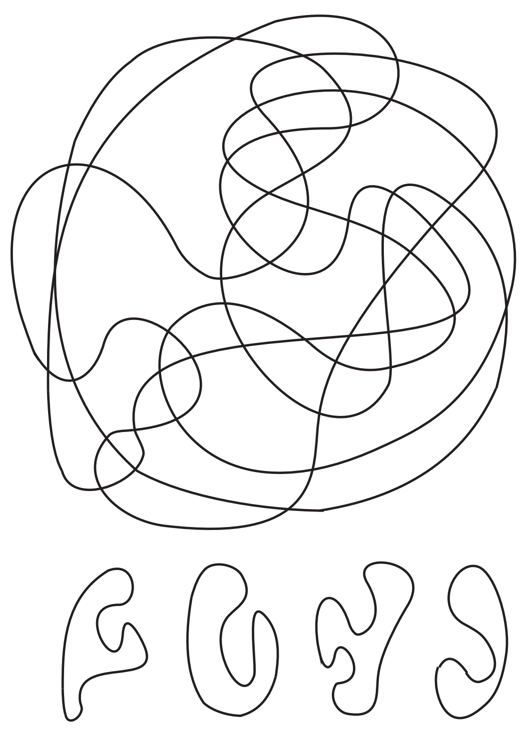


FIGURE 3. Two edge–4–critical Grötzsch–Sachs graphs with 48 vertices formed by four curves in the plane.

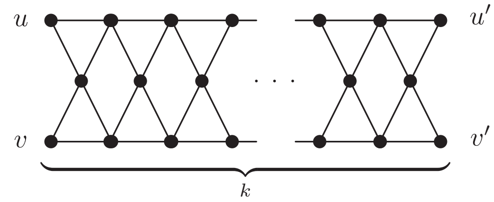


FIGURE 4. A ladder with k butterflies, LB_k.

These examples disproved Grötzsch–Sachs–Koester's conjecture on the chromatic number for graphs with $s = 4$ which stated that if $s = 4$ then $\chi(G) \le 3$ [5, 8, 11, 12, 17]. All edge–4–critical subgraphs of these graphs are described in [3]. Up to the present time, Koester's graph with class number $s = 6$ has been the only known example of an edge–4–critical Grötzsch–Sachs graph. Therefore, the problem arose whether for $s = 4, 5$ similar graphs exist, and if so, how to construct them [1]. Recent progress and open problems in this field have been discussed in [2, 14]. Two infinite families of edge–4–critical Grötzsch–Sachs graphs with $s = 4$ have been recently announced in [4]. In this paper, we present a complete proof of this result.

2. Main Result

Consider Grötzsch–Sachs graphs G and H on 48 vertices generated by four curves in the plane (see Figure 3). The corresponding curve sets are depicted below the diagrams of the graphs. We show that these graphs are edge–4–critical.

Starting from G and H, we construct infinite families of such graphs.

Theorem 1. *There exist two infinite families of edge–4–critical Grötzsch–Sachs graphs with class number 4 with* $48 + 12k$, $k \ge 0$, *vertices.*

3. Construction of graphs

To construct critical graphs, we use graphs of special type. A *butterfly* can be obtained by identifying a single vertex of two triangles. A *ladder of butterflies* (or briefly *ladder*), LB_k, consisting of k butterflies is shown in Figure 4. Vertices u, v, u', and v' will be called the *terminal* vertices of a ladder. These vertices are used for inserting a ladder into graphs. Edges of the shortest path between the terminal vertices u and u' (v and v') are called the *outer* edges of the ladder. All other ladder's edges are called the *inner* ones. Such a ladder has a useful property: the insertion of LB_{4k}, $k \ge 1$, into a Grötzsch–Sachs graph results again graphs of this class.

The regular structure of a ladder of butterflies implies the following result [13].

Lemma 1. *The chromatic number of a ladder* LB_k *is 3 for* $k \ge 1$. *For any 3–coloring* f *of* LB_k, *the following properties hold:*

1) If $f(u) = f(v)$ *then* $f(u') = f(v')$.

2) If $f(u) \ne f(v)$ *then* $f(u') \ne f(v')$. *Moreover, if k is even then* $f(u) = f(u')$ *and* $f(v) = f(v')$*; if k is odd then* $f(u) = f(v')$ *and* $f(v) = f(u')$.

Let G' be a Grötzsch–Sachs graph with the class number $s' \ge 4$. To insert a ladder of butterflies LB_{4k} into G', two non-adjacent vertices a and b lying on the boundary of some face F_0 of G' are used (see Figure 5). We assume that the generating curves crossing the vertices a and b belong to four different families and these curves pairwise intersect in G'. Vertices a and b are split into two copies a', a'' and b', b''. Next, four terminal vertices of LB_{4k} are identified with a', a'', b', and b''. For $k = 1$, this operation is illustrated in Figure 5. Let s'' be the class number of the resulting graph G''.

Lemma 2. *G'' is a Grötzsch–Sachs graph and $s'' = s'$.*

The periodic structure of LB_{4k} provides that G'' is a Grötzsch–Sachs graph for every $k \geq 1$ (see Figure 5). Since the generating curves crossing the vertices a and b pairwise intersect in G'', $s'' = s'$.

The edge critical graphs G and H are constructed from an edge non-critical 4–chromatic Grötzsch–Sachs graph G_0 of order 24 by inserting a pair of ladder LB_4 into different places. Note that after inserting a ladder into an edge critical graph, some of its edges may become non-critical.

4. Method of proof

The proof of Theorem 1 consists of two parts. First, we demonstrate that graphs G and H are 4–chromatic. Then we prove that every edge of these graphs is critical. To construct infinite families of such graphs, we use growing ladders of butterflies.

Part 1). By Brooks theorem, $\chi(G) \leq 4$ and $\chi(H) \leq 4$. Since G and H contains triangles, $\chi(G) \geq 3$ and $\chi(H) \geq 3$. Suppose that G and H are a 3–chromatic graph and try to color them. The initial step of the coloring procedure of H is to assign colors to vertices of some 5–face in all possible ways. Then we show that any extension of the initial coloring implies that it is impossible to color H by three colors.

Since the structure of the graphs is quite complicated, their coloring is not unique, i. e. we have to consider several cases. To reduce the number of cases for coloring G, another initial fragment is chosen instead of a 5–face.

Part 2). In order to prove that G and H are edge critical graphs, we apply the method of recoloring uniquely 2–colorable paths (paths may be closed). A graph coloring is called an *almost 3–coloring* if it has the unique edge with vertices of the same color. Such an edge is called *monochromatic*. Obviously, if an edge of a 4–chromatic graph is monochromatic in some almost 3–coloring, then this edge is critical.

Denote by $c(w)$ the color of vertex w. To construct a uniquely 2–colorable path, we start from a monochromatic edge $e = (u, v)$. Consider the three vertices (say, x, y, and z) distinct from u and adjacent to v. These vertices cannot all have the same color because, if they had, we could obtain a proper coloring by assuming a free color to vertex v. Without loss of generality, assume that $c(x)$ is distinct from $c(y) = c(z)$. Then a uniquely 2-colorable path, based on monochromatic edge e is defined by two colors $c(u) = c(v)$ and $c(x)$. Its initial part consists of the edges $e = (u, v)$ and (v, x). Next we consider the three vertices different from v and adjacent to x. If precisely one of then, say, r has the color of v, we add the edge (x, r) to our path, and so on. The process stops with an edge (s, t) if two of the three new vertices adjacent to t have the color of s (i.e., we cannot uniquely choose a new edge to add to the path), or when the vertex u is reached (in which case we have created a cycle of odd length).

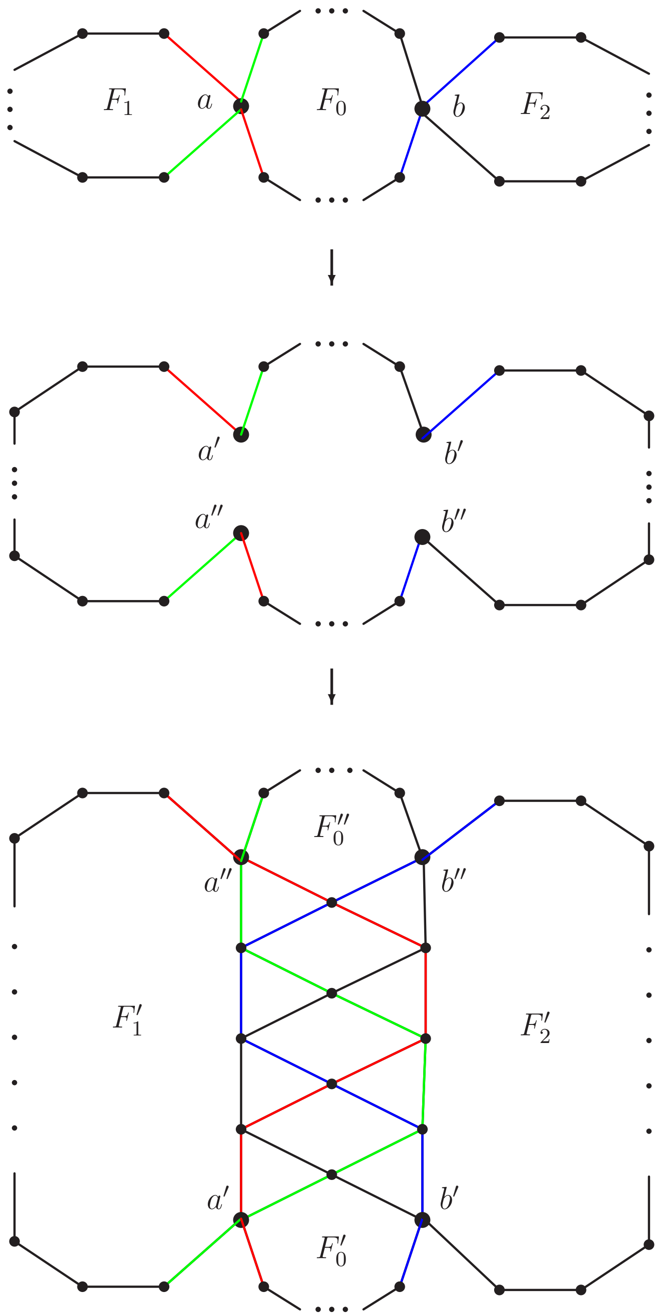


FIGURE 5. Inserting a ladder of butterflies LB_4.

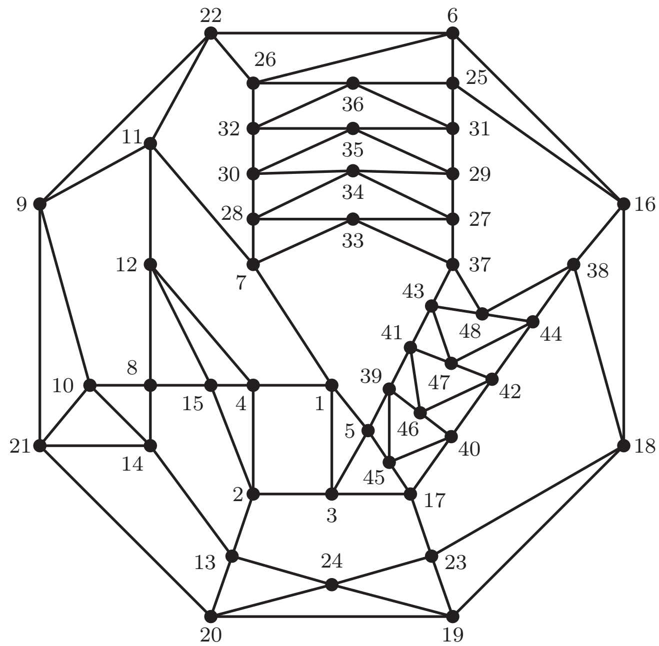


Figure 6. Vertex numbering of G.

If vertex v is given the color of vertex x then edge (v, x) becomes monochromatic and vertices u, v of edge e become properly colored. It is clear that any edge of the constructed path will be monochromatic after repeating such a recoloring. Therefore, our goal is to cover all edges of G and H by a set of uniquely 2–colorable paths with monochromatic edges in some almost 3–colorings.

5. G is 4–chromatic

Consider the vertex numbering of the graph G shown in Figure 6. This graph contains two ladders of butterflies. The first ladder LB_5 has terminal vertices 7, 37, 22, and 16 (this ladder is formed by attaching LB_4 with LB_1 of G_0). The second ladder LB_4 has terminal vertices 5, 17, 37, and 38.

Suppose that G is a 3–chromatic graph with coloring f. The initial step of the coloring procedure is to assign colors to two vertices of graph's ladder LB_4. If the color of some vertex can not be uniquely defined, a new branch of our proof arises. We depict vertex colors as a star, a triangle, or a diamond in pictures (see Figure 7) and as letters s, t, and d in the text, respectively. We consider eight cases of coloring extension named as A11, A12, A2, A3, B11, B12, B21, and B22. All these cases are illustrated in Figures 7, 8.

We choose the following initial cases of coloring of ladder's vertices 17 and 38:

Case A. $f(17) = f(38)$;
Case B. $f(17) \neq f(38)$.

Coloring for the case A.

Without loss of generality, we assume $f(17) = f(38) = t$ and $f(16) = d$ (see Figure 7 for the case A11). Colors of the following vertices are defined by colors of their neighbors: $f(18) = s$, $f(23) = d$, $f(19) = t$, $f(24) = s$, $f(20) = d$, and $f(13) = t$ (see Figure 7 for the case A11).

By Lemma 1, vertices 5 and 37 must have the same color. Consider all possible cases

Case A1. $f(5) = f(37) = t$.
Case A2. $f(5) = f(37) = d$.
Case A3. $f(5) = f(37) = s$.

Coloring for the case A1.

Consider three subcases with respect of possible color for ladder's terminal vertex 7.

Case A11. $f(7) = f(37) = t$.
Case A12. $f(7) \neq f(37)$ and $f(7) = d$.
Case A13. $f(7) \neq f(37)$ and $f(7) = s$. This subcase is impossible by Lemma 1.

Coloring for the case A11 (see Figure 7).

By Lemma 1 for LB_5, $f(22) = d$. Then we immediately obtain $f(11) = s$, $f(9) = t$, and $f(14) = t$ (vertices 9 and 14 belong to two triangles with common edge). Since (13, 14) is an edge, this coloring can not be proper.

Coloring for the case A12 (see Figure 7).

We have $f(1) = s$, $f(3) = d$, $f(2) = s$, and $f(12) = s$. By Lemma 1 for LB_5, $f(22) = t$. As a result, vertex 11 can not be properly colored.

Coloring for the case A2 (see Figure 7).

We have $f(3) = s$, $f(1) = t$, and $f(2) = d$. Then $f(4) = f(8) = s$ (see two triangles with common edge) and $f(14) = f(9) = d$. By Lemma 1 for LB_5, $f(7) = f(22) = d$. As a result, edge (9, 22) have endvertices with the same color.

Coloring for the case A3 (see Figure 7).

We have $f(3) = d$, $f(1) = t$, $f(2) = s$. Then $f(4) = f(8) = f(21) = d$. Both ends of edge (20, 21) have the same color.

Coloring for the case B.

By Lemma 1 for LB_4, $f(5) = f(17) = t$, and $f(37) = f(38) = d$ (see Figure 8 for the case B11). Further, case B can be divided into two subcases with respect of color for vertex 16.

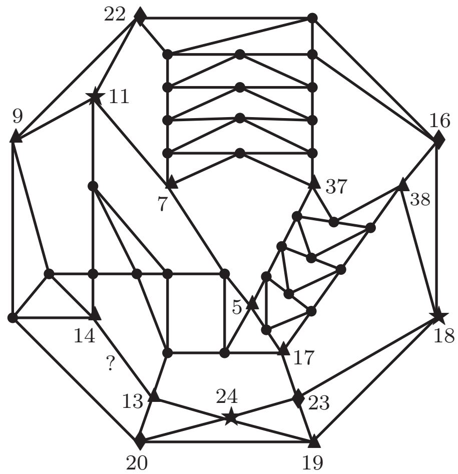
Case B1. $f(16) = t$.
Case B2. $f(16) = s$.

Coloring for the case B1 (see Figure 8 for the case B11).

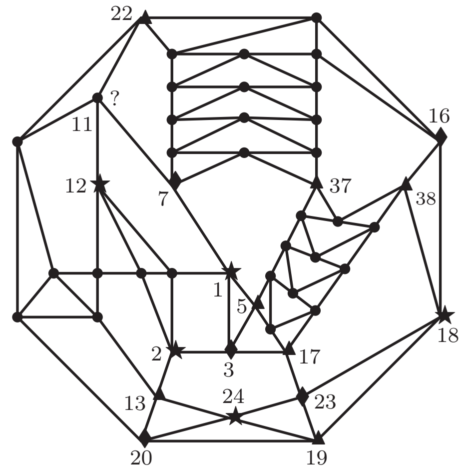
We have $f(18) = f(24) = s$, $f(23) = f(20) = d$, and $f(19) = f(13) = t$. By Lemma 1 for LB_4, $f(5) = f(17) = t$ and $f(37) = f(38) = d$. Vertex 7 can be colored in two colors.

Case B11. $f(7) = d$ (see Figure 8 for the case B11).

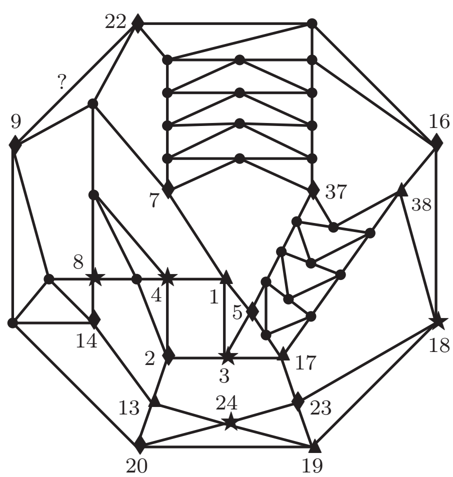
By Lemma 1 for LB_5, $f(22) = t$. $f(1) = s$, $f(3) = d$, and $f(2) = f(12) = s$. Vertex 11 can not be properly colored.



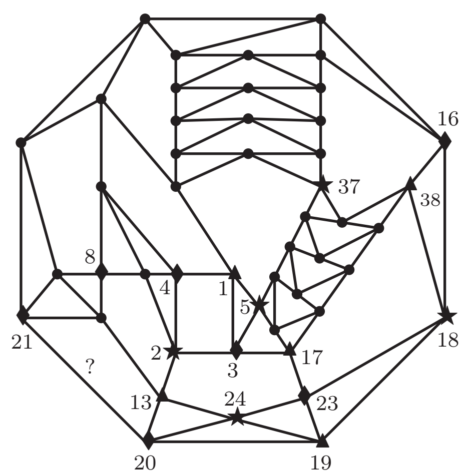
Case A11.



Case A12.

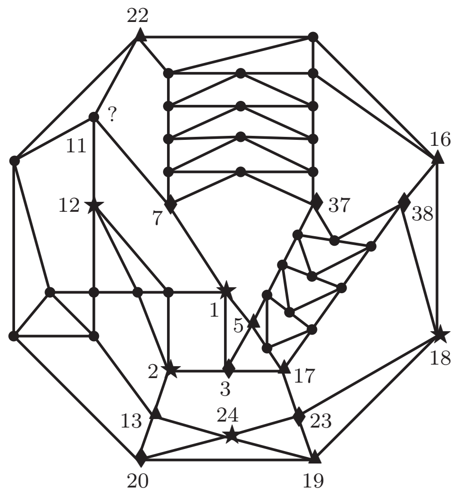


Case A2.

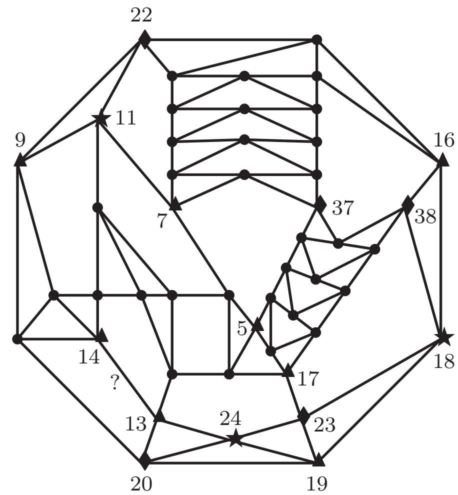


Case A3.

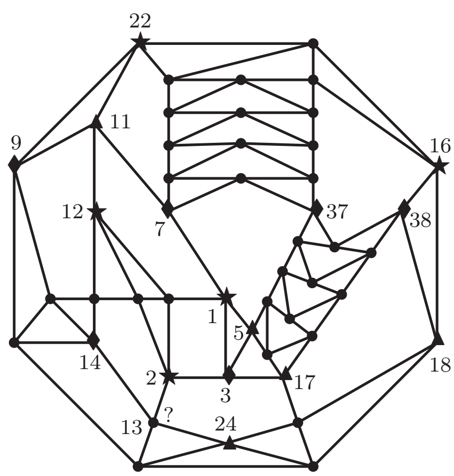
Figure 7. Extension of colorings for the case A.



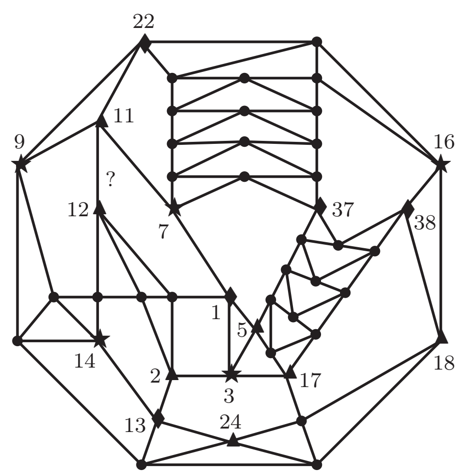
Case B11.



Case B12.



Case B21.



Case B22.

Figure 8. Extension of colorings for the case B.

Case B12. $f(7) = t$ (see Figure 8 for the case B12).

By Lemma 1 for LB_5, $f(22) = d$. Then $f(11) = s$, and $f(9) = f(14) = t$. Adjacent vertices 13 and 14 have the same color.

Coloring for the case B2.

We have $f(18) = f(24) = t$. Vertex 7 can be colored in two colors.

Case B21. $f(7) = d$. (see Figure 8 for the case B21).

By Lemma 1 for LB_5, $f(22) = s$. Then $f(11) = t$, $f(9) = f(14) = d$, and $f(1) = s$, $f(3) = d$. Since vertices 12 and 2 are adjacent with vertices with colors d and t, they must have the same color s. One can see that vertex 13 can not be properly colored.

Case B22. $f(7) = s$ (see Figure 8 for the case B22).

By Lemma 1 for LB_5, $f(22) = d$. Then $f(11) = t$, $f(9) = f(14) = s$ and $f(13) = d$. Next, $f(1) = d$, $f(3) = s$, and $f(2) = f(12) = t$. Vertices 11 and 12 are adjacent and have the same color t.

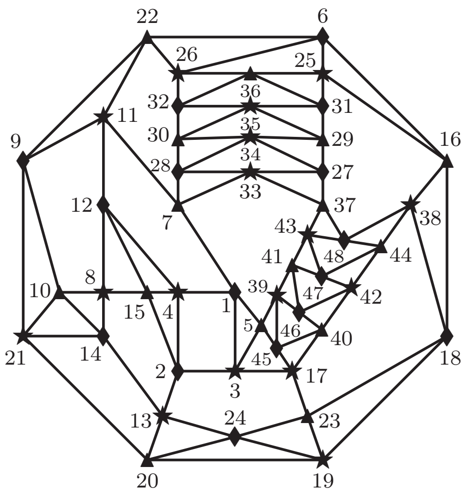
All possible cases of 3–coloring of the graph G are considered. This implies that G is a 4–chromatic graph.

6. G IS CRITICAL

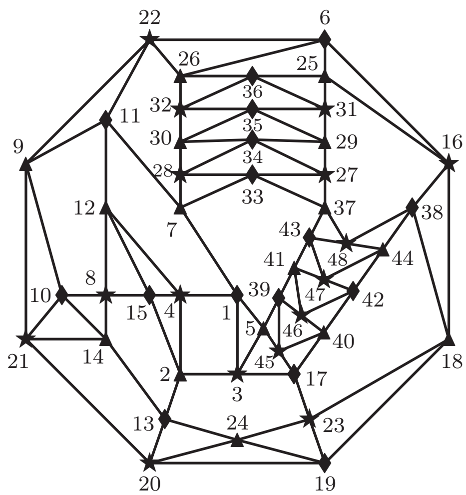
Consider four almost colorings of G named as A, B, C, and D shown in Figure 9. A list of vertices of uniquely 2–colorable paths is presented below for each almost coloring. Every path starts with its monochromatic edge. We will denote a uniquely 2–colorable path as $P_{ds}(u, v)$, where d and s are vertex colors in this path and u, v are endvertices of the path. Table 1 demonstrates that every edge of the graph is covered by uniquely 2–colorable paths from the corresponding almost colorings A–D.

TABLE 1. Edges of G and the corresponding almost 3–colorings.

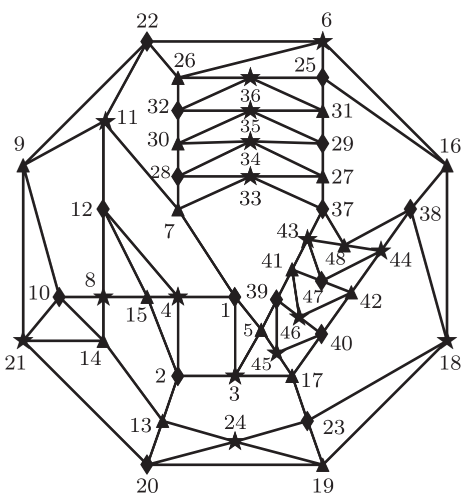
(1,3)	BD	(1,4)	B	(1,5)	C	(1,7)	CD	(2,3)	BD
(2,4)	BD	(2,13)	D	(2,15)	D	(3,5)	AB	(3,17)	AB
(4,12)	B	(4,15)	B	(5,39)	AC	(5,45)	B	(6,16)	C
(6,22)	D	(6,25)	D	(6,26)	C	(7,11)	AC	(7,28)	C
(7,33)	AC	(8,10)	B	(8,12)	B	(8,14)	B	(8,15)	B
(9,10)	C	(9,11)	C	(9,21)	C	(9,22)	C	(10,14)	C
(10,21)	B	(11,12)	D	(11,22)	AD	(12,15)	D	(13,14)	C
(13,20)	C	(13,24)	ACD	(14,21)	BC	(16,18)	BC	(16,25)	AB
(16,38)	A	(17,23)	C	(17,40)	AC	(17,45)	A	(18,19)	ACD
(18,23)	B	(18,38)	AD	(19,20)	C	(19,23)	C	(19,24)	ACD
(20,21)	B	(20,24)	B	(22,26)	AC	(23,24)	B	(25,31)	B
(25,36)	AD	(26,32)	C	(26,36)	AC	(27,29)	B	(27,33)	C
(27,34)	C	(27,37)	B	(28,30)	C	(28,33)	D	(28,34)	D
(29,31)	B	(29,34)	D	(29,35)	D	(30,32)	C	(30,34)	C
(30,35)	C	(31,35)	C	(31,36)	C	(32,35)	D	(32,36)	D
(33,37)	AD	(37,43)	AD	(37,48)	B	(38,44)	AD	(38,48)	A
(39,41)	AC	(39,45)	A	(39,46)	A	(40,42)	AC	(40,45)	B
(40,46)	B	(41,43)	AD	(41,46)	BD	(41,47)	BC	(42,44)	AD
(42,46)	AD	(42,47)	AC	(43,47)	A	(43,48)	A	(44,47)	B
(44,48)	B								



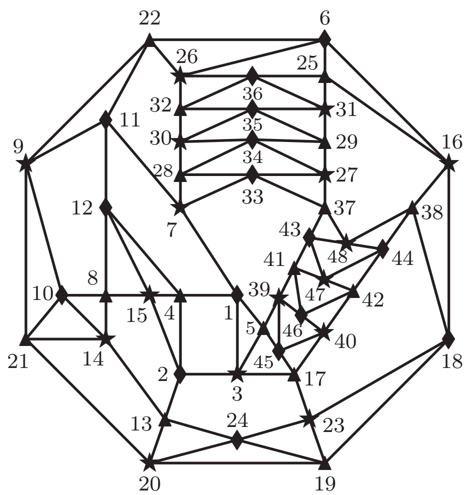
A. Almost 3–colorings of G with monochromatic edge (3,17)



B. Almost 3–colorings of G with monochromatic edge (20,21)



C. Almost 3–colorings of G with monochromatic edge (13,14)



D. Almost 3–colorings of G with monochromatic edge (11,12)

FIGURE 9. Covering paths for edges of G.

$$
\begin{aligned}
A:\ P_{st}(17,17) &= (17,3,5,39,41,43,37,33,7,11,22,26,36,25,16,38,44,42,\\
&\quad 40,17).\\
A:\ P_{sd}(3,13) &= (3,17,45,39,46,42,47,43,48,38,18,19,24,13).\\
B:\ P_{st}(21,21) &= (21,20,24,23,18,16,25,31,29,27,37,48,44,47,41,46,40,\\
&\quad 45,5,3,2,4,12,8,14,21).\\
B:\ P_{sd}(20,17) &= (20,21,10,8,15,4,1,3,17).\\
C:\ P_{td}(13,13) &= (13,14,10,9,22,26,32,30,28,7,1,5,39,41,47,42,40,17,\\
&\quad 23,19,20,13).\\
C:\ P_{ts}(14,14) &= (14,13,24,19,18,16,6,26,36,31,35,30,34,27,33,7,11,9,\\
&\quad 21,14).\\
D:\ P_{dt}(12,4) &= (12,11,22,6,25,36,32,35,29,34,28,33,37,43,41,46,42,\\
&\quad 44,38,18,19,24,13,2,4).\\
D:\ P_{ds}(11,7) &= (11,12,15,2,3,1,7).
\end{aligned}
$$

7. An infinite family based on G

To construct an infinite family of edge–4–critical Grötzsch–Sachs graphs, one can replace the ladders of butterflies LB_4 and LB_5 in G by longer ladders LB_{4i}, $i \geq 1$, and LB_{4j+1}, $j \geq 1$, respectively. The obtained graphs G_{ij} of order $24 + 12(i + j)$ are 4–chromatic graphs for every $i, j \geq 1$. The proof that $\chi(G_{ij}) = 4$ is exactly the same as for the graph G. Based on the previous considerations, it is easy to verify that all edges of new butterflies will belong to covering paths. Indeed, the paths of almost 3–coloring A covers all outer edges of LB_{4i} and the paths of almost 3–colorings A and B cover all inner edges of LB_{4i}. The paths of almost 3–colorings B and C cover all outer edges of LB_{4j+1} and the paths of almost 3–colorings C and D cover all inner edges of LB_{4j+1}.

As a result, an infinite family of edge–4–critical graphs G_n with order $48 + 12n$, $n \geq 0$, is obtained. The graph G is the initial member of this family.

The properties of ladders with $4k$ butterflies imply that all members of the constructed family are Grötzsch–Sachs graphs generated by four curve in the plane (class number $s = 4$).

8. H is 4–chromatic

Consider the vertex numbering of the graph H shown in Figure 10. This graph contains two ladders. The first ladder LB_4 has terminal vertices 7, 5, 26, and 25. The second ladder LB_5 has terminal vertices 17, 25, 38, and 37.

Suppose that the graph is 3–chromatic. We choose 5–face (2, 15, 8, 14, 13) for initial coloring. To color vertices of any 5–face in a 3–chromatic graph, one needs exactly 3 colors. One vertex of a 5–face has a color that is distinct from the colors of the other four vertices. Assume that this color is always depicted as a star. This vertex will be called *initial vertex*.

For H, it is sufficient to consider initial vertices 2, 15 and 8 only. Indeed, if the initial vertex is 14 or 13 then vertices 2 and 8 have the same color. This implies that adjacent vertices 4 and 12 must have the same color. Therefore, the main cases of coloring are

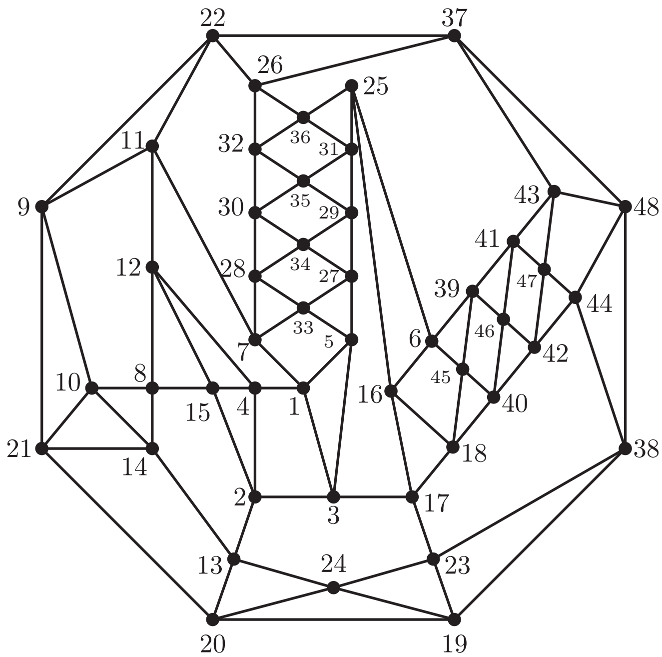


FIGURE 10. Vertex numbering of the graph H.

Case A. 2 — initial vertex;
Case B. 15 — initial vertex;
Case C. 8 — initial vertex.

8.1. **Coloring for the case A.** For the initial face, $f(2) = s$, $f(15) = d$, $f(8) = t$, $f(14) = d$, and $f(13) = t$ (see Figure 11 for the case A11). From triangles, we have $f(4) = t$, $f(12) = f(10) = s$, $f(21) = t$, $f(9) = d$, $f(11) = t$, and $f(22) = s$. Since vertices 13 and 19 belong to two triangles with common edge, $f(19) = t$. Vertex 1 can be colored in two colors d and s.

Case A1. $f(1) = d$;
Case A2. $f(1) = s$.

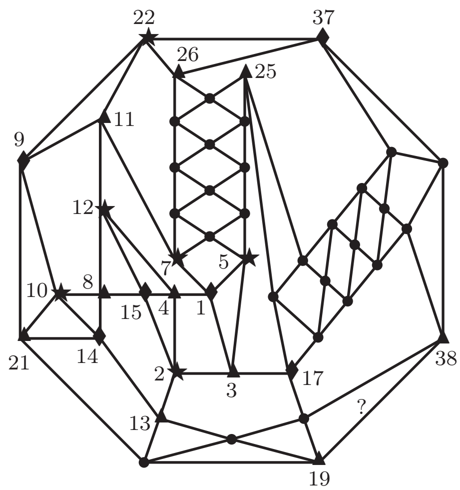
Coloring for the case A1 (see Figure 11 for the case A11).

Let $f(1) = d$. Then $f(7) = s$, $f(3) = t$, and $f(5) = s$. Vertex 26 can be colored in colors t and d.

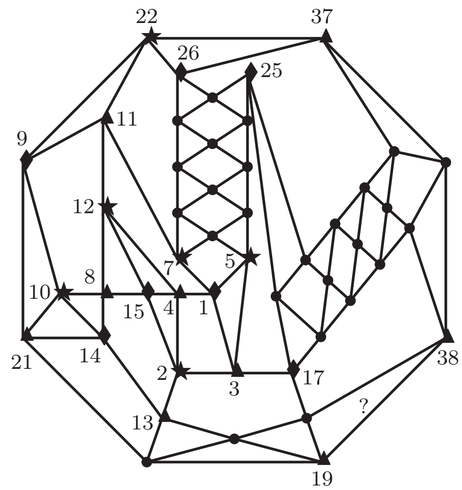
Case A11. $f(26) = t$;
Case A12. $f(26) = d$.

Coloring for the case A11 (see Figure 11).

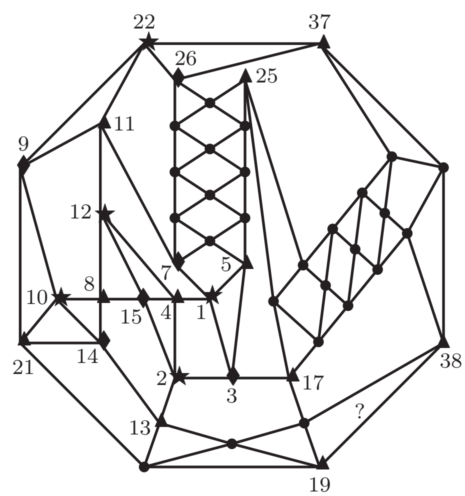
Let $f(26) = t$. Then $f(37) = d$. By Lemma 1 for LB_4, $f(25) = t$. By Lemma 1 for LB_5, $f(17) \in \{d, t\}$. Since vertex 17 is adjacent with a vertex colored as t, $f(17) = d$ and, therefore, $f(38) = t$. As a result, edge (19, 38) can not be properly colored.



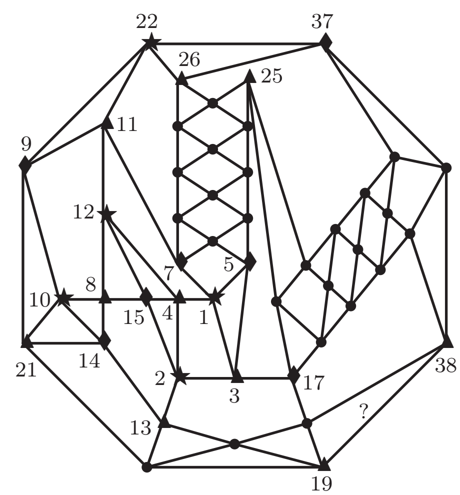
Case A11.



Case A12.



Case A21.



Case A221.

Figure 11. Extension of colorings for the case A.

Coloring for the case A12 (see Figure 11).

Let $f(26) = d$. Then $f(37) = t$. By Lemma 1 for LB_4, $f(25) = d$. By Lemma 1 for LB_5, $f(17) \in \{d, t\}$. Since vertex 17 is adjacent with a vertex colored by t, $f(17) = d$. Then $f(38) = t$. We see that edge (19, 38) can not be properly colored.

Coloring for the case A2 (see Figure 11 for the case A21).

Let $f(1) = s$. Then $f(7) = d$. Vertex 5 can be colored in two colors t and d.

Case A21. $f(5) = t$;
Case A22. $f(5) = d$.

Coloring for the case A21 (see Figure 11).

Let $f(5) = t$. Then $f(3) = d$. By Lemma 1 for LB_4, $f(26) = d$ and $f(25) = t$. Next, $f(37) = t$. By Lemma 1 for LB_5, $f(17) = f(38) = t$. Edge (19, 38) can not be properly colored.

Coloring for the case A22 (see Figure 11 for the case A221).

Let $f(5) = d$. Then $f(3) = t$. Vertex 26 can be colored in colors t and d.

Case A221. $f(26) = t$;
Case A222. $f(26) = d$.

Coloring for the case A221 (see Figure 11).

Let $f(26) = t$. Then $f(37) = d$. By Lemma 1 for LB_4, $f(25) = t$. By Lemma 1 for LB_5, $f(17) = d$ and $f(38) = t$. Endvertices of edge (19, 38) have the same color.

Coloring for the case A222 (see Figure 12).

Let $f(26) = d$. Then $f(37) = t$. By Lemma 1 for LB_4, $f(25) = d$. By Lemma 1 for LB_5, $f(17) = d$ and $f(38) = t$. Edge (19, 38) can not be properly colored.

8.2. **Coloring for the case B.** For the initial face, $f(15) = s$, $f(8) = t$, $f(14) = d$, $f(13) = t$, and $f(2) = d$ (see Figure 12 for the case B11). From triangles, we have $f(4) = t$, $f(12) = d$, and $f(19) = t$. Vertex 1 can be colored in two colors s and d.

Case B1. $f(1) = s$;
Case B2. $f(1) = d$.

Coloring for the case B1 (see Figure 12 for the case B11).

Let $f(1) = s$. Then $f(3) = t$ and $f(5) = d$. Vertex 23 can be colored in two colors s and d.

Case B11. $f(23) = s$;
Case B12. $f(23) = d$.

Coloring for the case B11 (see Figure 12).

Let $f(23) = s$. Then $f(17) = f(38) = d$. By Lemma 1 for LB_5, $f(25) = f(37) = d$. By Lemma 1 for LB_4, $f(26) = f(7) = t$. We have $f(22) = s$. Vertex 11 can not be properly colored.

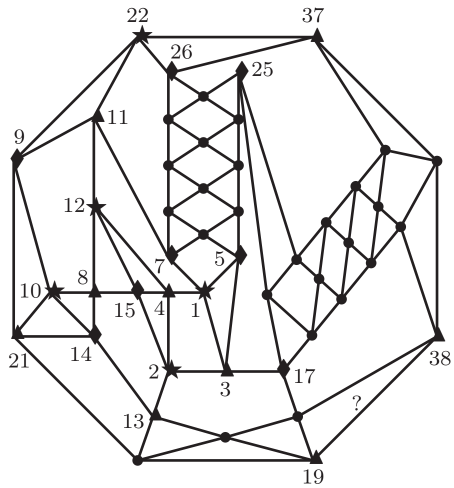
Coloring for the case B12 (see Figure 12).

Let $f(23) = d$. Then $f(17) = f(38) = s$. By Lemma 1 for LB_5, $f(25) = f(37) = s$. By Lemma 1 for LB_4, $f(26) = s$. Edge (26, 37) can not be properly colored.

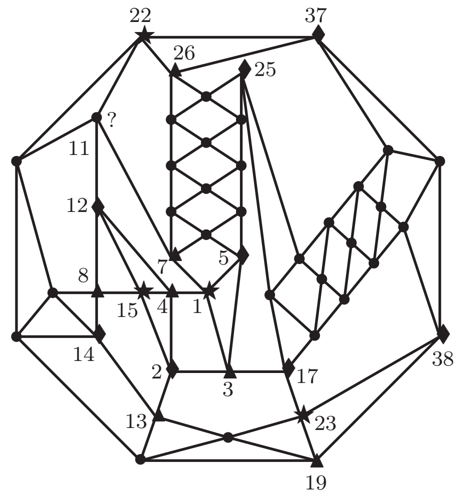
Coloring for the case B2 (see Figure 12 for the case B211).

Let $f(1) = d$. Vertex 11 can be colored in two colors t and s.

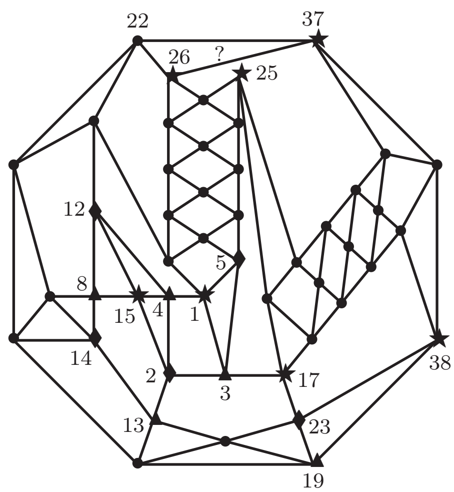
Case B21. $f(11) = t$;
Case B22. $f(11) = s$.



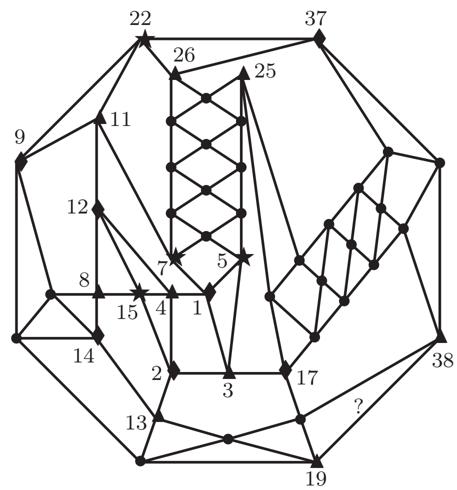
Case A222.



Case B11.



Case B12.



Case B211.

FIGURE 12. Extension of colorings for the cases A and B.

Coloring for the case B21.

Let $f(11) = t$. Then $f(7) = s$, $f(9) = d$ (see two triangles with common edge), and $f(22) = s$. Vertex 26 can be colored in two colors t and d.

Case B211. $f(26) = t$;
Case B212. $f(26) = d$.

Coloring for the case B211 (see Figure 12).

Let $f(26) = t$. Then $f(37) = d$. By Lemma 1 for LB_4, $f(25) = t$ and $f(5) = s$. Then $f(3) = t$. By Lemma 1 for LB_5, $f(17) = d$ and $f(38) = t$. Edge (19, 38) can not be properly colored.

Coloring for the case B212 (see Figure 13).

Let $f(26) = d$. Then $f(37) = t$. By Lemma 1 for LB_4, $f(25) = d$ and $f(5) = s$. Then $f(3) = t$. By Lemma 1 for LB_5, $f(17) = d$ and $f(38) = t$. Edge (19, 38) can not be properly colored.

Coloring for the case B22 (see Figure 13 for the case B221).

Let $f(11) = s$. Then $f(7) = t$, $f(9) = d$ (see two triangles with common edge), and $f(22) = t$. Vertex 26 can be colored in two colors s and d.

Case B221. $f(26) = s$;
Case B222. $f(26) = d$.

Coloring for the case B221 (see Figure 13).

Let $f(26) = s$. Then $f(37) = d$. By Lemma 1 for LB_4, $f(25) = s$ and $f(5) = t$. Then $f(3) = s$. By Lemma 1 for LB_5, $f(17) = d$ and $f(38) = s$. Vertex 23 can not be properly colored.

Coloring for the case B222 (see Figure13).

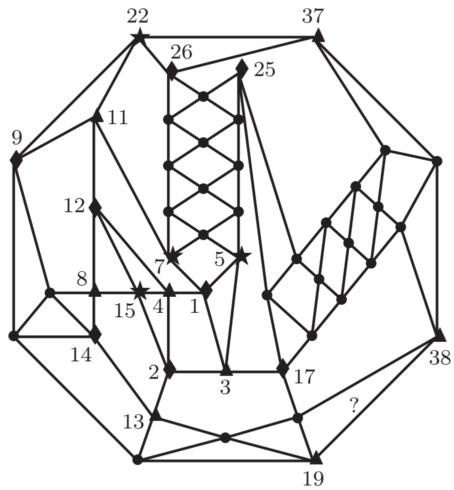
Let $f(26) = d$. Then $f(37) = s$. By Lemma 1 for LB_4, $f(25) = d$ and $f(5) = t$. Then $f(3) = s$. By Lemma 1 for LB_5, $f(17) = d$ and $f(38) = s$. Vertex 23 can not be properly colored.

8.3. **Coloring for the case C.** For the initial face, $f(8) = s$, $f(14) = d$, $f(13) = t$, $f(2) = d$, and $f(15) = t$ (see Figure 13 for the case C11). From triangles, we have $f(10) = t$, $f(21) = s$, $f(9) = d$, $f(4) = s$, $f(19) = t$, $f(20) = f(23) = d$, and $f(38) = s$. Vertex 3 can be colored in two colors s and t.

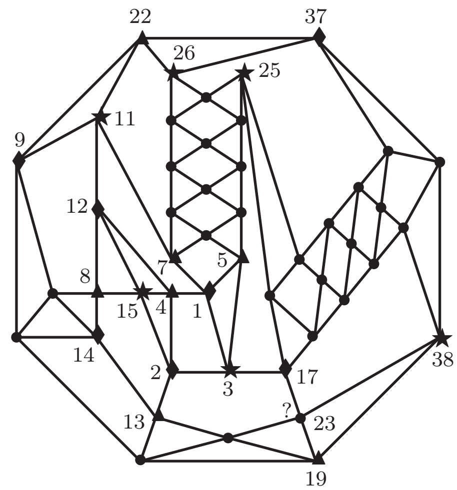
Case C1. $f(3) = s$;
Case C2. $f(3) = t$.

Coloring for the case C11 (see Figure 13).

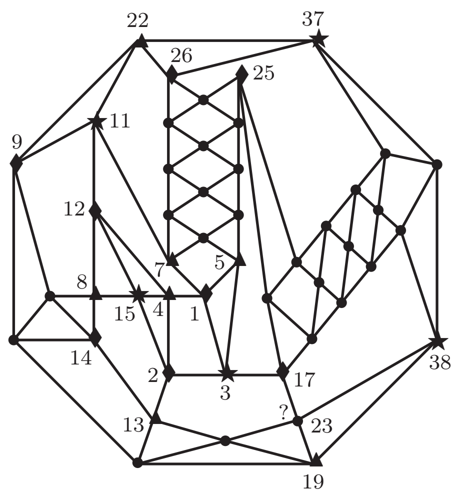
Let $f(37) = s$. Then $f(22) = t$ and $f(26) = d$. By Lemma 1 for LB_5, $f(25) = t$. By Lemma 1 for LB_4, $f(7) = d$ and $f(5) = t$. Vertex 1 can not be properly colored.



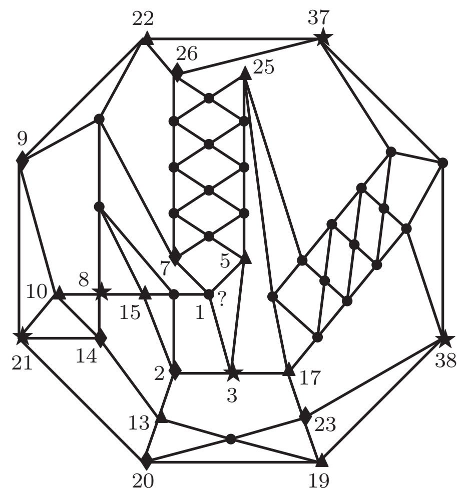
Case B212.



Case B221.

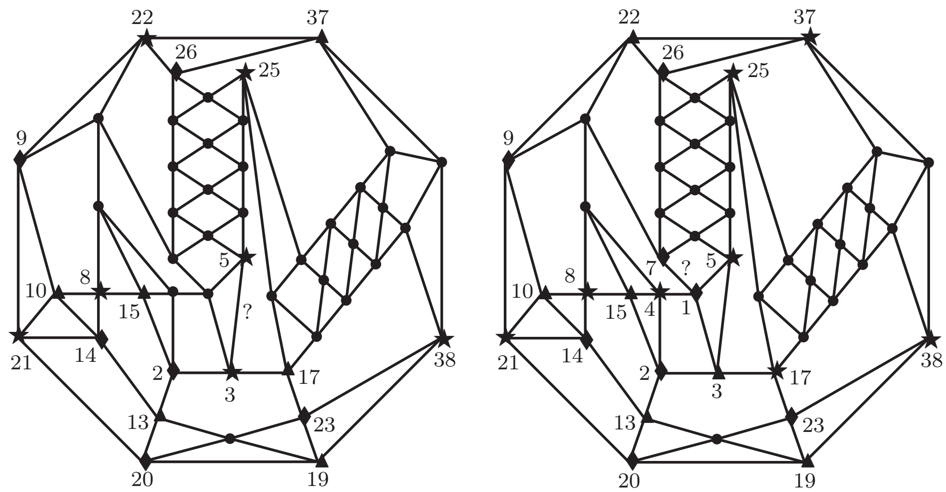


Case B222.



Case C11.

FIGURE 13. Extension of colorings for the cases B and C.



Case C12. Case C2.

Figure 14. Extension of colorings for the case C.

Coloring for the case C1 (see Figure 13 for the case C11).

Let $f(3) = s$. Then $f(17) = t$. By Lemma 1 for LB_5, vertex 37 can be colored in two colors s and t.

Case C11. $f(37) = s$;
Case C12. $f(37) = t$.

Coloring for the case C12 (see Figure 14).

Let $f(37) = t$. Then $f(22) = s$ and $f(26) = d$. By Lemma 1 for LB_5, $f(25) = s$. By Lemma 1 for LB_4, $f(5) = s$. Edge (3, 5) can not be properly colored.

Coloring for the case C2 (see Figure 14).

Let $f(3) = t$. Then $f(1) = d$ and $f(5) = f(17) = s$. By Lemma 1 for LB_5, $f(25) = f(37) = s$. Then $f(22) = t$ and $f(26) = d$. By Lemma 1 for LB_4, $f(7) = d$. Edge (1, 7) can not be properly colored.

All possible cases of 3–coloring of the graph H are examined. This implies that H is a 4–chromatic graph.

9. H is critical

Consider six almost colorings of H named as A, B, C, D, E, and F shown in Figures 15, 16. A list of vertices of uniquely 2–colorable paths is presented below for each almost coloring. Every path starts with its monochromatic edge. A uniquely 2–colorable path is denoted by $P_{ds}(u, v)$, where d and s are vertex colors in this path and u, v are endvertices of the path. Table 2 shows that every edge of the graph is covered by uniquely 2–colorable paths from the corresponding almost colorings A–F.

Table 2. Edges of H and the corresponding almost 3–colorings.

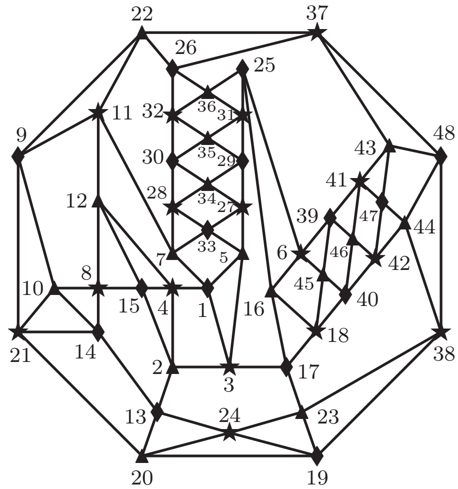
(1,3)	ACDE	(1,4)	ACE	(1,5)	E	(1,7)	DE	(2,3)	CDE
(2,4)	CF	(2,13)	F	(2,15)	E	(3,5)	BC	(3,17)	ABC
(4,12)	C	(4,15)	AC	(5,27)	BC	(5,33)	E	(6,16)	B
(6,25)	AE	(6,39)	AE	(6,45)	B	(7,11)	B	(7,28)	B
(7,33)	D	(8,10)	C	(8,12)	C	(8,14)	AC	(8,15)	AC
(9,10)	ABD	(9,11)	B	(9,21)	B	(9,22)	ABD	(10,14)	ABD
(10,21)	C	(11,12)	EF	(11,22)	EF	(12,15)	E	(13,14)	ABD
(13,20)	AB	(13,24)	ABF	(14,21)	BC	(16,17)	AB	(16,18)	C
(16,25)	AC	(17,18)	AB	(17,23)	AB	(18,40)	AB	(18,45)	C
(19,20)	AB	(19,23)	AB	(19,24)	ABF	(19,38)	ABF	(20,21)	C
(20,24)	C	(22,26)	AD	(22,37)	BEF	(23,24)	C	(23,38)	C
(25,31)	AC	(25,36)	AE	(26,32)	A	(26,36)	AD	(26,37)	A
(27,29)	ABC	(27,33)	AD	(27,34)	D	(28,30)	AB	(28,33)	AE
(28,34)	E	(29,31)	AC	(29,34)	E	(29,35)	BE	(30,32)	A
(30,34)	D	(30,35)	BD	(31,35)	D	(31,36)	D	(32,35)	E
(32,36)	E	(37,43)	EF	(37,48)	AB	(38,44)	BF	(38,48)	AC
(39,41)	AEF	(39,45)	CF	(39,46)	C	(40,42)	ABF	(40,45)	BF
(40,46)	B	(41,43)	EF	(41,46)	B	(41,47)	AB	(42,44)	BF
(42,46)	C	(42,47)	AC	(43,47)	C	(43,48)	C	(44,47)	B
(44,48)	B								

$$
\begin{array}{lcl}
A:\ P_{ds}(14,14) & = & (14,13,24,19,38,48,37,26,32,30,28,33,27,29,31,25,6,\\
 & & \ 39,41,47,42,40,18,17,3,1,4,15,8,14).\\
A:\ P_{dt}(13,13) & = & (13,14,10,9,22,26,36,25,16,17,23,19,20,13).\\
B:\ P_{ts}(14,14) & = & (14,13,24,19,38,44,42,40,18,17,3,5,27,29,35,30,28,7,\\
 & & \ 11,9,21,14).\\
B:\ P_{td}(13,13) & = & (13,14,10,9,22,37,48,44,47,41,46,40,45,6,16,17,23,19,\\
 & & \ 20,13).\\
C:\ P_{st}(21,21) & = & (21,20,24,23,38,48,43,47,42,46,39,45,18,16,25,31,29,\\
 & = & \ 27,5,3,2,4,12,8,14,21).\\
C:\ P_{sd}(20,17) & = & (20,21,10,8,15,4,1,3,17).\\
D:\ P_{sd}(2,13) & = & (2,3,1,7,33,27,34,30,35,31,36,26,22,9,10,14,13).\\
E:\ P_{dt}(12,4) & = & (12,11,22,37,43,41,39,6,25,36,32,35,29,34,28,33,5,\\
 & = & \ 1,4).\\
E:\ P_{ds}(11,7) & = & (11,12,15,2,3,1,7).\\
F:\ P_{dt}(12,12) & = & (12,11,22,37,43,41,39,45,40,42,44,38,19,24,13,2,4).
\end{array}
$$

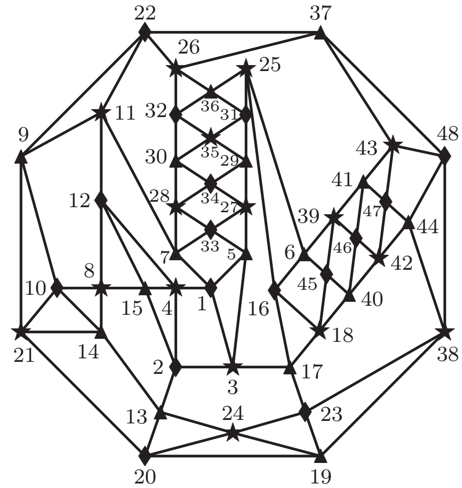
10. An infinite family based on H

To construct an infinite family of edge–4–critical Grötzsch–Sachs graphs, one can insert to graph H more long ladders of butterflies LB_{4i} and LB_{4j+1} instead of ladders LB_4 and LB_5, respectively. Based on the previous considerations, it is easy to verify that all edges of new butterflies will belong to covering paths. Indeed, the paths of almost 3–colorings A and B cover all outer edges of LB_{4i} and the paths of almost 3–colorings D and E cover all inner edges of LB_{4i}. The paths of almost 3–colorings B and E cover all outer edges of LB_{4j+1} and the paths of almost 3–colorings B and C cover all inner edges of LB_{4j+1}.

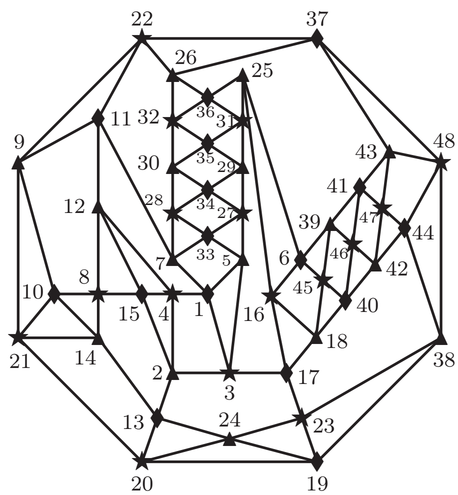
As a result, an infinite family of edge–4–critical graphs H_n with order $48+12n$, $n \geq 0$, is obtained. The graph H is the initial member of this family.



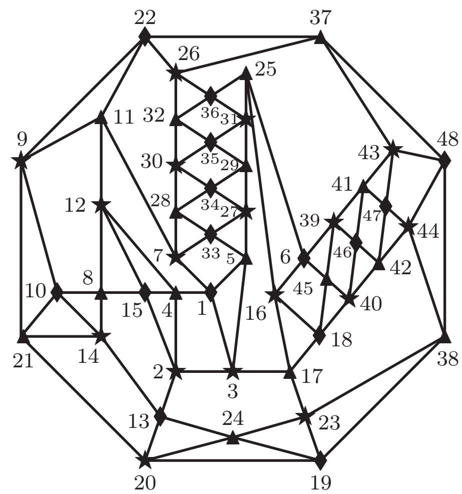
A. Almost 3–colorings of H with monochromatic edge (13, 14).



B. Almost 3–colorings of H with monochromatic edge (13, 14).

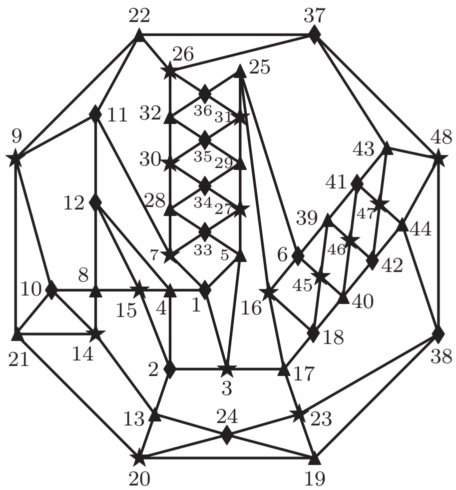


C. Almost 3–colorings of H with monochromatic edge (20, 21).

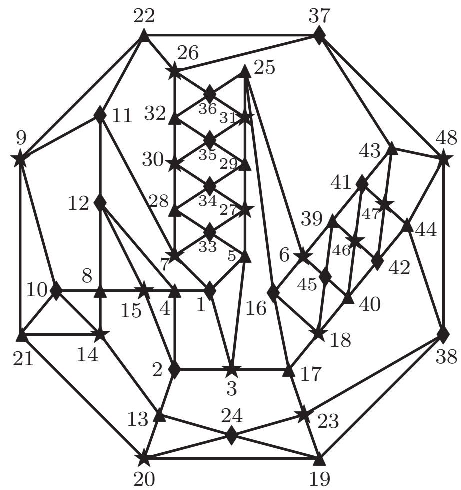


D. Almost 3–colorings of H with monochromatic edge (2, 3).

FIGURE 15. Covering paths for edges of H.



E. Almost 3–colorings of H with monochromatic edge (11, 12).



F. Almost 3–colorings of H with monochromatic edge (11, 12).

FIGURE 16. Covering paths for edges of H (*continue*).

The properties of ladders with $4k$ butterflies imply that all members of the constructed family are Grötzsch–Sachs graphs generated by four curve in the plane (class number $s = 4$).

11. Open problems

In the previous sections, edge–4–critical Grötzsch–Sachs graphs generated by four curves in the plane are obtained. They have the class number $s = 4$ and the minimal such graphs have 48 vertices.

Problem 1. *Find edge–4–critical Grötzsch–Sachs graphs with the class number* $s = 4$ *and order less than 48.*

Problem 2. *Is it true that for every* $s \geq 5$ *there exists an infinite family of edge–4–critical Grötzsch–Sachs graphs with the class number* s*?*

It is easy to see that a Venn diagram generates a Grötzsch–Sachs graph.

Conjecture 1. *Grötzsch–Sachs graphs that are Venn diagrams are 3–chromatic.*

Acknowledgement

The authors thank Prof. Horst Sachs and Prof. Gerhard Koester for their valuable comments and suggestions. We also grateful to the referee for improving the readability of the paper.

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