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PLANAR GRAPHS WITHOUT TRIANGULAR 4-CYCLES ARE 4-CHOOSABLE

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Abstract. It is known that not all planar graphs are 4-choosable (Margit Voigt, 1993), but those without 4-cycles are 4-choosable (Lam, Xu and Liu, 1999). We prove that all planar graphs without 4-cycles adjacent to 3-cycles are 4-choosable.

1. Introduction

As proved by Thomassen [9], every planar graph is 5-choosable, but Margit Voigt [11] constructed planar graphs that are not 4-choosable. Lam, Xu and Liu [8] proved that every planar graph without 4-cycles is 4-choosable.

The purpose of this paper is to provide a broader sufficient condition for the 4-choosability.

A cycle C is *triangular* if it has a common edge with a triangle T such that $T \neq C$. Note that the well-known Steinberg's conjecture (1976) that every planar graph without 4- and 5-cycles is 3-colorable was strengthened by Borodin, Glebov, Jensen and Raspaud [4] by asking whether every planar graph without triangular 4- and 5-cycles is 3-colorable. A relaxation of this Novosibirsk 3 Color Conjecture was proved in [4]: forbidding triangular cycles of length from 4 to 9 implies the 3-colorability.

Our main result is

Theorem 1. *Every planar graph without triangular 4-cycles is 4-choosable.*

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Recall that a graph G is called k-choosable if for any sets $L(v)$ of cardinality at least k at its vertices, one can choose a color for each vertex from its list $L(v)$ so that the colors of every two adjacent vertices were different. This extension of coloring was introduced by Vizing [10] and, independently, by Erdős, Rubin and Taylor [5] and has been intensively studied ever since.

Let $\delta(G)$ be the minimum degree of graph G. It follows from a result in [2] that if G is planar and has no triangular 4-cycles then $\delta(G) \leq 4$. By f_5^3 denote a 5-face adjacent to a 3-face. The proof of Theorem 1 is based on the following

Theorem 2. *Every connected planar graph G with $\delta(G) = 4$ and no triangular 4-cycles has an f_5^3 whose all vertices have degree 4.*

This is an extension of a similar result in [8], where all 4-cycles were forbidden rather than only triangular ones.

2. Deducing Theorem 1 from Theorem 2

Let G be a plane graph, and let $V(G)$, $E(G)$ and $F(G)$ be its sets of vertices, edges and faces, respectively. We consider only simple graphs. Denote the degree of a vertex v by $d(v)$ and the size of a face f by $r(f)$; a k-vertex is one of degree k. We write $\geq k$-vertex for a vertex of degree at least k, etc. Similar notation is used for faces; triangle is a synonym for 3-cycle. An edge is called triangular if it belongs to a 3-face. By a chord in a cycle C we mean an edge joining two nonconsecutive vertices of C.

Gallai's theorem [6] says that in every critical k-chromatic graph, every block of the subgraph induced by $(k-1)$-vertices is either a cycle or a complete graph. Erdős, Rubin and Taylor [5] extend Gallai's theorem to the case of choosability without changes (see also Corollary 6 in [3]). More generally, Borodin [1] (see also Theorem 6 in [3]) gives a necessary and sufficient condition for a connected graph to be nonchoosable w.r.t. a given list L such that $|L(v)| \geq d(v)$ for each vertex v.

Suppose G is a counterexample to Theorem 1 with the minimum number of vertices. Clearly, G is connected and $\delta(G) \geq 4$. Since $\delta(G) \leq 4$ by a result in [2], our G satisfies the hypothesis of Theorem 2.

Thus G has a 6-cycle $C_6^* = x_1 \ldots x_6$ with a triangular chord x_2x_6, which corresponds f_5^3. Note that the subgraph C_6^* induced by G on $\{x_1, \ldots, x_6\}$ has precisely 7 edges: this follows from the absence of triangular 4-cycles in G. Differently put, each of x_2 and x_6 is joined to $V(G) - \{x_1, \ldots, x_6\}$ by precisely one edge, whereas each x_1, x_3, x_4, and x_5, by two edges.

One readily sees that C_6^* is neither a cycle nor a complete graph; this contradicts the minimality of G due to the results in [1, 5].

3. Proof of Theorem 2

Suppose G is a counterexample to Theorem 2. Euler's formula $|V(G)|-|E(G)|+|F(G)| = 2$ for G may be rewritten as

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (r(f) - 4) = -8,$$

where $F(G)$ is the set of faces in G.

We set the *initial charge* $ch(x)$ of every $x \in V(G) \cup F(G)$ of G to be either $d(x) - 4$ or $r(x) - 4$, respectively. Clearly,

$$\sum_{x \in V(G) \cup F(G)} ch(x) = -8.$$

We now use a discharging procedure, leading to the *final charge* ch^*, defined by applying the following rules R0–R3, in which f is a ≥ 4-face with the boundary vertices $v_1 v_2 \dots$ (clockwise):

R0. Each 3-face receives $1/3$ from each adjacent face.

R1. Suppose f is adjacent to two 3-faces at v_2, $d(v_1) \geq 5$, $d(v_2) = 4$, and $d(v_3) \geq 5$. Then f sends charge $1/6$ across v_2 to the other ≥ 4-face incident with v_2.

R2. If $d(v_2) \geq 5$ and f is adjacent to two 3-faces at v_2, then f gets charge $2/3$ from v_2.

R3. Suppose $d(v_2) \geq 5$, $v_1 v_2$ belongs to a 3-face, while $v_2 v_3$ does not. Then f gets from v_2:
(a) $1/6$ if $d(v_3) \geq 5$;
(b) $1/3$ if $d(v_3) = 4$ and $v_3 v_4$ belongs to a 3-face.

Since the above procedure preserves the total charge, we have:

$$\sum_{x \in V(G) \cup F(G)} ch^*(x) = -8.$$

The rest of the proof consists in showing that $ch^*(x) \geq 0$ whenever $x \in V(G) \cup F(G)$, with an obvious final contradiction.

3.1. **Checking that new charge of $v \in V(G)$ is nonnegative.** Let $v_1, v_2 \dots$ be the neighbors of v in a cyclic order.

If $r(v) = 4$ then $ch^*(v) = ch(v) = 0$ since such vertices do not participate in R0–R3.

Now suppose $d(v) = 5$. If v participates in R2, giving away $2/3$ to a ≥ 4-face $v_1 v v_2 \dots$ between two triangles, then v gives to the other two ≥ 4-faces at most $1/3$ in total by R3: either $1/6 + 1/6$ (if $d(v_4) \geq 5$) or $1/3 + 0$ (if $d(v_4) = 4$ and v_4 is incident with a 3-face).

If v does not participate in R2, then it can only give charge of at most $1/3$ to at most two ≥ 4-faces by R3, so that $ch^*(v) \geq ch(v) - 2 \times 1/3 > 0$.

Finally, suppose $d(v) \geq 6$. Note that v can afford giving all incident faces a charge of at least $1/3$ on the average, for $d(v) - 4 - d(v)/3 = 2(d(v) - 6)/3 \geq 0$. So, we are done unless v participates in R2.

However, every donation of $2/3$ by v to a ≥ 4-face f_2 can be looked at as giving $1/3$ to that face and giving $1/6$ to each of 3-faces f_1 and f_3 that are clock- and counterclockwise neighbors of f_2 at v. As a result of this averaging, every face takes away from v at most $1/3 = 2/3 - 2 \times 1/6 = 2 \times 1/6$.

3.2. **Checking $ch^*(f) \geq 0$ whenever $f \in F(G)$.** Let $v_1, v_2 \dots$ be the vertices incident with f in a cyclic order. Recall that f can only loose charge of $1/3$ by R0 and of $1/6$ by R1.

If $r(f) = 3$ then $ch^*(f) = 3 - 4 + 3 \times 1/3 = 0$ by R0. If $r(f) = 4$ then $ch^*(f) = ch(f) = 0$ since such faces do not participate in R0–R3.

Now suppose $r(f) = 5$.

CASE 1. *f is incident with five 3-faces.* Thus R0 leaves f with a charge of $5 - 4 - 5 \times 1/3 = -2/3$.

If f is incident with a ≥ 5-vertex then we are done, unless f participates at least once in R1, giving away $1/6$, in which case f gets another $2/3$ by R2 and $ch^*(f) \geq 1 - 5 \times 1/3 + 2 \times 2/3 - 3 \times 1/6 > 0$.

If f is completely surrounded by 4-vertices, then due to the absence of f_5^3 in G, we have $ch^*(f) \geq 1 - 5 \times 1/3 + 5 \times 1/6 > 0$ by R1.

CASE 2. *f is incident with precisely four 3-faces.* Now R0 leaves f with a charge of $5 - 4 - 4 \times 1/3 = -1/3$. Let edge v_1v_2 be not incident with a 3-face.

If $d(v_1) \geq 5$ and $d(v_2) \geq 5$, then f gets $2 \times 1/6 = 1/3$ from v_1 and v_1 by R3a. If $d(v_1) = 4$ and $d(v_2) \geq 5$, then f gets $1/3$ from v_2 by R3b. In both cases, we are already done, unless f donates at least once $1/6$ by R1. In this case, it suffices to observe that f at least once gets $2/3$ by R2, whereas it donates $1/6$ by R1 at most twice, so that $ch^*(f) > 0$.

Now suppose $d(v_1) = d(v_2) = 4$. If at least one of v_3, v_4, v_5 has degree at least 5, then f has a superfluous $1/3$ after applying R0 and R2. Again, $ch^*(f) > 0$ since R1 is applicable to f (as a donator) at most once. Finally, suppose $d(v_1) = \ldots = d(v_5) = 4$; then due to the absence of f_5^3 in G, we have $ch^*(f) \geq 1 - 4 \times 1/3 + 3 \times 1/6 > 0$ by R1 applied to f as a receiver.

CASE 3. *f is incident with at most three 3-faces.* This time, R0 leaves f with a nonnegative charge. Suppose f makes a donation of $1/6$ by R1; say, via v_2. Since such a donation may be done only once, it would be suffice for f to get at least $1/6$ by R2 or R3. Suppose otherwise; then none of v_3v_4 and v_5v_1 belongs to a 3-face due to R2 and hence $d(v_4) = d(v_5) = 4$ due to R3a. Now, we see from R3b that v_4v_5 is not incident with a 3-face, which implies that $ch^*(f) \geq 1 - 2 \times 1/3 - 1/6 > 0$.

Finally, suppose $r(f) > 5$. Since $r(f) - 4 - r(f)/3 = 2(r(f) - 6)/3 \geq 0$, an argument similar to that given above in Case 3 of $r(f) = 5$ works. Namely, if v_2 causes troubles and v_3v_4 belongs to a 3-face, then v_3 brings additional $2/3$ to f, and $1/6$ of this $2/3$ may be assigned to v_2. Otherwise, the nontriangular edge v_3v_4 saves $1/3$ from the expenditure of f caused by R0, so again, $1/6$ may be attributed to v_2.

This completes the proof of Theorem 2.

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References

[1] O.V.Borodin, *Criterion of chromaticity of a degree prescription*, Abstracts of IV All-Union Conf. on Theoretical Cybernetics, Novosibirsk, 1977, 127–128 (in Russian).

[2] O.V.Borodin, *Structural properties of plane graphs without adjacent triangles and an application to 3-colorings*, J. of Graph Theory, **21**: 2 (1996), 183–186.

[3] O.V.Borodin, A.V.Kostochka and B.Toft, *Variable degeneracy: extensions of Brooks' and Gallai's theorems*, Discrete Math., 214 (2000), 101–112.

[4] O.V.Borodin, A.N.Glebov, T.R.Jensen, A.Raspaud, *Planar graphs without triangles adjacent to cycles of length from 3 to 9 are 3-colorable*, Siberian Electronic Math. Reports, http://semr.math.nsc.ru 3 (2006), 428–440.

[5] P.Erdős, A.Rubin and H.Taylor, *Choosability in graphs*, Congr. Numer., 26 (1979), 125–157.

[6] T.Gallai, *Kritische Graphen I*, Publ. Math. Inst. Hungar. Acad. Sci., 8 (1963), 165–192.

[7] T.R.Jensen and B.Toft, *Graph coloring problems*, Wiley Interscience (1995).
[8] P.C.B. Lam, B.Xu, J.Liu, *The 4-choosability of plane graphs without 4-cycles*, J. Combin. Theory, B 76 (1999), 117–126.
[9] C.Thomassen, *Every planar graph is* 5-*choosable*, J. Combin. Theory, B 62, 1 (1994), 180–181.
[10] V.G.Vizing, *Coloring the vertices of a graph with assigned colors*, Diskret. Analiz, Novosibirsk, 29 (1976), 3–10.
[11] M.Voigt, *List colourings of planar graphs*, Discrete Math., 120 (1993), 215–219.

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