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PERFECT COLORINGS OF RADIUS $r > 1$ OF THE INFINITE RECTANGULAR GRID

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Abstract. A coloring of vertices of a graph G with n colors is called *perfect of radius* r if the number of vertices of each color in a ball of radius r depends only on the color of the center of this ball. Perfect colorings of radius 1 have been studied before under different names including equitable partitions. The notion of perfect coloring is a generalization of the notion of a perfect code, in fact, a perfect code is a special case of a perfect coloring. We consider perfect colorings of the graph of the infinite rectangular grid. Perfect colorings of the infinite rectangular grid can be interpreted as two-dimensional words over a finite alphabet of colors. We prove that every perfect coloring of radius $r > 1$ of this graph is periodic.

Keywords: perfect coloring, equitable partition, perfect code, graph of the infinite rectangular grid.

1. Introduction

Let $G = (V, E)$ be a graph, r a positive integer. The distance between two vertices $\mathbf{x}$ and $\mathbf{y}$, denoted by $d(\mathbf{x}, \mathbf{y})$, is defined as the usual graph metric. By an *r-neighborhood of a vertex* $\mathbf{x}$ we mean the following set of vertices:

$$N_r(\mathbf{x}) = \{\mathbf{y} \in V \mid 1 \leq d(\mathbf{x}, \mathbf{y}) \leq r\}.$$

A vertex *coloring* of the graph G with n colors is a mapping:

$$\varphi : V \to \{1, ..., n\}.$$

Let $A = (a_{ij})_{i,j=1}^n$ be an integer nonnegative matrix. If the number of vertices of a color j in the r-neighborhood of a vertex $\mathbf{x}$ of a color i does not depend on the

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choice of $\mathbf{x}$ and is equal to a_{ij}, then the coloring is called *perfect of radius* r with the matrix A. Properties of perfect colorings of radius 1 have been studied under different names, e. g., equitable partitions [5].

We consider perfect colorings of the graph $G(\mathbb{Z}^2)$, that is the infinite rectangular grid. We say that a matrix A is *admissible* if there exists a perfect coloring of $G(\mathbb{Z}^2)$ with the matrix A for the appropriate r.

The aim of this paper is to prove that every perfect coloring of radius $r \geq 2$ of the infinite rectangular grid is periodic. Perfect colorings of the infinite rectangular grid can be interpreted as two-dimensional words over the finite alphabet of colors. We use the technique of R-prolongable words, which was introduced in [10] and used for studying another type of two-dimensional words called centered functions.

Notice that the case $r \geq 2$ is completely different from the case $r = 1$. There exist non-periodic perfect colorings of radius 1. However, in [8] it was proved that for any admissible matrix of a perfect coloring of radius 1 a periodic perfect coloring exists. Moreover, all non-periodic perfect colorings of radius 1 can be obtained from periodic ones by switchings of binary diagonals. A binary diagonal is a diagonal that consists of two alternating colors, its switching is switching colors inside the diagonal.

In [1], Axenovich classified all admissible matrices and all perfect colorings of radius 1 with 2 colors of the infinite rectangular grid and found some necessary conditions for a matrix of perfect coloring to be admissible for radius $r \geq 2$.

The notion of a perfect coloring is a generalization of the notion of a perfect code. Let $G = (V, E)$ be a graph, $C \subseteq V$. A *perfect 1-error correcting code* is a subset C of the set of vertices V satisfying the following condition: every ball of radius 1 contains exactly one vertex from C. Indeed, a perfect 1-error correcting code in an n-regular graph can be considered as a set of vertices of a color 1 of a perfect coloring with the matrix $\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}$. Some other codes also can be represented as perfect colorings with 2 or more colors.

Golomb and Welch considered perfect codes in $\mathbb{Z}^n$ [6], [7]. They proved, for any length n, the existence of perfect single-error correcting codes in Lee metric. Such codes can be considered either as regular periodic tilings of the euclidean space $\mathbb{R}^n$ by Lee spheres of radius 1 or as periodic tilings of the grid $\mathbb{Z}^n$ by balls of radius 1. The authors also considered perfect codes of radii greater than 1 and obtained some results about nonexistence of such codes.

2. Definitions and notation

Let $G = (V, E)$ be a graph. A *ball* $B_r(\mathbf{x})$ of radius r with the center at the vertex $\mathbf{x}$ is defined in the following way:

$$B_r(\mathbf{x}) = \{\mathbf{y} \in V |\ d(\mathbf{x}, \mathbf{y}) \leq r\}.$$

Similarly, a *sphere* $S_r(\mathbf{x})$ is given by

$$S_r(\mathbf{x}) = \{\mathbf{y} \in V |\ d(\mathbf{x}, \mathbf{y}) = r\}.$$

Notice that the r-neighborhood of a vertex $\mathbf{x}$ consists of all vertices of the ball of radius r centered in $\mathbf{x}$ except $\mathbf{x}$ itself:

$$N_r(\mathbf{x}) = B_r(\mathbf{x}) \backslash \mathbf{x}.$$

We are interested in perfect colorings of the graph $G(\mathbb{Z}^2)$, that is the infinite rectangular grid. This graph is 4-regular, its vertices are all possible ordered pairs of integers. Two vertices $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are adjacent if $|x_1 - y_1| + |x_2 - y_2| = 1$. Denote $\|\mathbf{x}\| = d(\mathbf{x}, \mathbf{0})$, where $\mathbf{0} = (0, 0)$.

Examples of perfect colorings with 2 colors are shown in Fig. 1. In the pictures we color cells instead of vertices, i.e. actually consider the graph dual to $G(\mathbb{Z}^2)$. It just makes pictures more illustrative.

3. Constructions and examples

Construction A. One of the methods of constructing perfect colorings is the orbit method. Let G be a graph with the automorphism group H, and H' be a subgroup of H. If we color each orbit of V by an action of H' with its own color, we obtain a perfect coloring of radius $r \in \mathbb{N}$ of G (see [3]).

Construction B. Another method of constructing perfect colorings is based on gluing colors. Construction B is explained by the following lemma:

Lemma 1. *Let φ be a perfect coloring of radius r with n colors with a matrix A and a coloring ψ be obtained from φ by gluing colors together into m groups L_1, ..., L_m. Then the coloring ψ is perfect with m colors of radius r if and only if the matrix A satisfies the following condition: for every $i, j \in \{1, ..., m\}$, $i \neq j$ and for every $p, s \in L_i$,*

$$\sum_{q \in L_j} a_{pq} = \sum_{q \in L_j} a_{sq}.$$

The matrix of the perfect coloring ψ is $B = (b_{ij})_{i,j=1}^m$, where $b_{ij} = \sum_{q \in L_j} a_{pq}$ for any $p \in L_i$.

The proof is straightforward.

Example 1. Orbit colorings with two colors.

There exist 9 orbit colorings with two colors (see Fig. 1). These colorings are contained in the set of perfect colorings of radius 1, which were described by Axenovich [1].

Example 2. Translation colorings.

Let H' be a group of translations generated by two noncollinear vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. If we color each orbit of $\mathbb{Z}^2$ by the action of H' with its own color, we obtain a translation coloring. This coloring is perfect of any radius with $|u_1 v_2 - u_2 v_1|$ colors. The number of colors is equal to the number of vertices in the parallelogram spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$.

Example 3. Perfect code and colorings obtained from it by joining colors.

Consider a translation coloring generated by vectors $(r + 1, r)$ and $(r, -r - 1)$. This coloring is perfect of radius r with $n = 2r^2 + 2r + 1$ colors, the corresponding matrix is

$$\begin{pmatrix} 0 & 1 & ... & 1 \\ 1 & 0 & ... & 1 \\ & ... & & \\ 1 & 1 & ... & 0 \end{pmatrix}.$$

By Lemma 1, we can join colors and get perfect coloring with the matrix

$$\begin{pmatrix} k & n-k \\ k+1 & n-k-1 \end{pmatrix}.$$

For $k = 0$ it is a perfect code with the minimum distance $2r + 1$.

1.

1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1

2.

1	0	1	0	1	0	1	0	1
0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1
0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1
0	1	0	1	0	1	0	1	0

3.

1	1	0	0	1	1	0	0	1
1	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	0
0	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	1
1	0	0	1	1	0	0	1	1

4.

1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1
0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1
0	1	0	0	1	0	0	1	0

5.

0	0	0	0	1	0	0	0	0
0	1	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	1
1	0	0	0	0	1	0	0	0
0	0	1	0	0	0	0	1	0
0	0	0	0	1	0	0	0	0

6.

1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
0	0	1	1	0	0	1	1	0
0	0	1	1	0	0	1	1	0
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1

7.

1	0	0	1	0	0	1	0	0
1	0	0	1	0	0	1	0	0
1	0	0	1	0	0	1	0	0
1	0	0	1	0	0	1	0	0
1	0	0	1	0	0	1	0	0
1	0	0	1	0	0	1	0	0

8.

1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1

9.

1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1
0	1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1

Fig. 1. Orbit colorings with two colors.

Example 4.

M. Axenovich [1] considered perfect colorings of an arbitrary radius of the flat infinite rectangular grid with two colors. In that paper colorings were split into two types depending on the 1-neighborhood of the vertex. A perfect coloring is of type A if some vertex has an odd number of neighbors of each color or has its horizontal neighbors in one color and its vertical neighbors in the other color. A perfect coloring is of type B if every vertex **v** has all its neighbors of the same color or has two neighbors of each color that do not lie on a line through **v**. All colorings of type B are described, they are of diagonal type. For colorings of type A it is proved that $|a_{11} + 1 - a_{21}| \leq 4$. In that paper the author conjectured that for colorings of type A the following condition holds: $|a_{11} + 1 - a_{21}| \leq 2$.

Here is a counterexample to this conjecture.

1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1

This coloring is perfect for every radius r. For $r \equiv 0, 3 (\mathrm{mod}\ 4)$ the corresponding matrix is

$$\begin{pmatrix} r^2+r & r^2+r \\ r^2+r & r^2+r \end{pmatrix}.$$

For $r \equiv 1, 2 (\mathrm{mod}\ 4)$ the matrix of the coloring is

$$\begin{pmatrix} r^2+r+1 & r^2+r-1 \\ r^2+r-1 & r^2+r+1 \end{pmatrix}.$$

In the second case we have $|a_{11} + 1 - a_{21}| = 3$.

4. Periodicity

In this section we consider the periodicity of perfect colorings of radius $r \geq 2$ on the graph $G(\mathbb{Z}^2)$.

A coloring φ is $\mathbf{v}$-*periodic* (or $\mathbf{v}$ is a vector of periodicity of a coloring φ) if $\varphi(\mathbf{x}+\mathbf{v}) = \varphi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^2$. A perfect coloring that is $\mathbf{v}$- and $\mathbf{u}$-periodic for some noncollinear $\mathbf{v}$ and $\mathbf{u}$ is called *periodic*. A *fundamental parallelogram* is a set of vertices in the parallelogram spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$. Note that we can always choose the vectors $\mathbf{u}$ and $\mathbf{v}$ to be horizontal and vertical ($\mathbf{u} = (a, 0)$, $\mathbf{v} = (0, b)$). In this case we say 'rectangle' instead of 'parallelogram'.

Colorings of the infinite rectangular grid can be interpreted as two-dimensional words over the finite alphabet of colors $\{1, ..., n\}$. Notation $\omega|_{B_R(\mathbf{x})} = \omega|_{B_R(\mathbf{z})}$ means that $\omega(\mathbf{x}+\mathbf{y}) = \omega(\mathbf{z}+\mathbf{y})$ for any $\mathbf{y}$ such that $\|\mathbf{y}\| \leq R$. We say that a two-dimensional word ω is *R-prolongable* if for any $\mathbf{x}, \mathbf{z} \in \mathbb{Z}^2$ the equality $\omega|_{B_R(\mathbf{x})} = \omega|_{B_R(\mathbf{z})}$ implies $\omega|_{B_{R+1}(\mathbf{x})} = \omega|_{B_{R+1}(\mathbf{z})}$.

Proposition 1. *If ω is R-prolongable, then it is R'-prolongable for every $R' \geq R$.*

The proof is straightforward.

Lemma 2. [10] *Let ω be a two-dimensional word over a finite alphabet. If ω is R-prolongable for some $R \geq 0$, then ω is periodic.*

Proof. Since the alphabet is finite, there exist two balls $B_R(\mathbf{x})$ and $B_R(\mathbf{y})$ such that $\omega|_{B_R(\mathbf{x})} = \omega|_{B_R(\mathbf{y})}$. Denote $\mathbf{v} = \mathbf{y} - \mathbf{x}$. We will prove that $\mathbf{v}$ is a vector of periodicity. By Proposition 1, ω is R'-prolongable for every $R' \geq R$. So $\omega|_{B_{R'}(\mathbf{x})} = \omega|_{B_{R'}(\mathbf{y})}$ for every integer R'. This means that $\omega(\mathbf{x}+\mathbf{z}) = \omega(\mathbf{y}+\mathbf{z})$ for every vector $\mathbf{z}$. Consider an arbitrary vertex $\mathbf{w}$. Take $\mathbf{z} = \mathbf{w} - \mathbf{x}$, so $\omega(\mathbf{w}) = \omega(\mathbf{w}+\mathbf{v})$, which means $\mathbf{v}$-periodicity.

Let $\mathbf{u}$ be a vector noncollinear to $\mathbf{v}$. Consider the infinite set of balls $\{B_R(k\mathbf{u}) | k \in \mathbb{Z}\}$. There exist two balls $B_R(k_1\mathbf{u})$ and $B_R(k_2\mathbf{u})$, $k_1 \neq k_2$, from this set such that $\omega|_{B_R(k_1\mathbf{u})} = \omega|_{B_R(k_2\mathbf{u})}$. Arguing as above we conclude that $(k_2 - k_1)\mathbf{u}$ is a vector of periodicity. So ω is periodic. □

Remark. From the proof of this lemma we see that vectors of periodicity can be chosen as follows: $\mathbf{u} = (a, 0)$ and $\mathbf{v} = (0, b)$, where $a, b \leq n^{2R^2+2R+1}$ (here n is the number of elements in the alphabet, $2R^2 + 2R + 1$ is the number of vertices in a ball of radius R). So the number of vertices in the fundamental rectangle $a \times b$ is at most $n^{2(2R^2+2R+1)}$.

Theorem 1. *Let $\varphi : \mathbb{Z}^2 \to \{1, ..., n\}$ be a perfect coloring of radius $r \geq 2$ of the infinite rectangular grid. Then φ is periodic.*

Proof. Due to Lemma 2 it is sufficient to prove that φ is R-prolongable for some $R \geq r$. We prove that φ is R-prolongable for $R \geq 2r^2 + 5r + 1$.

Consider two arbitrary balls $B_R(\mathbf{x})$ and $B_R(\mathbf{z})$ such that $\varphi|_{B_R(\mathbf{x})} = \varphi|_{B_R(\mathbf{z})}$. We will prove that $\varphi|_{S_{R+1}(\mathbf{x})} = \varphi|_{S_{R+1}(\mathbf{z})}$. Without loss of generality we suppose that $\mathbf{x} = \mathbf{0}$.

We need some auxiliary notation.

For any $\mathbf{y} \in \mathbb{Z}$ let us define $\mathbf{y}' = \mathbf{y} + \mathbf{z}$, where $\mathbf{z}$ is as above. It means that $\mathbf{y}'$ is a translation of $\mathbf{y}$ by the vector $\mathbf{z}$. Accordingly, for any subset M of $\mathbb{Z}^2$ we define $M' = \{\mathbf{y}' \mid \mathbf{y} \in M\}$.

Let M be an arbitrary subset of $\mathbb{Z}^2$. Denote the number of vertices of a color k in M by $I_k(M)$. The *vector of color spectrum* of the set M is $I(M) = (I_1(M), ..., I_n(M))$. Notice that $e_j = (0, ..., 1, 0, ..., 0)$ is a vector of color spectrum of a vertex of a color j.

Let M, $N \subseteq \mathbb{Z}^2$. Define the following operation on the vectors $I(M)$ and $I(N)$:

$$I(M) + I(N) = (I_1(M) + I_1(N), ..., I_n(M) + I_n(N)).$$

Analogously we define componentwise operations $\min(I(M), I(N))$ and $I(M) - I(N)$. Note that if $N \cap M = \emptyset$, then $I(M \cup N) = I(M) + I(N)$; if $N \subseteq M$, then $I(M \setminus N) = I(M) - I(N)$.

Let k be an integer, $1 \leq k \leq 2r + 1$. For the balls $B_R(\mathbf{0})$ and $B_r(\mathbf{y})$, where $\mathbf{y}$ is an arbitrary vertex in the sphere $S_{R-r+k}(\mathbf{0})$, we define the *k-outside set* $O_k(B_r(\mathbf{y}))$ as follows: $O_k(B_r(\mathbf{y})) = B_r(\mathbf{y}) \setminus B_R(\mathbf{0})$. In other words, the k-outside set is a set of vertices that belong to the small ball and do not belong to the large ball, k is a number of 'layers' of vertices in k-outside set. The example in Fig. 2 is for $R = 5$, $r = 2$, the boundaries of the balls are marked by bold, centers of the balls are marked by points. The set $O_1(B_2(2, -2))$ consists of three vertices $\mathbf{v_2}$, $\mathbf{v_3}$, $\mathbf{v_4}$.

Proposition 2. *If $k \leq r$, $\mathbf{y} \in S_{R-r+k}(\mathbf{0})$, then*

$$I(O_k(B_r(\mathbf{y}))) = I(O'_k(B_r(\mathbf{y}))).$$

Proof. The proof follows from the definition of perfect coloring and the equality $\varphi|_{B_R(\mathbf{0})} = \varphi|_{B'_R(\mathbf{0})}$. □

We will need only 1- and 2-outside sets, so the inequality $k \leq r$ holds. Note that for $r = 1$ and $k = 2$ this inequality does not hold and we cannot apply Proposition 2. This is a reason why the situation for $r = 1$ is completely different from the situation for $r \geq 2$.

Now we proceed to the proof of the theorem. We should prove that

$$\varphi|_{S_{R+1}(\mathbf{0})} = \varphi|_{S'_{R+1}(\mathbf{0})}.$$

Each of the spheres $S_{R+1}(\mathbf{0})$, $S'_{R+1}(\mathbf{0})$ consists of five sets of vertices: $S_{R+1}(\mathbf{0}) = \bigcup_{i=1}^{5} M_i$, where

$M_1 = \{(j, j - R - 1) | j = 1, 2, ..., R + 1\}$,
$M_2 = \{(-j, j - R - 1) | j = 1, 2, ..., R + 1\}$,
$M_3 = \{(j, R + 1 - j) | j = 1, 2, ..., R + 1\}$,
$M_4 = \{(-j, R + 1 - j) | j = 1, ..., R + 1\}$,
$M_5 = \{(0, -R - 1), (0, R + 1), (R + 1, 0), (-R - 1, 0)\}$.

(see Fig. 3, where the cells in M_i are denoted by i).We will prove $\varphi|_{M_i} = \varphi|_{M_i'}$ $i = 1, ..., 5$ for each set separately.

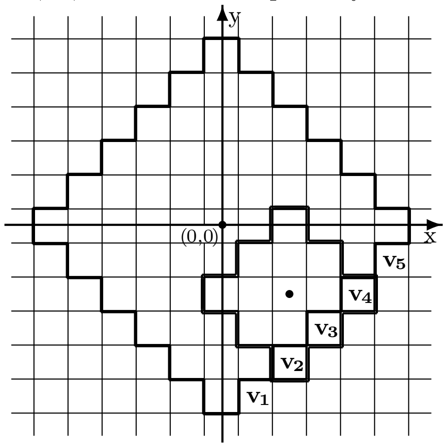

Fig. 2. The ball $B_R(\mathbf{0})$ and one of the balls $B(i)$ for $R = 5$, $r = 2$.

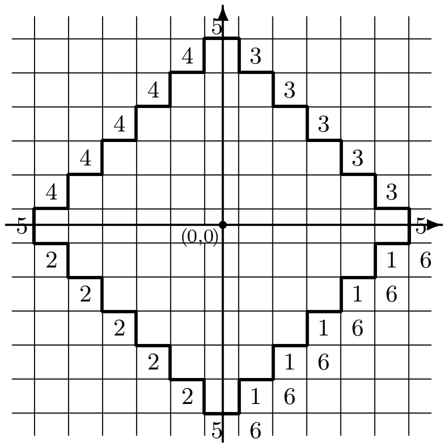

Fig. 3. The ball $B_5(\mathbf{0})$ and the sets M_1–M_6.

Proposition 3. *It holds $\varphi|_{M_1} = \varphi|_{M_1'}$.*

Proof. Denote by $\mathbf{v_j}$ the vertex $(j, j-1-R)$, where $1 \le j \le R$, then $M_1 = \{\mathbf{v_1}, ..., \mathbf{v_R}\}$ (see Fig. 2, where $R = 5$, $r = 2$). Suppose, by contradition, that there exists i such that $\varphi(\mathbf{v_i}) \neq \varphi(\mathbf{v_i'})$. Denote $\varphi(\mathbf{v_i}) = a$ and $\varphi(\mathbf{v_i'}) = b$.

The proof of the proposition consists of two parts.

In the first part of the proof we will prove that $\varphi(\mathbf{v_i}) = \varphi(\mathbf{v_{i+k(r+1)}}) = a$, $\varphi(\mathbf{v_i'}) = \varphi(\mathbf{v'_{i+k(r+1)}}) = b$ for $k \in \mathbb{Z}$, $1 \le i + k(r+1) \le R$. This fact will be used in the second part of the proof. Let us prove it for $k = 1$ (if $i + r + 1 > R$, then it can be proved for $k = -1$ by the same reasoning). Consider the balls $B(i) = B_r(i, i-1-R+r)$, $B'(i)$ and 1-outside sets $O_1(B(i))$, $O_1'(B(i))$ of these balls. Due to Proposition 2, $I(O_1(B(i))) = I(O_1'(B(i)))$. Then,

$$I(O_1(B(i))\backslash\mathbf{v_i}) = I(O_1(B(i))) - e_a,$$

$$I(O_1'(B(i))\backslash\mathbf{v_i'}) = I(O_1'(B(i))) - e_b.$$

Denote

$$P = min(I(O_1(B(i))\backslash\mathbf{v_i})), I(O_1'(B(i))\backslash\mathbf{v_i'}).$$

Then

$$I(O_1(B(i))\backslash\mathbf{v_i})) = P + e_b,$$

$$I(O_1'(B(i))\backslash\mathbf{v_i'})) = P + e_a.$$

Now consider the balls $B(i+1)$ and $B'(i+1)$. Since $O_1(B(i)) \bigcap O_1(B(i+1)) = O_1(B(i))\backslash\mathbf{v_i} = O_1(B(i+1))\backslash\mathbf{v_{i+r+1}}$, it follows that

$$I(O_1(B(i+1)) \setminus \mathbf{v_{i+r+1}}) = I(O_1(B(i))\backslash\mathbf{v_i}) = P + e_b, \tag{1}$$

$$I(O_1'(B(i+1)) \setminus \mathbf{v'_{i+r+1}}) = I(O_1'(B(i))\backslash\mathbf{v_i'}) = P + e_a. \tag{2}$$

By Proposition 2, $I(O_1(B(i+1))) = I(O_1'(B(i+1)))$. Therefore, $\varphi(\mathbf{v_{i+r+1}}) = a$, $\varphi(\mathbf{v'_{i+r+1}}) = b$, so $\varphi(\mathbf{v_i}) = \varphi(\mathbf{v_{i+r+1}})$, $\varphi(\mathbf{v_i'}) = \varphi(\mathbf{v'_{i+r+1}})$. Analogously, $\varphi(\mathbf{v_i}) =$

$\varphi(\mathbf{v_{i+k(r+1)}}) = a$, $\varphi(\mathbf{v'_i}) = \varphi(\mathbf{v'_{i+k(r+1)}}) = b$ for $k \in \mathbb{Z}$, $1 \le i + k(r+1) \le R$. This completes the first part of the proof of the proposition.

In the second part of the proof we deal with the spheres $S_{R+2}(\mathbf{0})$ and $S'_{R+2}(\mathbf{0})$. Consider the balls

$$C(k) = B_r(i + 1 + k(r + 1), i - 1 - R + r + k(r + 1))$$

and $C'(k)$, where $k \in \mathbb{Z}$, $1 \le i + 1 + k(r + 1) \le R - r + 2$, and 2-outside sets of these balls. In Fig. 4 one can see a part of the ball $B_R(\mathbf{0})$ of radius $R = 17$ and five balls $C(k)$ of radius $r = 2$, centers of these balls are marked by points. Due to Proposition 2,

$$I(O_2(C(k))) = I(O_2(C'(k)). \tag{3}$$

Consider the sets

$$A_k = O_2(C(k))\backslash(O_1(B(i + k(r + 1))) \setminus \mathbf{v_{i+k(r+1)}})$$

and A'_k (in Fig. 4 one of 2-outside sets $O_2(C(k))$ is marked by black and white circles, the corresponding set A_k is marked by white circles). In fact, the set A_k is a part of the 2-outside set $O_2(C(k))$. Using (1) and (2), we get that

$$I(O_2(C(0))) = I(O_1(B(i)) \setminus \mathbf{v_i}) + I(A_0) = P + e_b + I(A_0)$$
$$I(O_2(C'(0)) = I((O_1(B'(i)) \setminus \mathbf{v'_i})) + I(A'_0) = P + e_a + I(A'_0)$$

Combining it with (3), we obtain

$$I(A_0) + e_a = I(A'_0) + e_b.$$

Similarly,

$$I(A_k) + e_a = I(A'_k) + e_b. \tag{4}$$

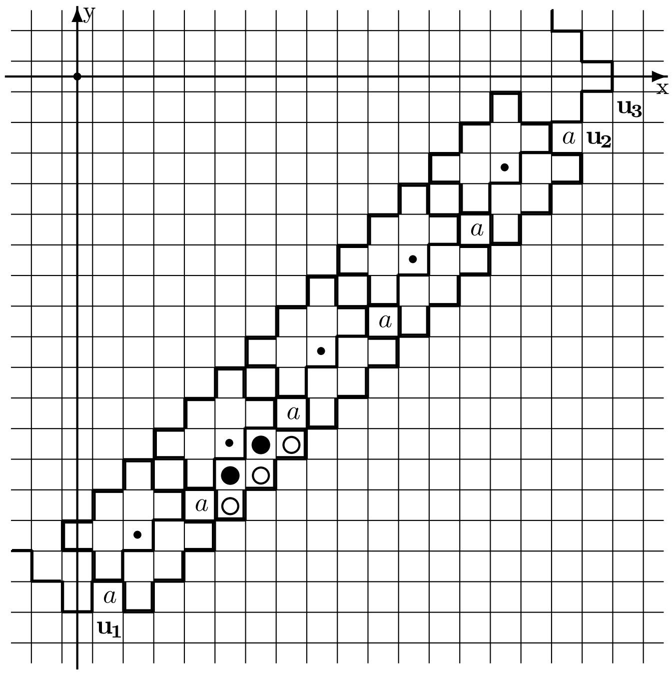

Fig. 4. A part of the ball $B_R(\mathbf{0})$ and the balls $C(i)$, $R = 17$, $r = 2$.

Define the set

$$M_6 = \{(j, j - R - 2) \mid j = 1, 2, ..., R + 1\}$$

(see Fig. 3). This set can be represented as a union of disjoint sets A_k and the set D of vertices in M_6 that do not belong to one of the sets A_k (this set appears because of boundary effects):

$$M_6 = \bigcup_k A_k \cup D,\ 1 \le i + k(r+1) \le R - r + 2.$$

In Fig. 4 the set D consists of three vertices $\mathbf{u_1}$, $\mathbf{u_2}$, $\mathbf{u_3}$. The number of elements in the set D is at most $2r$: $|D| \le 2r$. Similarly

$$M_6' = \bigcup_k A_k' \cup D',\ 1 \le i + k(r+1) \le R - r + 2,$$

where $|D'| \le 2r$. Using (4), we obtain

$$I(\bigcup_k A_k) + ke_a = I(\bigcup_k A_k') + ke_b.$$

It means that we have the following condition on the number of vertices of color a in the sets $\bigcup_k A'_{i+k(r+1)}$ and $\bigcup_k A_{i+k(r+1)}$:

$$I_a(\bigcup_k A_k') + k = I_a(\bigcup_k A_k).$$

So, if we take $k \ge 2r + 1$ (therefore, $R \ge (2r+1)(r+1) + 2r = 2r^2 + 5r + 1$), then the set M_6' contains more vertices of the color a (and, similarly, fewer vertices of color b), than the set M_6.

Now, there exists j such that $\varphi(v_j') = a$ and $\varphi(v_j') \ne \varphi(v_j)$. Arguing as above we get that the set M_6 contains more vertices of color a, than the set M_6'. A contradiction. The proposition is proved. □

So, we proved that $\varphi|_{M_1} = \varphi|_{M_1'}$. The proof is similar for the sets M_2, M_3, M_4. Now, $\varphi(0, -R-1) = \varphi(0, -R-1)'$, because colors of all other vertices in the balls $B_r(0, r - R - 1)$ and $B_r(0, r - R - 1)'$ are the same. For other elements of the set M_5 the proof is symmetric.

Thus, we have $\varphi|_{S_{R+1}(\mathbf{0})} = \varphi|_{S_{R+1}(\mathbf{x}')}$, therefore, φ is R-prolongable for $R \ge 2r^2 + 5r + 2$. By Lemma 2, φ is periodic. This completes the proof of Theorem 1. □

Corollary 1. *Let φ be a perfect coloring of radius $r \ge 2$ of the infinite rectangular grid with n colors. Then there exists a fundamental rectangle with at most $n^{2(2(2r^2+5r+1)^2+2(2r^2+5r+1)+1)}$ vertices.*

The proof follows from the proof of Theorem 1 and Remark to Lemma 2.

Note that if $\mathbf{v}$ and $\mathbf{u}$ are vectors of periodicity of a perfect coloring φ, then φ can be obtained by joining colors (Construction B) from the translation coloring generated by the vectors $\mathbf{v}$ and $\mathbf{u}$ (Construction A, example 2).

Theorem 1 and Corollary 1 yield an upper bound for the number of colors in the corresponding translation coloring: this number is at most

$$n^{2(2(2r^2+5r+1)^2+2(2r^2+5r+1)+1)}.$$

So, we found a way to obtain all perfect colorings of radius $r \geq 2$ with n colors, but it requires checking a huge number of cases. A raw upper estimate for the number of cases to check is $n^{n^{Cr^4}}$, where C does not depend on r and n. Indeed, the number of vertices in the fundamental rectangle is less than or equal to n^{Cr^4} and there exist at most $n^{n^{Cr^4}}$ possibilities to split vertices of the fundamental rectangle into n groups. So, this method is not appropriate for computer experiments and we still do not have even a list of all perfect colorings of radius 2 with 2 colors.

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