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NEAR AND FAR. A CENTENNIAL TRIBUTE TO FRIGYES RIESZ

SOMASHEKHAR (SOM) NAIMPALLY

Abstract. This article gives a short history of the concept of near and far during the last one hundred years. It was first formulated by Frederick Riesz at the International Mathematical Congress in Rome (1908). It has been extended and studied by several mathematicians in teaching and research. Applications to general relativity and digital images have been found. A comprehensive bibliography is given.

Keywords: topology, proximity, nearness, calculus teaching, extension of continuous functions, Taimanov theorem, Wallman compactification, hyperspaces, Vietoris topology, proximal topology, Hausdorff metric, general relativity, digital images.

"Frederic Riesz made significant suggestions as to how the axiomatic foundations of general topology might be formulated. These suggestions are contained in two articles which appeared, respectively, in 1906 and 1908 [Math. Naturwiss. Ber. Ungarn 24 (1906), 309-353; per bibl; Atti IV Congr. Intern. Mat. II (1908), 18-24; JFM 40.0098.07]. Unfortunately, they were generally overlooked at that time and their importance was appreciated only after they were rediscovered much later. Even now they are not as well known as they deserve to be".

W.J. Thron [Th]

Frigyes (Frederic) Riesz, a leading mathematician of the last century, was a founder of functional analysis on which subject he wrote the well known book with his student B Szökefalvi-Nagy [RN]. He is famous for the following important discoveries: Riesz representation theorem, F. and M. Riesz theorem, Riesz–Fischer

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theorem, Riesz space. Frigyes' younger brother Marcel Riesz (1886–1969) was also a well known mathematician. Frigyes' work is fundamental in physics-especially in quantum theory. But, as Thron remarks in the above quote, the concept of near that he suggested as a foundation of topology/analysis is not as much known as his other achievements.

Frigyes Riesz was born on January 22, 1880 in Györ, Hungary. He studied at Budapest, Göttingen and Zurich obtaining a doctorate in Budapest in 1902. After a couple of years of teaching in schools he got a university appointment. In 1911 Riesz was appointed to a chair in Kolozsvár and in 1922 he set up the János Bolyai Mathematical Institute at Szeged with Haar. Riesz became editor of the newly founded journal of the Institute Acta Scientiarum Mathematicarum which became a leading periodical. He was appointed to the chair of mathematics in the University of Budapest in 1945. Among the many honours which Riesz received for his work are election to the Hungarian Academy of Sciences and the Paris Academy of Sciences and honorary doctorates from the universities of Szeged, Budapest and Paris. He died on February 28, 1956 in Budapest, Hungary. (For more details please see the biography by J. O'Connor and E. Robertson [OR]).

One hundred years ago Frigyes (Frederic) Riesz presented to the International Congress of Mathematicians in Rome the concept of nearness of two sets now known as proximity. *This concept, near-far is one of the rare concepts in the whole of mathematics that is at once intuitive and which can be made rigorous with little or practically no effort.*

(1) It can be used as an effective teaching tool in calculus, advanced calculus, analysis and topology to explain clearly the concepts of continuity, limits etc.

(2) By replacing intersection by near one can get general results with simpler proofs; this makes it valuable in research and in unification of results scattered in the literature.

In this article we propose to give a few simple examples to illustrate the above two cases. A few references are given at the end wherein further ones can be found by those readers interested in pursuing the matter further.

What is topology?

The concepts near and far are used in daily life and are so simple that they can be used to explain topology/continuity even to non-mathematicians. Consider a typical family mother, father, son, daughter. One may say that a person is near the family if that person is blood related to some member of the family. Of course, every member of the family is near their family! Grandparents, aunts, uncles, cousins... are persons near the family though they are not in the family. There are many other ways of defining such nearness relations, e.g. one may say that a person is near the family if the person helps the family in some way. In this definition the family physician, the plumber, the mailman, ... are near the family. This concept is axiomatized with a few simple obvious conditions and one gets the abstract concept of a topological space. This approach, suggested by Riesz, was formalised by Kuratowski in 1922 [Ku]. Nowadays it is customary for texts in topology to begin with the definition of a topology as a family of open sets satisfying a few conditions but in 1935 Alexandroff and Hopf [AH] used the Kuratowski closure axioms as a starting point of their treatise in topology.

With a topological space, is associated another concept that of a continuous transformation. Suppose five years ago P was a family physician of the family F. That is, the Physician P was near the family F under the rule that P helped the family in some way. Today, after five years, both the Physician P and the family F have changed. If the Physician P is still near the family F today, we say that it is a continuous relationship in day-to-day life and the same is said in mathematics. If for any reason, the Physician P ceases to be the family doctor of F, we say that the relationship is discontinuous. Thus a continuous transformation is one in which the nearness of a point to a set remains unchanged under that transformation.

To recapitulate, in topology we have nearness relations between points and sets together with continuous transformations which preserve these nearness relations.

At the next level of abstraction we talk about nearness between two families, technically called proximity. This idea, already present in Riesz's work, was thoroughly studied by Efremović around 1940 and published in 1951 [Ef]. This idea was further developed by Smirnov [Sm] and others. Again, this nearness between two families, can occur in several ways:

(a) two families can be near because a daughter from one family has married a son from another, or

(b) two families have a common friend (i.e., a person near both families), or

(c) two families are interested in music and meet at a concert thus getting near each other.

Moreover, in this subject one studies transformations which preserve nearness between pairs of sets. These are called proximally continuous transformations.

To recapitulate, in proximity we have nearness relations between pairs of sets together with proximally continuous transformations which preserve these nearness relations.

Subsequent levels of abstraction deal with nearness of finite sets called contiguity by Ivanov in 1959 [Iv] and nearness of sets of arbitrary cardinality called nearness by Herrlich in 1964 [He].

Teaching calculus and topology

In mathematics we often find that intuitive ideas are not rigorous and rigorous ideas are not intuitive. As an example consider the concept of continuity in calculus. The usual epsilon-delta definition is rigorous but is neither intuitive nor easy to teach or understand; for a detailed analysis of the difficulties involved see the Mathematical Association of America website of Devlin [De].

Intuitively, calculus teaching usually begins with the statement: a function f is continuous at a point p if whenever a point x is near p then $f(x)$ is near $f(p)$. This statement is intuitively clear but is not rigorous. Moreover, it is not easy to go from the above intuitive statement to the rigorous epsilon-delta definition. The intuitive statement goes from the domain to the range. On the other hand, the epsilon-delta definition goes in the opposite direction! However, the ideas of near and far are intuitive and they can be made rigorous with a little effort [CHN]. We replace "a point x is near p" by "a set E of points is near p" meaning the infimum of the distances between points of E and p is zero. So the precise definition of continuity of f at p means: f takes any set E that is near p to the set $f(E)$ that is near $f(p)$. The contra positive of the above statement is: if a set $f(E)$ is not near $f(p)$ in the range, then E is not near p in the domain; this is, of course, equivalent to the epsilon-delta definition! This approach to continuity is easily extended to other concepts, for example, limit point:

p is a *limit point* of a set E if p is near $E \setminus \{p\}$.

This approach can be used in teaching courses in calculus, advanced calculus, metric spaces, analysis and general topology [CHN, MN].

The next step is that of an abstract topological space. Suppose X is a nonempty set. We define a relation "a point p is near a set E" with axioms based on the relation described above in a metric space. Obviously,

(1) a point cannot be near an empty set;

(2) every point of a nonempty set is near it;

(3) a point is near a union of two sets if and only if it is near at least one of them;

(4) if a point p is near a set B and each point of B is near a set C, then p is near C.

The above four conditions are precisely the Kuratowski closure axioms wherein "p is near a set E" is equivalent to "p is a closure point of E".

Proximity

"The significant problems we face cannot be solved by the same level of thinking that created them."

Albert Einstein

Initially the word "proximity"was used in the literature to denote the one which satisfies the axioms discovered by Efremović. Among its many generalizations, one with an axiom analogous to the fourth Kuratowski closure axiom "closure of a set is idempotent"was discovered in 1964 by Leader (non-symmetric) [Le2] and Lodato (symmetric) [Lo]. In this article we deal with symmetric proximities only and use the terms "Efremović proximity" and "Lodato proximity" to distinguish them. We use the term "proximity" to denote nearness of two sets.

As explained above, topology deals with the structure "a point is near a set" and continuous functions which preserve this structure. Proximity is a structure finer than topology and is concerned with "one set is near another set". The Kuratowski closure axioms, presented in the above section, enable us to generalise it to proximity which deals with the nearness of two sets. From any proximity we get a topology when one of the sets is a singleton; the proximity and the topology related in this manner are called *compatible*. Thus a proximity induces a unique compatible topology. However, given a topology there are, in general, many proximities which induce the same compatible topology. For example, on the set of real numbers with the usual topology, there are infinitely many compatible proximities. Here are three:

(1) A is near B if and only if the infimum of the distance between points of A and points of B is zero (*the metric proximity*);

(2) A is near B if and only if their closures intersect (*the fine proximity*);

(3) A is near B if and only if their closures intersect or are not compact (*the Alexandroff proximity*).

We will show how this finer structure *proximity*, as per Einstein's above statement, is helpful in solving purely topological problems. The axioms that a proximity should satisfy are easily obtained by generalizing "a point is near a set" to "one set is near another set" in the Kuratowski closure axioms given above.

(1) If two sets are near then they are not empty.

(2) If two sets intersect then they are near, (thus nearness is a generalization of intersection).

(3) a set is near a union of two sets if and only if it is near at least one of them.

4) If a set A is near a set B and each point of B is near a set C, then A is near C. (This condition is equivalent to: two sets are near if and only if their closures are near.)

One question remains: if A is near B then is B near A? Both possibilities "yes" and "no" are useful. Thus there are symmetric as well as non-symmetric proximities, each having an extensive literature. In this article we deal with only symmetric proximities associated with a T_1 topological space that is a space in which distinct points are not near each other.

Proximities satisfying condition (4) in the above paragraph were studied by Leader (non-symmetric) [Le2] and Lodato (symmetric) [Lo] in 1964. The proximity that Efremović discovered in 1940 satisfies a condition stronger than the above [Ef]. The motivation is provided by a metric space. Consider a metric space X with metric d. Metric proximity can be defined by: subsets A and B of X are far (not near) if there is a positive such that the distance between each point of A and each point of B is greater than or equal to ε. Let E denote the set of points of X that are at least at a distance $\varepsilon/2$ from points of A. Then

$(4)'$ A is far from E and $(X \setminus E)$ is far from B. (*Efremović axiom*)

This axiom is a vestigial form of the triangle inequality in a metric space.

It is easy to show that $(4)'$ is stronger than (4); that is each Efremović proximity is a Lodato proximity. In a metric space the metric proximity is generally different from the fine proximity; for example on the space of real numbers the set $\mathbb{N}$ of natural numbers is near the set $\{n - 1/n : n \in \mathbb{N}\}$ in the metric proximity but not near in the fine proximity; but both proximities induce the same usual topology on real numbers

A question arises: which topological spaces have compatible Efremović proximities. These are Tychonoff spaces in which a closed set and a point not in it can be separated by a continuous function into the closed interval $[0, 1]$. Each Tychonoff space has compatible Efremović proximities which, in turn, have compatible uniformities. There is a vast literature on this topic [NW]. There are also many other proximities but we will not consider them in this article.

Extensions of spaces: Wallman compactification

One of the most important topics in topology is compactification, which is to embed a non-compact space into a compact one. The two well known examples in analysis are

(1) the Riemann sphere which is the one-point-compactification of the space of complex numbers, and

(2) the two-points compactification of the space of real numbers with the addition of $\{-\infty, \infty\}$.

There are many compactifications in the literature, for example Alexandroff, Stone-Ćech, Freudenthal, Wallman-Frink, Smirnov, Bohr, Samuel etc.. One of them is due to Wallman [Wa] which is given as a problem in the well known text by Kelley ([Ke, p. 167]). We show now that by merely changing intersection into near and using a compatible Lodato proximity with maximal bunches (see below the definition of a bunch) we can get a generalization which includes all T_1, and ipso facto, all

Hausdorff compactifications as special cases. Recall the important property of an ultrafilter in a nonempty set X : each finite family of subsets of the ultrafilter have nonempty intersection. It is possible to reduce this property to the intersection of just any two by showing that an ultrafilter $\mathcal{L}$ in X is a family of nonempty subsets of X characterized by three conditions:

(1) any two sets in $\mathcal{L}$ intersect,

(2) union of two sets is in $\mathcal{L}$ if and only if at least one of them is in $\mathcal{L}$,

(3) if a set intersects every member of $\mathcal{L}$ then it belongs to $\mathcal{L}$.

If X is a proximity space and we replace *intersection* by *near* we get the definition of a *cluster* σ [Le1]:

(a) any two members of σ are near,

(b) union of two sets is in σ if and only if at least one of them is in σ,

(c) if a set is near every member of σ then it belongs to σ.

For example the family of all points near a point p in X is a cluster called point cluster σ_p. If we replace condition (c) by

(c)′ a set belongs to σ if and only if its closure belongs to it, we get a *bunch* [Lo]. So bunches and clusters are proximal cousins of ultrafilters.

Let X be a T_1 topological space. The *Wallman compactification*: wX is the set of all closed ultrafilters in X with a suitable topology ([Ke, p. 167]). A subset E of X is said to *absorb* $\mathcal{P} \subset wX$ iff $E \in \mathcal{L}$ for each ultrafilter $\mathcal{L} \in \mathcal{P}$.

(α) The *Wallman topology* on wX is given by the Kuratowski closure operator cl defined by a closed ultrafilter $\mathcal{L} \in \mathrm{cl}\mathcal{P} \subset wX \Leftrightarrow$ whenever a subset E of X absorbs P, it belongs to $\mathcal{L}$.

(β) There is a natural map $\phi : X \to wX$, given by $\phi(x) = \mathcal{L}_x$, the closed ultrafilter containing $\{x\}$. The map ϕ is a homeomorphism on X onto $\phi(X) \subset wX$, i.e. ϕ is an embedding of X into wX.

(γ) $\phi(X)$ is dense in wX.

(δ) wX with the Wallman topology is a compact T_1 space. That is the space wX is a T_1 compactification of X.

We now begin with a T_1 topological space with a compatible Lodato proximity. Replacing wX, the set of all closed ultrafilters, by the family of all maximal bunches in the proximity space X and, following Wallman in his construction step by step, we arrive at a T_1 compactification that includes *all* T_1 compactifications as special cases [GN1]. Further if we begin with a Tychonoff space, each maximal bunch becomes a cluster and we get the Smirnov compactification which includes all Hausdorff compactifications as special cases. This shows that the Wallman's method, generalized with proximity, makes it all the more valuable. Instead of being relegated to a mere exercise, it deserves to be considered the most important compactification.

Extensions of continuous functions from dense subspaces: generalized Taimanov's theorem

An important problem in analysis and topology is the continuous extension of a continuous function from a dense subspace to the whole space. This is used, for example, in partially ordering compactifications. A well known result on this topic says that sufficient conditions are: the function is uniformly continuous and the range space is complete ([Ke, p. 195]). One of the most beautiful and interesting results is due to A. D. Taimanov. Taimanov proved the following [Ta]:

Theorem. *Let D be a dense subset of a T_1 space X and let Y be a compact Hausdorff space. Then a continuous function f on D to Y has a continuous extension F on X to Y if and only if for every pair of disjoint closed subsets A, B of Y, closures in X of $f^{-1}(A), f^{-1}(B)$ are disjoint.*

From the section on proximity it is clear that if X and Y are assigned the fine proximities, then Taimanov's condition is equivalent to: f is proximally continuous. Using this observation the following generalization of Taimanov's theorem can be proven [GN1]:

Generalized Taimanov Theorem. *Let D be a dense subset of a T_1 space X and let Y be a Tychonoff space with Smirnov compactification ηY. Let X and ηY be assigned the fine proximities. Then a continuous function f on D to Y has a continuous extension F on X to Y if and only if f is proximally continuous.*

This result can be used to simplify the proofs of many results involving compactifications such as for example: the Stone- ?ech compacification is maximal.

Hyperspaces

Let X be a metric space, a topological space or a topological vector space. Let $CL(X)$ denote the family of all nonempty closed or closed convex or compact subsets of X. Then there is a need to put a suitable topology on $CL(X)$ known as *hyperspace*. There is a vast ever growing literature in hyperspaces and there are numerous applications to mathematical economics, optimization, convex analysis, functional analysis, mines, etc. In this section we give a brief introduction to the role of proximity in hyperspaces. A standard reference on this subject is [Be].

One of the early attempts, in a very general setting, was made by Vietoris in 1922 [Vi]. Let $CL(X)$ denote the family of all nonempty closed subsets of a T_1 space X with a compatible Lodato proximity. A typical member of a base for the *lower Vietoris topology* on $CL(X)$ consists of members of $CL(X)$ which intersect or *hit finitely* many open sets (called *hit* sets). A typical member of a base for the *upper Vietoris topology* on $CL(X)$ consists of members of $CL(X)$ which are disjoint from a closed set i.e., they *miss* the closed set. The *Vietoris topology* is the join of the lower and upper Vietoris topologies. and can be obviously called a *hit-and-miss* topology.

Next we keep the lower Vietoris topology the same and change the upper Vietoris topology by requiring a typical member of the base to be *far* from a closed set. In this way we get a new upper topology called the *upper-far topology* which is coarser than the upper Vietoris topology. The join of the lower Vietoris hit topology and the upper far topology is called the *proximal topology* which is really a *hit-and-far* topology. The Vietoris topology is, of course, a special case of the proximal topology when the space has the fine proximity. Proximal topology, introduced in 1988 [DCNS], has been found to be quite useful.

The Hausdorff metric topology was first discovered in an equivalent form by Pompeiu in 1905 and by Hausdorff in 1927 (see [Be]). A widely known hypertopology ([Ke, p. 131]), it has been used in function space topologies, approximation theory, optimization etc. Let (X, d) be a metric space. First we recall, with a slight difference, the Hausdorff metric d' as given in Kelley's text; suppose A, B are nonempty closed subsets of X. For $\varepsilon > 0$, let $S(A, \varepsilon)$ denote the set of all points x of X such that

$\text{dist}(x, A) < \varepsilon$. Then

$$d'(A,B) = \inf\{\varepsilon > 0 : B \subset S(A,\varepsilon) \text{ and } A \subset S(B,\varepsilon)\},$$
$$= \infty \text{ if no such } \varepsilon \text{ exists.}$$

It is a simple exercise to show that the above definition of the Hausdorff metric is equivalent to:

$$d'(A,B) = \sup\{|d(x,A) - d(x,B)| : x \in X\}.$$

It is clear from the above definition that in the Hausdorff metric topology, the sequence (A_n) of closed sets converges to a closed set A if and only if the sequence of distance functionals $(\text{dist}(x, A_n))$ *converges uniformly* on X to the distance functional $\text{dist}(x, A)$.

For tackling problems in convex analysis, Wijsman [Wi] replaced uniform convergence by *pointwise convergence*, that is (A_n) converges to A if and only if the sequence $(\text{dist}(x, A_n))$ converges to $\text{dist}(x, A)$ for each point x of X. *Wijsman topology* has been found to be useful in many ways, especially as building blocks of other hypertopologies [Be]. Wijsman topology can also be split into two parts. The lower Wijsman topology is the same as the lower Vietoris topology. The upper Wijsman topology can be described as follows:

(A_n) converges to A if and only if for every $0 < \varepsilon < \alpha$, and for each $x \in X$, if A misses the open ball $S(x, \alpha)$, then eventually, A_n misses $S(x, \varepsilon)$.
Obviously, "A misses the bigger open ball $S(x, \alpha)$" implies "A is far from the smaller concentric ball $S(x, \varepsilon)$".

So the upper Wijsman topology is a *far-miss* topology. A typical neighbourhood of A in $CL(X)$ in the upper Wijsman topology consists of closed sets E which miss a ball B that is far from A. Here the far-miss sets can be open or closed balls, since two sets are far iff their closures are far. Thus the Wijsman topology is a *hit-and-far-miss* topology. This representation is useful in giving simple conceptual proofs of many results in place of long epsilonetic proofs [DN].

There are many other hypertopologies. They have lower parts similar to the lower Vietoris topology, sometimes containing discrete or locally finite families of open sets as hit sets. The upper parts are similar to the upper Vietoris topology, and the miss sets or far sets can be balls in a metric space or, in general, any family Δ, a subset of $CL(X)$, as Poppe first studied (see [Po]).

Applications

"By and large it is uniformly true that in mathematics there is a time lapse between a mathematical discovery and the moment it becomes useful. This lapse can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things, which are useful."

John von Neumann

So far we have given some details of applications of proximity to topological problems. There are several others such as (a) unified approach to metrisation, (b) proximities and Mozzochi uniformities associated with developable spaces; Brown's problem, (c) hyperconvergence which generalizes continuous convergence, (d) function space topologies, (e) real compactifications, (f) open and uniformly open maps: Pettis' problem, (g) duality dealing with Ascoli-Arzelá theorems etc. [Na].

Besides, there are many applications of proximity outside general topology. We give some details about two of them.

(I) Application of proximity and Mozzochi uniformity to general relativity [MN, Mo, DL, HKM].

(II) Digital images [LP, PK].

General Relativity. The solution of the metrisation problem by Bing, Nagata and Smirnov, led to a surge in the study of generalized metric spaces. A generalized metric space is a topological space which is not metrisable but satisfies some condition fulfilled by a metric space. A semi-metric satisfies the conditions of a metric with the possible exception of the triangle inequality. The topology is related to the semi-metric d by the condition:

x is in the closure of $A \;\Leftrightarrow\; \inf\{d(x,a) : a \in A\} = 0$.

A *developable space* is a semi-metric space which arose naturally in the metrisation problem. Developable spaces were recognized to be an important class of topological spaces. They lie midway between general topological spaces and metrisable spaces. A Moore space is a regular developable space and many distinguished mathematicians worked on this topic. The following are equivalent (solution to a problem posed by Brown [Br] published in [GN2]):

(1) A topological space X is developable;

(2) X has a compatible upper semi-continuous semi-metric;

(3) X has a compatible Mozzochi uniformity with a countable base in which every convergent sequence is Cauchy.

In a semi-metric space, open balls need not be open, convergent sequences need not be Cauchy etc. These properties are unusual; it is a pleasant surprise to see that these spaces find application in general relativity [DL].

Digital Images. The concept of *nearness* enters as soon as one starts studying digital images. The digital image of a photograph should resemble, as accurately as possible, the original i. e. it should be *globally close*. Here topology alone does not suffice since it is *local* and a digital image, which is topologically equivalent to a picture, might be just a *cartoon*! Since proximity deals with *global* properties, it is appropriate for this study. The quality of a picture depends on Lodato proximity. A digital image put into a computer consists of *finitely* many pixels, which try to give a faithful representation of a *continuous* original. So this involves a transformation from an uncountable infinite set E in the Euclidean space $\mathbb{R}^n$ into a finite set in $\mathbb{Z}^n$ where $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers. Thus, there are problems that need new concepts in topology and proximity [LP, PK].

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