S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 5, стр. 417–426 (2008) УДК 519.172.2

MSC 05C15

CIRCULAR $(5,2)$-COLORING OF SPARSE GRAPHS

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Abstract. We prove that every triangle-free graph whose subgraphs all have average degree less than $\frac{12}{5}$ has a circular $(5,2)$-coloring. This includes planar and projective-planar graphs with girth at least 12.

Keywords: triangle-free graph, circular (k,d)-coloring, projective-planar graph.

1. Introduction

A circular (k,d)-*coloring* of a graph G, introduced by Vince [9], is a map $\varphi : V(G) \longrightarrow \{0,\dots,k-1\}$ such that $d \le |\varphi(u) - \varphi(v)| \le k-d$ for every edge $uv \in E(G)$. Such a coloring is "circular" in the sense that we may view the k colors as points on a circle, where the colors on adjacent vertices must be at least d positions apart on the circle. Note that a circular $(k,1)$-coloring is an ordinary proper k-coloring.

Clearly, G has a circular $(2t+1,t)$-coloring if and only if it has a homomorphism into the cycle C_{2t+1}. A relaxation for planar graphs of a conjecture of Jaeger [5] on nowhere-zero flows states the following:

Conjecture 1. *For every positive integer t, every planar graph with girth at least $4t$ has a circular $(2t+1,t)$-coloring.*

O.V.Borodin, S.G.Hartke, A.O.Ivanova, A.V.Kostochka, D.B.West, Circular (5,2)-Coloring of Sparse Graphs.

The first and third authors were supported by the grants 06-01-00694 and 08-01-00673 of the Russian Foundation for Basic Research, the third author was also supported by the President of Russia grant for young scientists MK-2302.2008.1, the fourth author by NSF Grant DMS-0400498, and the fifth by NSA Award No. H98230-06-1-0065.

Received August 14, 2008, published October 29, 2008.

When $t = 1$, Conjecture 1 reduces to Grötzsch's Theorem. The conjecture is sharp if true, as shown by DeVos [10].

Nešetřil and Zhu [7] and Galuccio, Goddyn, and Hell [4] proved a relaxation of Conjecture 1 with girth at least $10t - 4$, which bound was improved by Zhu [11] to $8t-3$. Recently, Borodin, Kim, Kostochka, and West [1] further lowered this bound to $\frac{20t-2}{3}$.

One of results in the present paper is a partial step (for $t = 2$) towards Conjecture 1:

Theorem 2. *If G is a planar graph G with girth at least 12, then G has a circular $(5, 2)$-coloring.*

By $mad(G)$ denote the *maximum average degree* of all subgraphs of G. This is a conventional measure of sparseness of arbitrarily graphs (not necessarily planar). For planar graphs, the sparseness is usually expressed in terms of girth; the *girth* $g(G)$ of G is the minimum length of cycles in G.

The following is an easy application of Euler's formula $|V| - |E| + |F| = 2$:

Remark 3. *Every plane graph G with girth g satisfies $mad(G) < \frac{2g}{g-2}$.*

Indeed,

$$\frac{g(G)-2}{2}2|E| - g(G)|V| + 2|E| - g(G)|F| =$$
$$= \sum_{v\in V}\Big(\frac{g(G)-2}{2}d(v) - g(G)\Big) + \sum_{f\in F}(r(f) - g(G)) = -2g(G).$$

Thus, a planar graph G with a large girth has a low $mad(G)$. On the other hand, a nonplanar graph G may have arbitrarily large $g(G)$ and $mad(G)$, as follows from the Erdös theorem [3] on the existence of k-chromatic graphs with arbitrarily large girth.

Often, coloring theorems on planar graphs can be extended to arbitrary graphs. In particular, the result in [1] reads as follows:

Theorem 4. *Every graph G with girth at least $6t - 2$ and $mad(G) < 2 + \frac{3}{5t-2}$ has a circular $(2t + 1, t)$-coloring.*

Our main result improves the special case $t = 2$ of Theorem 4:

Theorem 5. *Every triangle-free graph G with $mad(G) < \frac{12}{5}$ has a circular $(5, 2)$-coloring.*

It follows from Remark 3 that Theorem 5 implies Theorem 2; i.e., it proves the cyclic $(5, 2)$-colorability of plane graphs with girth at least 12, whereas Theorem 4 can be applied only to plane graphs of girth at least 13.

Note that the case $t = 2$ in Conjecture 1 deserves attention because the case $t = 2$ of the general conjecture by Jaeger [5] on nowhere-zero flows implies Tutte's 5-Flow Conjecture [8] (see [6, p. 209]).

We would like to mention a novel technical feature of the proof of Theorem 5, which perhaps could be used in further research; namely, the global character of discharging: portions of charge in our proof are sometimes sent from vertices to arbitrarily remote vertices. Note that in [1] charge is also moved far away but only along paths consisting solely of vertices with degree 2. The first example of a nontrivial global discharging is given in the paper [2] on oriented colorings.

Clearly, a triangle does not admit a circular $(5, 2)$-coloring. Also note that the bound on average degree in Theorem 5 is sharp.

Indeed, consider the following graph G_D. It has adjacent vertices a, b and paths through 2-vertices: aa_zz, aa_ww, bb_zz, bb_ww, and $wxyz$. So, $mad(G_D) = \frac{12}{5}$. Suppose G_D has a (5,2)-coloring φ. W.l.o.g., we can assume that $\varphi(a) = 0$, $\varphi(b) = 2$. Due to paths aa_ww and bb_ww, the only possible color for w is 1. The same is true for z. However, then we cannot color both of x and y.

2. Proof of Theorem 5

Let G be a counterexample to Theorem 5 with the fewest edges. Since $mad(G) < < \frac{12}{5}$, we have

$$\sum_{v \in V} (5d(v) - 12) < 0, \tag{1}$$

where V is the set of vertices of G and $d(v)$ is the degree of vertex v.

Let the *charge* $\mu(v)$ of each vertex v of G be $5d(v) - 12$. Note that the charge of 2-vertex is -2, the charge of 3-vertex is 3, for 4-vertex it is 8, etc.

We shall describe a number of structural properties of G which make it possible to vary the charges so that the new charge of every vertex becomes nonnegative. Since the sum of charges does not change, we shall get a contradiction with (1), which will complete the proof of Theorem 5.

2.1. **Basic properties of the minimal counterexample.**

Lemma 1. $\delta(G) \geq 2$. □

Lemma 2. *G has no 4-cycle $wxyz$, where $d(w) = 2$.* □

Proof. We delete the edges incident with w, color the graph obtained, and color w the same as y. □

In what follows, by *k-thread* we mean a path consisting of precisely k vertices of degree 2.

Lemma 3. *If an end vertex of a k-thread P, where $k \leq 3$, is colored then the other end of P gets $3 - k$ forbidden colors along P.*

Proof. Let $v_0, v_1, \ldots, v_k, v_{k+1}$, $k \leq 3$, be a k-thread, where $v_1, v_2, \ldots, v_k$ are vertices degree 2, while v_0 and v_{k+1} have degree at least 3. Let α be the color of vertex v_0. Note that v_1 has two colors ($\alpha + 2$ and $\alpha - 2$) admissible from v_0, while v_2 has three colors ($\alpha - 1$, α and $\alpha + 1$), and for v_3 all colors are admissible except for α. □

Corollary 4. *G has no ≥ 3-thread.*

Proof. We delete an edge between the 2-vertices v_1, v_2, v_3 of such a path, take a circular coloring of the graph G' obtained (clearly, $mad(G') \leq mad(G)$), discolor the 2-vertices and color them in this order: v_3, v_2, v_1. □

In what follows, while proving the reducibility of configurations, we will simply delete vertices, color the graph G' obtained, since $mad(G') \leq mad(G)$ will always hold, and extend a coloring of G' to G.

Corollary 5. *If the end vertices of a* 2*-thread are colored differently then its* 2*-vertices can be colored.* □

Lemma 6. *If an end vertex of an edge has two admissible colors, while its other end vertex has either three or two nonconsecutive admissible colors, then one can choose a color at every end vertex so that the two colors differ by* 2 (*mod* 5). □

Corollary 7. *If an end vertex of a* 1*-thread has either three or two nonconsecutive admissible colors, while its other end vertex is already colored, then one can find an admissible color so that the* 2*-vertex of this path can be colored.* □

Lemma 8. *If the vertices of* 1*-thread* $P = abc$ *have lists* A, B, C *of admissible colors, then* P *can be colored in each of the following cases:*

(*i*) $|A| \geq 1, |B| \geq 5, |C| \geq 3$,
(*ii*) $|A| \geq 1, |B| \geq 4, |C| \geq 4$,
(*iii*) $|A| \geq 2, |B| \geq 3, |C| \geq 4$,
(*iv*) $|A| \geq 2, |B| \geq 4, |C| \geq 3$,
(*v*) $|A| \geq 3, |B| \geq 3, |C| \geq 3$.

Proof. In cases (ii) and (v) we first color vertices a and b using Lemma 6, then we can color c. In the other cases, we first erase the colors forbidden for b along ab; then use Lemma 6 for coloring b and c, and then we can color vertex a. □

Lemma 9. *If the vertices of* 5*-cycle* $C = uwpz't$ *have lists* U, W, P, Z', T *of admissible colors, where* $|P| = |T| = 5$*, then* C *can be colored in each of the following cases:*

(*i*) $|U| \geq 4, |W| \geq 4, |Z'| \geq 4$,
(*ii*) $|U| \geq 4, |W| \geq 5, |Z'| \geq 3$,
(*iii*) $|U| \geq 4, |W| \geq 3, |Z'| \geq 5$,
(*iv*) $|U| \geq 5, |W| \geq 3, |Z'| \geq 4$.

Proof. (i) Observe that any admissible color at z' brings two restrictions to each of the vertices w, u, so that each of them is left with at least two admissible colors. By properly choosing a color for z', we can leave w with either three or two nonconsecutive colors. Then w and u can be colored by Lemma 6. Indeed, suppose α is forbidden for z', while β, at w. We first try to color z' with $\alpha + 2$. We are done unless w has forbidden colors $\beta+1, \beta+2$ or $\beta+3, \beta+4$. In the first case we color z' with $\alpha+3$; in the second, with $\alpha+1$. Then w has nonconsecutive admissible colors $\beta+1, \beta+4$, as desired.

(ii) We first color z', then u and w by Lemma 6, and finally, p and t.

(iii) We first color u and w.

(iv) Is equivalent to (ii). □

Lemma 10. *If the vertices of* 5*-cycle* $C = uwptz$ *have lists* U, W, P, T, Z *of admissible colors, where* $|P| = |T| = 5$*, then* C *can be colored in each of the following cases:*

(*i*) $|W| \geq 4, |U| \geq 4, |Z| \geq 4$,
(*ii*) $|W| \geq 5, |U| \geq 3, |Z| \geq 4$,
(*iii*) $|W| \geq 3, |U| \geq 5, |Z| \geq 4$,
(*iv*) $|W| \geq 3, |U| \geq 4, |Z| \geq 5$.

Proof. (i) Observe that any admissible color at w leaves three colors at z along $wptz$. To apply Lemma 6, it suffices to have two colors at u. Suppose α is forbidden for u, then either α or $\alpha+1$ is suitable.

(ii)–(iv) Follow by Lemma 9. □

By $(k_1, k_2, \ldots)$-*vertex* we mean a vertex that is incident with k_1-, k_2-, … threads.

Lemma 11. *G has no* $(\geq 1, 2, 2)$-*vertex.*

Proof. Let v be a $(\geq 1, 2, 2)$-vertex. We delete v and 2-vertices of the paths incident with it. By Lemma 3, v has 2 restrictions along the 1-thread and 1 restriction along every 2-thread. It follows that we can find a color for v which allows a circular 5-coloring of 2-vertices of the paths incident with v. □

Lemma 12. *G has no cycles that consists of* $(\geq 1, \geq 1, \geq 1)$- *and* 2-*vertices.*

Proof. Suppose the contrary, and let C be the shortest among such cycles. Let $v_1, v_2, \ldots, v_k$ be the vertices of degree 3 in C in the clockwise order. Note that two paths incident with each v_i belong to C, while the third, *outside*, path cannot end in C due to Lemma 2 and the minimality of C. By w_i denote the 2-vertex not in C that is adjacent to v_i, where $1 \leq i \leq k$, and let s_i be the vertex different from v_i and adjacent to w_i. We delete all the vertices of C along with all the w_i, and color the graph obtained.

Let s_i be colored with α_i. According to Lemma 3, v_i has 3 colors, $\alpha_i - 1$, α_i and $\alpha_i + 1$, which are admissible from the outside; i.e., for each of them used at v_i we can find a suitable color for w_i. It remains to color all v_i's with those admissible colors so that we could color the 2-vertices of C itself. We shall prove this fact for C having arbitrary length.

Observe that 3-vertices of C can be joined both by 1-threads and also by 2-threads.

Case 1. The 3-vertices in C are joined by 1-threads only.

Subcase 1.1. There are two consecutive 3-vertices in C, say v_1, v_2, such that their central colors α_1 and α_2 differ by at most 1.

Then α_1 belongs to the triple of colors admissible for v_2. We color v_2 with α_1. Then, using Corollary 7, we color v_3 so that the 2-vertex of C that lies between v_2 and v_3 could be also colored. Similarly we color C until v_1. Since none of the colors admissible at v_1 differs from the color α_1 of the vertex v_2 more than by 1, we can now color the 2-vertex between v_1 and v_2.

Subcase 1.2. There are three consecutive 3-vertices in C, say v_1, v_2, v_3, such that $\alpha_1 = \alpha$, $\alpha_2 = \alpha + 2$, and $\alpha_2 = \alpha + 4$. Then we color v_3 with $\alpha + 3$ and go along the cycle until v_k. Since $c(v_3) = \alpha + 3$ and $c(s_2) = \alpha + 2$, we have two choices for v_2: $\alpha + 2$ or $\alpha + 3$. Thus, v_2 forbids for v_1 only color α, while s_1 forbids $\alpha + 2$ and $\alpha + 3$. Hence, $\alpha + 1$ and $\alpha + 4$ are allowed for v_1, so that at least one of them does not contradict any color of v_k.

Subcase 1.3. All s_i's are colored alternatively with α and $\alpha + 2$. Then we color all v_i's with $\alpha + 1$.

Case 2. C has a 2-thread, say, v_1xyv_2.

Let, for example, v_1 have admissible colors 0, 1, 2. If v_2 has the same admissible colors, then we color v_2 with 1 and delete 1 from the list of colors admissible for v_1; otherwise we color v_2 with a color different from 0, 1, 2. Then we color cycle C

clockwise, using Corollaries 5 and 7. Note that when we color v_1, there is no conflict with vertex v_2. □

Lemma 13. *Let P be a path $s'_0y_0x_0v_0x_1v_1x_2\dots v_kx_{k+1}v_{k+1}$, where $v_1, v_2, \dots, v_k$ are $(1,1,1)$-vertices, v_0 is a $(2,1,1)$-vertex, and x_i's and y_0 are 2-vertices. Let w_i be the 2-vertex not in P that is adjacent to v_i and s_i be the vertex different from v_i and adjacent to w_i. If s_i's and s'_0 are colored then vertex v_{k+1} gets just one restriction along P.*

Proof. Observe that Lemma 13 extends the case $k = 2$ of Lemma 3. So, v_0 has 2 restrictions from s_0 (along 1-thread), and 1 restriction from s'_0 (along 2-thread). Hence, there exist two admissible colors for vertex v_0 such that for any color chosen at v_0, we can always color the 2-vertices w_0, x_0, y_0. Having two admissible colors for v_0, we have only one restriction at v_1 that comes from v_0. Along with the two restrictions from s_1, there are two admissible colors left for v_1 (for any color chosen at v_1, we can always color v_0 and the 2-vertices w_1 and x_1). This way, v_{k+1} will get one restriction along P. □

Lemma 14. *G has no path $v_0x_1v_1x_2\dots x_{k+1}v_{k+1}$, where $v_1, v_2, \dots, v_k$ are $(1,1,1)$-vertices, v_0 and v_{k+1} are $(2,1,1)$-vertices and x_i's are 2-vertices.*

Proof. Clearly, Lemma 14 extends Lemma 11 (with $k = -1$). Suppose such a path P exists. By Lemma 12, v_0 and v_{k+1} are different. By s'_0 and s'_{k+1} denote the ends of 2-threads that are incident with v_0 and v_{k+1}, respectively. It is not excluded that $s'_0 = s'_{k+1}$, but due to Lemma 12, we have $s'_0 \neq v_{k+1}$ and $v_0 \neq s'_{k+1}$. It also follows from Lemma 12 that vertices $s_0, s_1, \dots, s_{k+1}$ do not belong to P. We assume that path P is embedded into a path $P' = y_0x_0v_0x_1v_1x_2\dots x_{k+1}v_{k+1}x_{k+2}y_{k+2}$, where $x_0, y_0, x_{k+2}, y_{k+2}$ are 2-vertices.

We delete from G all vertices of P' along with all w_i's and color the graph obtained. By Lemma 13, v_{k+1} gets one restriction along P'. By Lemma 3, v_{k+1} gets one restriction from s'_{k+1} and two restrictions from s_{k+1}; i.e., v_{k+1} can be colored. Then we color the vertices of P with admissible colors in the opposite order, and finally we color the 2-vertices of P' and all the w_i's. □

2.2. **Initial rules of discharging and their consequences.**

R1: Every 2-vertex that belongs to a 1-thread gets charge 1 from its end vertices, while 2-vertex that belongs to 2-thread gets charge 2 from the neighbor vertex of degree greater than 2.

Note that after applying R1, the charge of every 2-vertex vanishes, while the charges of (2,1,1)- and (2,2,0)-vertices are equal to -1 and the charges of the other vertices are nonnegative.

We now introduce the concept of *sponsor* as follows. Every (2,1,1)-vertex v_0 gets a *feeding path* $FP = v_0x_1v_1x_2\dots x_{k+1}v_{k+1}$, where all $v_1, \dots, v_k$ are (1,1,1)-vertices, whereas v_{k+1} is not; moreover, FP is a shortest path with these properties. Such a path exists due to the finiteness of G combined with Lemma 12. By Lemma 14, vertex v_{k+1} is not a (2,1,1)-vertex. Then v_{k+1} is called *the sponsor* for the (2,1,1)-vertex v_0. It also follows from Lemma 14 that the feeding paths of any two different (2,1,1)-vertices have no (1,1,1)-vertex in common, which implies that at most one feeding path enters any 1-thread of any sponsor.

R2: Every (2,1,1)-vertex gets charge 1 along the feeding path from its sponsor.

After applying R2, the charge of every (2,1,1)-vertex vanishes, the charge of every (2,2,0)-vertex is -1, while the charges of all other vertices, except for (2,1,0)-vertices that are sponsors and (1,1,0)-vertices that are double sponsors (entered by two feeding paths) are nonnegative.

Indeed, if $d(v) \geq 4$ then every path takes away from v at most two units of charge by R1 and R2 (note that it takes precisely two only in the cases of 2-threads or those 1-threads that belong to a feeding path because feeding paths do not branch), while $5d(v) - 12 \geq 2d(v)$. But if $d(v) = 3$ then it suffices to observe that (1,1,1)-vertices have charge 0, while any other 3-vertex, except for (2,1,1)-vertex, is incident with a 0-thread and therefore, having initial charge 3, can end up with a negative charge only if each of its two other paths takes away two units of charge.

A path that takes away two units of charge (i.e. being either a 2-thread or a 1-thread that belongs to a feeding path), is called *loaded*.

Lemma 15. *If all the boundary vertices of a loaded path P are colored then its end vertex gets just one restriction along P.*

Proof. Follows directly from Lemmas 3 and 13. □

Let $P = s'_0 y_0 x_0 v_0 x_1 v_1 x_2 \dots v_k x_{k+1} v_{k+1}$ be a loaded path going out of an overloaded vertex v_{k+1}, where v_0 is a (2,1,1)-vertex, $v_1, \dots, v_k$ are (1,1,1)-vertices, while $x_0, \dots, x_k$ and y_0 are 2-vertices. Then 2-vertices not from P adjacent to v_i, where $0 \leq i \leq k$, are denoted by w_i, while a vertex adjacent to w_i and different from v_i, by s_i. Vertex s'_o and all s_i's will be called *the boundary* vertices of the loaded path P; in particular, s'_o will be called *terminal*, while all s_i's, *side* vertices. Also, the (1,1,1)-vertices $v_1, \dots, v_k$ will be called *internal* for P. If a loaded path is simply a 2-thread, then this corresponds to the case $k = -1$, when it has neither internal nor side vertices.

Lemma 16. *Let the terminal vertex of a loaded path L_1 be the* $(2,1,1)$*-vertex for another loaded path L_2. Suppose L_2 does not degenerate into* 2*-thread and all the side vertices of paths L_1, L_2 are colored. Then the end vertex of L_2 can get no restrictions along L_2 at price that every* 3*-vertex of L_1 gets two restrictions along L_1.*

Proof. Let a be the (2,1,1)-vertex for the (feeding) path L_2, and let b be terminal vertex of L_2 that belongs to L_1. Let A be the triple of admissible colors at a left by the side vertex. Let B be the set of admissible colors at b. We erase from B the two extreme colors in A and erase the central color from A.

Now observe that we can color path L_2 in the last place, from the sponsor to a, whatever color of vertex b. Indeed, now a has no restrictions along the 2-thread from b, so that it can be colored in the last place due to Corollary 5 since it has either three or two nonconsecutive admissible colors. In the same fashion, the whole path L_2 can be colored by Corollary 7 from the beginning to end. Thus, L_2 does not impose restrictions to the choice of color on its initial vertex.

As for L_1, we have nothing to prove if L_1 is a 2-thread. If L_1 is a feeding path, then b has at least one admissible color. The next 3-vertex of L_1 gets three admissible colors from its side vertex and two restrictions from b, and so can be colored; etc. □

A 3-vertex is called *overloaded* if it is incident with a 0-thread and two loaded paths.

Lemma 17. *G has no edge between overloaded vertices.*

Proof. Let u and w be adjacent overloaded vertices incident with loaded paths P_1, P_2 and P_3, P_4, respectively. Recall that no two feeding paths can have a (1,1,1)-vertex in common due to Lemma 14. In particular, this implies that neither u nor w can be a boundary vertex for any P_i. Indeed, then a side path of P_i belongs to another feeding path P_j, which is impossible.

Paths P_3 and P_4 at w, as well as paths P_1 and P_2 at u, are called *related.* We say that paths P_i, P_j are *closed* on each other if the terminal vertex of P_i is the (2,1,1)-vertex of P_j.

We delete u, w, internal vertices of P_1, P_2, P_3, P_4 and all the other 2-vertices of the incident 1-threads.

Case 1. There are no closings between unrelated paths.

By Lemmas 13 and 16, each of u, w gets two restrictions along corresponding P_1, P_2, P_3, P_4, and we are done by Lemma 6.

Case 2. P_1 is closed with P_3.

With the same argument as in Case 1, we see that u and w have lists of admissible colors of the following cardinalities: $(\geq 1, \geq 5)$; $(\geq 2, \geq 4)$; $(\geq 3, \geq 3)$. $\square$

Corollary 18. *In particular, G has no two adjacent $(2,2,0)$-vertices u, w.*

2.3. **Final discharging.**

R3: Every overloaded vertex gets charge 1 from the end of incident 0-thread.

By Lemma 17, applying Rule R3 does not result in a collision; so its remains to prove that now every vertex has a nonnegative charge.

Suppose the contrary; i.e., the charge of u is negative. Then u must be adjacent to an overloaded vertex w. It is also clear that $d(u) = 3$, since $5d(u) - 12 \geq 2d(u)$ whenever $d(u) \geq 4$. But since u sends charge 1 along every edge uw, it follows that it sends 2 along one of the two other paths, P_2, and sends 1 along the other (it cannot send 2 along each of them since it is not overloaded due to Lemma 17). The path, P_1, that takes charge 1 from u, is either a 1-thread utz' (in which case z' is considered as a side vertex of P_1) or 0-thread (an edge) that leads to an overloaded vertex z, in which case the loaded paths going out of z are denoted by P_{11} and P_{12}. By P_3 and P_4 we denote the loaded paths going out of w.

Let us prove that such an edge uw is impossible; this will complete the proof of Theorem 2. We proceed along the lines of the proof of Lemma 17.

Note that since G has neither triangles nor 4-cycles incident with a 2-vertex due to Lemma 2, it follows that while considering possible closings among loaded paths we should concern only about creating cycles of lengths at least 5.

Case 1. P_1 is 1-thread.

Subcase 1.1. No boundary vertex of paths P_1, P_2, P_3, P_4 coincides with u, w or an internal vertex of paths P_2, P_3, P_4.

We delete u, w, the internal vertices of paths P_2, P_3, P_4, and also the 2-vertices of paths P_1, P_2, P_3, P_4. Now u gets $2 + 1$ restrictions from P_1, P_2, and w gets $1 + 1$

restrictions from P_3, P_4. Hence, we can first color u and w by Lemma 6, and then color paths P_1, P_2, P_3, P_4 from their beginnings to ends.

Subcase 1.2. Vertex z' is internal for at least one of paths P_2, P_3, P_4.

Suppose z' belongs to one of the feeding paths $P_i \in \{P_2, P_3, P_4\}$. Since P_i is the shortest and due to Lemma 2, it follows that P_i starts at w and reaches z' through a 1-thread wpz'. We can assume that P_i is P_4.

If paths P_2, P_3, P_4 are not closed between each other, then each of them brings one restriction to vertices u, w and z', respectively. It remains to color the vertices of 5-cycle $z'pwut$; this follows by (i) of Lemma 9. If paths P_2, P_3, P_4 are closed between each other, the proof follows by (ii)–(iv) of Lemma 9.

Subcase 1.3. Vertex z' is internal for none of paths P_2, P_3, P_4.

Now consider possible closings between paths P_2, P_3, P_4. If P_2 is closed with P_3 then, as above, either P_3 is "destroyable" and P_2 is colorable, in which case w has only one restriction (along P_4) and thus can be colored after u, or vice versa P_2 is "destroyable" and P_3 is colorable, in which case w has three restrictions while u has two restrictions, so we are done by Lemma 6.

To complete the proof of Case 1, it suffices to assume that P_4 is closed with P_3. By Lemma 16, w has three admissible colors, while u has two, so we are done again by Lemma 6.

Case 2. P_1 is 0-thread.

Delete w, u, z, the internal vertices of paths P_3, P_4, P_{11}, P_{12}, and their 2-vertices.

Subcase 2.1. There is a 2-thread P between w and z.

Then we have paths P_2, P_4, P_{12} ($P_{11} = P_3 = P$). If P_2, P_4, P_{12} are independent, then each brings one restriction at u, w and z, respectively, by Lemma 13, and we are done by (i) of Lemma 10. If these paths are closed, then we are done by the rest of Lemma 10 combined with Lemma 16. Observe that the case then there are two 2-threads between w and z is even easier because one can use the same colors for the 2-vertices of the second of them as for the first.

Subcase 2.2. There is no 2-thread P between w and z.

Whatever closings among paths $P_2, P_3, P_4, P_{11}, P_{12}$, the numbers of admissible colors at w, u, z are sufficient to apply Lemma 8.

So, after discharging according to Rules R1–R3, the charges of all vertices are nonnegative, which contradicts (1). This completes the proof of Theorem 5.

Acknowledgement

The authors are thankful to Aleksey Glebov for useful remarks on the proof.

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