

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

<http://semr.math.nsc.ru>

Том 6, стр. 272–311 (2009)

УДК 512.5

MSC 13A99

FILIPPOV–NAMBU n -ALGEBRAS RELEVANT TO PHYSICS

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ABSTRACT. Gauge symmetry based on Lie algebra has a rather long history and it successfully describes electromagnetism, weak and strong interactions in the nature. Recently the Filippov–Nambu 3-algebras have been in the focus of interest since they appear as gauge symmetries of new superconformal Chern–Simons non-Abelian theories in $2 + 1$ dimensions with the maximum allowed number of $\mathcal{N} = 8$ linear supersymmetries. These theories explore the low energy dynamics of the microscopic degrees of freedom of coincident M2 branes and constitute the boundary conformal field theories of the bulk $AdS_4 \times S_7$ exact 11-dimensional supergravity backgrounds of supermembranes. These mysterious new symmetries, the Filippov–Nambu 3-algebras represent the implementation of non-associative algebras of coordinates of charged tensionless strings, the boundaries of open M2 branes in antisymmetric field magnetic backgrounds of M5 branes in the M2–M5 system. A crucial input into this construction came from the study of the M2–M5 system in the Basu–Harvey’s work where an equation describing the Bogomol’nyi–Prasad–Sommerfield (BPS) bound state of multiple M2-branes ending on an M5 was formulated. The Filippov–Nambu 3-algebras are either operator or matrix representation of the classical Nambu symmetries of world volume preserving diffeomorphisms of M2 branes. Indeed at the classical level the supermembrane Lagrangian, in the covariant formulation, has the world volume preserving diffeomorphisms symmetry $SDiff(M_{2+1})$. The Filippov–Nambu 3-algebras presumably correspond to the quantization of the rigid motions in this infinite dimensional group, which describe the low energy excitation spectrum of the M2 branes. It emphasizes the Filippov–Nambu n -algebras as the mathematical framework for describing symmetry properties of classical and quantum mechanical systems.

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The work is supported by RFBR 09-02-00078, 08-02-00334-a and NSh - 2553.2008.2.

Received July, 8, 2009, published October, 16, 2009.

Keywords: Filippov n -algebra, Nambu bracket, supersymmetry, super p-branes.

1. INTRODUCTION

Mathematics provides us with a language in which we formulate the laws that govern phenomena observed in the nature. This language has proven to be both powerful and effective.

A foundation of physics cannot be built solely on this ground, however; an even more essential ingredient is experiment, and any substantial progress in physics eventually has to lead to predictions that can be tested experimentally. Nevertheless, the quest for a deeper understanding of fundamental physical issues, such as the interactions among elementary particles or the structure of space-time, as results us to theories which are even harder to put to observational tests. In this situation, mathematical conciseness and internal consistency of a physical theory become increasingly important guidelines in the evolution of physics. In studies of physical phenomena we want to discover hidden mathematical structures which govern underlying processes. These structures can be either known or new but in any case new studies in physics pose new mathematical problems even in old classical areas of mathematics. In its turn, studies of mathematical structures relevant to the physical phenomena lead to new developments of physical theories. Textbook example providing an important role in physics is an algebra and representation theory. They are the basis of major progress in string theory, conformal and topological quantum field theory, and integrable systems. Conversely, ideas from these areas are directly related to new developments in mathematics.

In recent years, novel issues such as aspects of three-dimensional topological field theories which are expected to be relevant to topological quantum computing and string/ \mathcal{M} theory have merged in mathematical physics, especially in quantum field theory. Accordingly, additional areas of mathematics have become influential and, in turn, been influenced themselves by the developments in physics.

Any search for generalizations of synthesis of quantum mechanics and relativity theory has to have a well defined motivation. One possible general starting point is provided by the observation that the evolution of fundamental physical theories, characterized by appearance of new dimensionful parameters (new constants of nature), can be mathematically understood from the point of view of deformation theory. In particular, relativity theory, quantum mechanics and quantum field theory can be understood mathematically as deformations of unstable structures [1]¹. An example of an unstable algebraic structure is non-relativistic classical mechanics. By deforming an unstable structure, such as classical non-relativistic mechanics, via dimensionful deformation parameters, the speed of light c and the Planck constant \hbar , one obtains new stable structures - special relativity and quantum mechanics. Likewise, relativistic quantum mechanics (quantum field theory) can be obtained through a double (c and \hbar) deformation. It is natural to expect that there is a further deformation via one more dimensionful constant, the Planck length l_P . The resulting structure could be expected to form a stable structural basis for a quantum theory of gravity. A closely related idea has appeared in open string field theory, as originally formulated by Witten [2]. There, the deformation parameters are the

¹An algebraic structure is called stable (or rigid) for a class of deformations if any deformation in this class leads to an equivalent (isomorphic) structure.

tension in the string α' and \hbar . The classical open string field theory Lagrangian is based on the use of the string field (which involves an expansion to all orders in α') and a star product which is defined in terms of the world-sheet path integral, also involving α' . The full quantum string field theory is thus, in principle, an example of a one-parameter (α') deformation of quantum mechanics. String theory is well known to be the leading prospect for quantizing gravity and unifying it with other interactions. One may also take a broader view of string theory as a description of string-like excitations that arise in many different physical systems, such as the superconducting flux tubes or the chromo-electric flux tubes in non-Abelian gauge theories. From the point of view of quantum field theories describing the physical systems where these string-like objects arise, they are "emergent" rather than fundamental.

In string theory [3] the graviton and all other elementary particles are one-dimensional objects: strings, rather than points as in quantum field theory. String theory may resemble the real world in its broad outlines, but a decisive test still seems to be far away. The main problem is that while there is a unique theory, it has an enormous number of classical solutions, even if we restrict attention to the solution with four large flat dimensions. Upon quantization, each of these solutions is a possible ground state for the theory and the four-dimensional physics is different in each of these. Until recently, our understanding of different versions of string theory was limited to perturbation theory, that corresponds to small numbers of strings interacting weakly. It was not known even how to define the theory at strong coupling. String theory was revolutionized by the discovery of D -branes [4]. The understanding of these nonperturbative objects allowed to uncover a deep connection between non-Abelian gauge theories and string theory. This resulted in the Maldacena correspondence [5] where a fascinating duality between gauge theories and string/gravitational theories is of great importance. The origin of the non-Abelian degrees of freedom came from the open strings extending between different D -branes and becoming massless when the D -branes coincide. In addition, thanks to the AdS/CFT correspondence and its extensions, we now know that at least some field theories have dual formulations in terms of string theories in curved backgrounds. In these examples, the strings that are "emergent" from the field theory point of view are dual to fundamental or D -strings in the string theoretic approach. Besides being of great theoretical interest, such dualities are becoming a useful tool for studying strongly coupled gauge theories. These ideas also have far-reaching implications for building connections between string theory and the real world.

In the past few years a great progress on these issues, we observed connected largely with the systematic application of the constraints imposed by supersymmetry. It was found that the strongly coupled limit of any string theory is described by a dual weakly coupled string theory, or by a new eleven-dimensional theory known as \mathcal{M} -theory whose low energy limit is an eleven dimensional supergravity. The extended objects are no longer strings but membranes and five-branes. All the different string theories are different compactification limits of this single theory, as such \mathcal{M} -theory unified string theories. The five different versions of string theory are just \mathcal{M} -theory expanded around different vacua. This \mathcal{M} -theory web then explained the nonperturbative dualities that had been conjectured in string theory some years before. Understanding the structure of \mathcal{M} -theory as an underlying theory of all

known string theories was one of the major efforts during the past decade. Although several approaches have been found, we still don't have any clear picture of this theory. A true formulation of \mathcal{M} -theory away from the low energy limit is still a far away dream. Fortunately, in the last few years, some ground breaking ideas were invented for describing the dynamics of fundamental objects of \mathcal{M} -theory based on the Filippov 3-algebras [6], i.e. membranes and five branes. The original motivation was a search for a theory describing coincident M2-branes [7] (see also [8] for the references).

While string theories are based on two-algebra structure, recent advances in M2-theory started by Bagger–Lambert–Gustavsson (BLG) [10] suggest that full description of \mathcal{M} -theory may require a generalized Lie-algebra structure: namely three algebra or even higher, n -algebra structure. In fact, the digits, two and three, appear to have intriguing associations with string and \mathcal{M} -theory respectively, first of all, two is the dimension of string worldsheet while three is that of membrane worldvolume. In the mathematical literature, the Lie 3-algebra (a term originally coined by Filippov [6], following earlier work of Nambu [9]) is not new, and its structure has been studied to some extent. In the long history of the study of the Nambu brackets their relation to the supermembranes or \mathcal{M} -theory especially interestingly. There have been many attempts to quantize the classical the Nambu bracket towards this direction. However, since the quantization is difficult and does not seem to be unique, we need to understand which properties are essential from the physical viewpoint. Recently there was some progress in constructing a $2 + 1$ -dimensional local quantum field theory with $SO(8)$ superconformal symmetry [10]. This is a useful significant step to obtain a world-volume Lagrangian description for coincident M2-branes. Crucial for the construction is the use of 3-algebras which are built around antisymmetrized product of three operators. In general, n -algebras as a natural generalization of Lie algebras are defined by a multi-linear map $[\star, \dots, \star]: \mathcal{A}^n \rightarrow \mathcal{A}$ on a linear space $\mathcal{A} = \sum v_a T_a, v_a \in \mathbb{C}$.

More than three decades ago, Nambu [9] proposed a generalization of classical Hamiltonian mechanics. In his formalism, he replaced the usual pair of canonical variables of the Hamiltonian mechanics by a triplet of coordinates in an odd dimensional phase space. Furthermore, he formulated his dynamics using a ternary operation, the Nambu bracket, as opposed to the usual binary Poisson bracket. Yet the fundamental principles of a canonical form of the Nambu's mechanics, similar to the invariant geometrical of the Hamiltonian mechanics, have only recently been discovered [11]. The re-emergence of this little known theory is possibly due to its relevance to the recent mathematical structures having their basis in the classical motion of topological open membranes as well as maximally superintegrable systems, such as the Hydrogen atom and so on that are controlled by classical Nambu brackets. Since the basic idea of the Nambu mechanics is to extend the usual binary operation on the phase space to multiple operations of higher order, this theory may also give some insights into the theory of higher order algebraic structures and their possible physical significance. In any case, both the Nambu's and Filippov's works motivated and inspired a lengthly survey of these ideas by Takhtajan [11] (also see [12]) along with many other studies.

In these review we have described main features and problems of a consistent \mathcal{M} -theory and pointed out some ways using the concepts of n -algebras along which it has been developed over last years. This survey is intended for mathematicians

who are non-specialists in the field of theoretical physics. Therefore, technical details are kept to a minimum and we refer to various other literature throughout for the relevant formalisms. There are many other topics in this vast field that are not touched upon in this review. Furthermore, the bibliography is not meant to be exhaustive, and we apologize in advance to those concerned for the omissions.

2. n -LIE ALGEBRAS

We will briefly recall the definition of metric n -Lie algebras as introduced by Filippov in [6], of which ordinary Lie algebras ($n=2$) and the 3-algebras appearing in the BLG theory and the Nambu 3-algebras [9] are special cases.

Define a (complex) n -Lie algebra as an algebra with an n -ary map $[\cdot, \dots, \cdot]: \mathcal{A}^n \rightarrow \mathcal{A}$ such that:

(a) $[\cdot, \dots, \cdot]$ is totally antisymmetric, i.e.

$$(1) \quad [T_1, \dots, T_n] = (-1)^{\varepsilon(\sigma)} [T_{\sigma(1)}, \dots, T_{\sigma(n)}], \quad T_i \in \mathcal{A},$$

for all $T_1, \dots, T_n \in \mathcal{A}$ and $\varepsilon(\sigma)$ is the parity of a permutation σ .

(b) any $(n-1)$ -plet acts via $[\cdot, \dots, \cdot]$ as a derivative, i.e. the bracket satisfies the fundamental identity for all $T_i, R_i \in \mathcal{A}$

$$(2) \quad [T_1, \dots, T_{n-1}, [R_1, \dots, R_n]] = \sum_{i=1}^n [R_1, \dots, R_{i-1}, [T_1, \dots, T_{n-1}, R_i], R_{i+1}, \dots, R_n],$$

which preserves main properties of the Jacobi identity. It means that the bracket $[T_1, \dots, T_{n-1}]$ acts as a derivative on \mathcal{A} , and it may be represented a symmetry transformation. In terms of the basis, n -algebra is expressed in terms of structure constants

$$(3) \quad [T_{a_1}, \dots, T_{a_n}] = i f_{a_1 \dots a_n}^b T_b.$$

The fundamental identity implies a bilinear relation between the structure constants

$$(4) \quad \sum_c f_{b_1 \dots b_p}^c f_{a_1 \dots a_{p-1} c}^d = \sum_i \sum_c f_{a_1 \dots a_{p-1} b_i}^c f_{b_1 \dots c \dots b_p}^d.$$

The adjoint action of $\wedge^{n-1} \mathcal{A}$ on \mathcal{A} is defined as follows

$$ad_\Lambda v = f^{a_1 \dots a_p}_b \Lambda_{a_1 \dots a_{p-1}} v_p T^b,$$

where $v = v_a T^a$ and $\Lambda = \Lambda_{a_1 \dots a_{p-1}} T^{a_1} \wedge \dots \wedge T^{a_{p-1}}$. The fundamental identity is equivalent to the statement that the adjoint action acts as a derivative on the bracket

$$ad_\Lambda([v_1, \dots, v_p]) = [(ad_\Lambda v_1), \dots, v_p] + \dots + [v_1, \dots, (ad_\Lambda v_p)].$$

The derivatives ad_Λ obviously form a Lie algebra $ad_\Lambda ad_{\bar{\Lambda}} - ad_{\bar{\Lambda}} ad_\Lambda = ad_{[\Lambda, \bar{\Lambda}]}$.

The n -Lie algebra can be equipped with an invariant inner product as a bilinear map from $\mathcal{A} \times \mathcal{A}$ to C

$$(5) \quad \langle T_a, T_b \rangle = h_{ab}.$$

We will refer to the symmetric tensor h_{ab} as the metric. As a generalization of the Killing form in Lie algebra, we require that the metric is invariant under any transformation generated by the bracket $[T_{a_1}, \dots, T_{a_{n-1}}, \cdot]$:

$$(6) \quad \langle [T_{a_1}, \dots, T_{a_{n-1}}, T_b], T_c \rangle + \langle T_b, [T_{a_1}, \dots, T_{a_{n-1}}, T_c] \rangle = 0.$$

This implies a relation for the structure constants

$$(7) \quad h_{cd}f_{a_1\dots a_{n-1}b}{}^d + h_{bd}f_{a_1\dots a_{n-1}c}{}^d = 0 ,$$

therefore the tensor

$$(8) \quad f_{a_1\dots a_n} \equiv f_{a_1\dots a_{n-1}}{}^b h_{ba_n} ,$$

is totally antisymmetrized. For applications to physics, it is very important to have a nontrivial metric h_{ab} in order to write down the Lagrangian or physical observables which are invariant under transformations defined by n -brackets. Assuming the positivity of metric h leads to severe restrictions on the structure constants of n -algebra [13]. The adjoint action of a n -algebra with an invariant metric can be described alternatively through the matrix action on \mathcal{A} . The element $\Lambda \in \wedge^{n-1}\mathcal{A}$ can be mapped to $Mat_{n \times n}$, $n = \dim \mathcal{A}$ as follows $\lambda^c{}_b = f^{a_1\dots a_{n-1}c}{}_b \Lambda_{a_1\dots a_{n-1}}$, such that λ 's satisfy the following properties

$$f^{a_1\dots a_n}{}_c \lambda^c{}_b = f^{ca_2\dots a_n}{}_b \lambda^{a_1}{}_c + \dots + f^{a_1\dots a_{n-1}c}{}_c \lambda^{a_n}{}_c, \quad \lambda^a{}_c h^{cb} = -h^{ac} \lambda^b{}_a .$$

Another mathematical structure of physical importance is the Hermitian conjugation. A natural definition of the Hermitian conjugate of an n -bracket is

$$(9) \quad [A_1, \dots, A_n]^\dagger = [A_n^\dagger, \dots, A_1^\dagger] .$$

This relation determines the reality of the structure constants. If we choose the generators to be Hermitian for the usual Lie algebra, the structure constants $f_{ab}{}^c$ are real numbers, and if the generators are anti-Hermitian, the structure constants are imaginary. This is not the case for 3-brackets. The structure constants are always imaginary when the generators are all Hermitian or all anti-Hermitian. In general, for n -brackets, the structure constants are real if $n = 0, 1 \pmod{4}$. They are imaginary if $n = 2, 3 \pmod{4}$ for the Hermitian generators. The structure constants are multiplied by a factor of $\pm i$ when we replace the Hermitian generators by anti-Hermitian ones only for even n .

Simple examples are given by the n -Lie algebras [6]. In particular, it is shown that vector multiplication of vectors of the $(n + 1)$ -dimensional Euclidean space and the Jacobian $|\partial f_i / \partial x_j|$ of the polynomials $f_1, \dots, f_n \in \mathcal{A}$ algebra in n variables x_1, \dots, x_n on an oriented n -dimensional manifold can be taken as canonical examples of the operations in \mathcal{A} . However, it is not only that the complete classification of the n -algebra does not exist, but there are very few explicit examples in the literature.

2.1. Three algebras and $\mathcal{N} = 8, 6$ Chern–Simons gauge theories. The branes of \mathcal{M} -theory are important but still are quite mysterious objects. Recently the construction of superconformal Chern–Simons–matter theories in three dimensions has attracted a lot of attention in string/ \mathcal{M} -theory community, because they are natural candidates for the dual gauge description of M2 branes in \mathcal{M} theory [14]. Briefly, there are eleven bosonic degrees of freedom corresponding to the embedding of the membrane. The reparameterization invariance of the worldvolume gauges away three of these so that there are eight bosonic degrees of freedom at the end. The fermions start out as thirty-two component spinors. The mass-shell condition and the κ -symmetry each have the available degrees of freedom. The bosonic and fermionic degrees of freedom are then organized as an $\mathcal{N} = 8$ multiplet of the three-dimensional worldvolume theory.

As is well known, generically Chern–Simons gauge theories in three dimensions are conformally invariant, both for pure gauge theories and for theories coupled

to massless matter fields. This remains true even at the quantum level (in spite of a quantum shift at one loop order), the Chern–Simons gauge coupling does not run at all, because its β function vanishes, as shown both by an explicit two-loop calculations for theories with matter and by formal proof up to all orders in perturbation theory for pure gauge theories. In order to construct the dual gauge description of M2 branes, the relevant issue is then how to incorporate extended supersymmetries into Chern–Simons-matter theories, since extended supersymmetry plays a crucial role in \mathcal{M} -theory as it does in superstring theory.

In a series of recent papers [10] a non-Abelian model of multiple M2-branes based on an 3-algebra as the internal symmetry has been proposed. The theory living on an M2 brane is conformal. So the fields acquire the length dimensions

$$(10) \quad \{A_m, X^a, \Psi, \epsilon\} = \{-1, -\frac{1}{2}, -1, \frac{1}{2}\}.$$

One may ask what requirements come from supersymmetry. The fermionic field Ψ is a Majorana spinor in $10 + 1$ dimensions satisfying the chirality condition $\Gamma_{012}\Psi = -\Psi$. As result Ψ has 16 real fermionic components, equivalent to 8 bosonic degrees of freedom X^I . In $2 + 1$ dimensions a gauge potential usually has one propagating degree of freedom. However, here the gauge potential has no canonical kinetic term, but only the Chern–Simons term, and hence it has no propagating degrees of freedom. Simple dimensional analysis suggests that in the supersymmetry variations include product of two as well as of three fields. It is of course desirable that all products of our fields are such that they close on some internal algebra. The way do that is to making the minimal assumption that there is a multiplication of two and three fields which belong to some set of fields, that we denote as \mathcal{A} , such that the product of three elements in \mathcal{A} must yield back an element in \mathcal{A} . Then we see what requirements of closure of the supersymmetry transformations impose these new type multiplications.

Based on the totally antisymmetric Filippov 3-brackets, the maximally (i.e. $\mathcal{N}=8$) supersymmetric Chern-Simons-matter theory in d3 with $SO(4)$ gauge group and $SO(8)$ R-symmetry, was constructed [10] as the dual gauge description of two M2 branes. It was also shown that in possible to overcome the obstacles of no-go theorem [13] only for three-algebra which has a symmetric and positive defined metric is either $so(4)$ or direct sum of a number of $so(4)$'s. In this sense the BLG theory is rather unique and can describe only two coincident M2-branes. The BLG theory is based on 3-algebras. A 3-algebra \mathcal{A} is an N dimensional vector space endowed with a trilinear skew-symmetric product $[A, B, C]$, which satisfies the so called fundamental identity

$$(11) \quad [A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]].$$

If we let $\{T^a\}_{1 \leq a \leq N}$ to be a basis of \mathcal{A} , then the 3-algebra will be specified by the structure constants f^{abc}_d of \mathcal{A} :

$$(12) \quad [T^a, T^b, T^c] = f^{abc}_d T^d.$$

The fundamental identity (11) is expressed as:

$$(13) \quad f^{abg}_h f^{cde}_g = f^{abc}_g f^{gde}_h + f^{abd}_g f^{cge}_h + f^{abe}_g f^{cdg}_h.$$

Classifying 3-algebra \mathcal{A} requires classifying the solutions of the fundamental identity (13) for the structure constants f^{abc}_d . In order to derive from the Lagrangian description the equations of motion of the BLG theory a bi-invariant non-degenerate

metric (5) that arises by postulating a bilinear scalar product $\text{Tr}(\cdot, \cdot)$ on the 3-algebra is needed

$$(14) \quad h^{ab} = \text{Tr}(T^a, T^b).$$

The Lagrangian of the BLG theory is completely specified once a collection of structure constants f_{ab}^c and a bi-invariant metric h^{ab} are given. The BLG theory encodes the interactions of a three dimensional $\mathcal{N} = 8$ multiplet, consisting of eight scalar fields X^I and their fermionic superpartners Ψ , and a non-propagating gauge field A_{mb}^a . Matter fields in this theory take values in \mathcal{A} , so that $X^I = X_a^I T^a$, $\Psi = \Psi_a T^a$. The indices I, J, K run in $1, \dots, 8$, and they specify the transverse directions of M2-brane; we denote the world-volume of the membrane as \mathcal{M} and its longitudinal directions as x^m where m, n run in $0, 1, 2$. The indices a, b, c take values in $1, \dots, N$ where N is the number of generators of the Lie 3-algebra specified by a set of structure constants f_{ab}^c . The fermionic field Ψ is a Majorana spinor in $10 + 1$ dimensions and $\Gamma^M = \{\gamma^m, \Gamma^I\}$ are eleven-dimensional gamma matrices satisfying the Clifford algebra $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$. As result Ψ has 16 real fermionic component, equivalent to 8 bosonic degrees of freedom.

The BLG Lagrangian is given by [10]

$$(15) \quad \mathcal{L} = -\frac{1}{2} \mathcal{D}_m X_{aI} \mathcal{D}^m X_a^I + \frac{i}{2} \bar{\Psi}^a \gamma^m \mathcal{D}_m \Psi_a + \frac{i}{4} f_{abcd} \bar{\Psi}^b \Gamma^{IJ} X^{cI} X^{dJ} \Psi^a \\ - \frac{1}{12} (f_{abcd} X^{aI} X^{bJ} X^{cK}) (f_{efg}^d X^{eI} X^{fJ} X^{gK}) + \frac{1}{2} \varepsilon^{mnl} (f_{abcd} A_m^{ab} \partial_n A_l^{cd}) \\ + \frac{2}{3} f_{aef}^g f_{bcdg} A_m^{ab} A_n^{cd} A_l^{ef},$$

where:

$$(16) \quad \mathcal{D}_m X^{aI} = \partial_m X^{aI} + f_{bcd}^a A_m^{cd} X^{bI}.$$

The theory is invariant under the gauge transformations

$$(17) \quad \delta X^{aI} = -f_{bcd}^a \Lambda^{bc} X^{dI}, \\ \delta \Psi^a = -f_{bcd}^a \Lambda^{bc} \Psi^d, \\ \delta(f_{ab}^{cd} A_m^{ab}) = f_{ab}^{cd} \mathcal{D}_m \Lambda^{ab},$$

and under the following supersymmetry transformations

$$(18) \quad \delta X^{aI} = i\bar{\epsilon} \Gamma^I \Psi^a, \\ \delta \Psi^a = \mathcal{D}_m X^{aI} \gamma^m \Gamma^I \epsilon + \frac{1}{6} f_{bcd}^a X^{bI} X^{cJ} X^{dK} \Gamma^{IJK} \epsilon, \\ \delta(f_{ab}^{cd} A_m^{ab}) = i f_{ab}^{cd} X^{aI} \bar{\epsilon} \gamma_m \Gamma_I \Psi^b,$$

where Ψ and ϵ are 16-component Majorana spinors satisfying the projection condition $\gamma_{012} \epsilon = \epsilon$ and $\gamma_{012} \Psi^a = -\Psi^a$ respectively.

When h^{ab} is positive definite the only one known example of this algebraic structure was given in [10]. In this case, the vector space is \mathbb{R}^4 and we can take

$$(19) \quad h^{ab} = \delta^{ab}, \quad f^{abcd} = f \epsilon^{abcd},$$

for some constant f . The usual constraint that arises by demanding invariance under large gauge transformations requires us to choose $f = \frac{2\pi}{\kappa}$ where the level κ is an integer. Then the triple product is the natural generalization to four dimensions of the usual cross product: it gives a new vector perpendicular to the vectors in the product whose length is the signed of the parallelepiped spanned by the vectors.

Recently, there have been several attempts to relax these assumptions and construct additional field-theory models of multiple M2-branes. There have been interesting proposals in which the metric h^{ab} has a Lorentzian (indefinite) signature [15]. This allows one to construct an associated 3-algebra for any Lie algebra, and the corresponding $\mathcal{N} = 8$ superconformal at the classical level Lagrangian. Although these models are built on a 3-algebra without a positive norm and have pathologic ghost-type fields (fields with negative kinetic energy), the corresponding quantum theories have been argued to be unitary and they have some encouraging features [16]. Choosing the Lorentzian metric, one finds an infinite class of 3-algebras $\mathcal{A}_{\mathcal{G}}$ with an underlying Lie algebra structure [16]. For any Lie algebra \mathcal{G} , $[T^i, T^j] = f^{ij}_k T^k$ with structure constants f^{ij}_k and Killing form h^{ij} one can define the corresponding 3-algebra as follows. Let the generators T^a of the 3-algebra be denoted by T^-, T^+, T^i ($a = +, -, i; i = 1, \dots, \dim \mathcal{G}$), where T^i are in one-to-one correspondence with the generators of the Lie algebra. Then the basic 3-algebra relations are chosen to be

$$[T^-, T^a; T^b] = 0, \quad [T^+, T^i; T^j] = f^{ij}_k T^k, \quad [T^i, T^j; T^k] = -f^{ijk} T^-.$$

This set of totally antisymmetric structure constants solves the fundamental identity, where f^{ijk} are structure constants of a compact semi-simple Lie algebra \mathcal{G} of dimension n . The invariant inner product is defined as follows

$$\langle T^\mp, T^\mp \rangle = 0, \quad \langle T^-, T^+ \rangle = 1, \quad \langle T^\mp, T^i \rangle = 0, \quad \langle T^i, T^j \rangle = h^{ij}.$$

The Lagrangian based on $\mathcal{A}_{\mathcal{G}}$, is gauge and supersymmetry transformations free out the gauge fields A_m^{+-} and A_m^{+b} , therefore they are not part of the theory. Similarly, A_{mij} appears only through the combination $f^{ijk} A_{mjk} = B_m^i$, so $A_m^i = A_m^{-i}, B_m^i$ will be viewed as the fundamental gauge fields in the theory. The BLG Chern–Simons term reduces, in this case of 3-algebra with Lorentzian signature to a three dimensional BF term. The peculiar form of the interactions makes this model resemble, in some aspects the Yang–Mills theories based on non semi-simple gauge groups [17]. In particular, the quantum effective action contains only 1-loop term with the divergent part that can be eliminated by a field redefinition. The on-shell scattering amplitudes are thus finite (scale invariant). This is a consequence of the presence of a null direction in the field space metric: one of the field components is a Lagrange multiplier which ‘freezes out’ quantum fluctuations of the ‘conjugate’ field. The non-positivity of the metric implies that these theories are apparently non-unitary. However, the special structure of interaction terms (degenerate compared to non-compact Yang–Mills theories) suggests that there may exist a unitary ‘truncation’.

Another option is to look for theories with a reduced number of supersymmetries. In [18] a class of Chern–Simons Lagrangians with $\mathcal{N} = 4$ supersymmetry was constructed. Of special interest is the work [19] in which an infinite class of brane configurations on the $\mathbb{C}^4/\mathbb{Z}_\kappa$ orbifold was given whose low energy effective Lagrangian is the Chern–Simons superconformal theory with $SO(6)$ R-symmetry and $\mathcal{N} = 6$ supersymmetry was constructed. The field content of the ABJM model is given by four complex scalar and spinor fields which live in the bifundamental representation of the $U(N) \times U(N)$ gauge group while the gauge fields are governed by Chern–Simons actions of levels κ and $-\kappa$, respectively.

Thus it is of interest to generalize the construction based on 3-algebras on a complex vector space to the case of $\mathcal{N} = 6$ supersymmetry [20]. This can be

accomplished by relaxing the conditions on the triple product

$$(20) \quad [T^a, T^b; \bar{T}^c] = f^{abc} T^d,$$

so that it is no longer real and antisymmetric in all three indices. Rather it is required to satisfy

$$(21) \quad f^{abc\bar{d}} = -f^{ba\bar{c}d}, \quad f^{abc\bar{d}} = f^{\star\bar{c}dab}.$$

The triple product is also required to satisfy the fundamental identity

$$(22) \quad f^{ef\bar{g}} f^{cb\bar{a}}_d + f^{fe\bar{a}} f^{cb\bar{g}}_d + f^{\star\bar{g}\bar{a}f} f^{ce\bar{b}}_d + f^{\star\bar{a}\bar{g}e} f^{cf\bar{b}}_d = 0.$$

To construct a gauge invariant Lagrangian it is necessary to have an inner product

$$(23) \quad h^{\bar{a}b} = \text{Tr}(\bar{T}^{\bar{a}}, T^b).$$

Then further restrictions of $\mathcal{N} = 6$ supersymmetry, scale invariance, $SU(4)$ R-symmetry, and a global $U(1)$ give the conditions on the structure constants $f^{abc\bar{d}}$. We use complex notation in which the supercharges ϵ_{AB} ($A = 1, \dots, 4$) are in the representation $\mathbf{6}$ of the algebra $SU(4)$ with vanishing $U(1)$ charge. They satisfy the reality condition $\epsilon^{AB} = \frac{1}{2}\epsilon^{ABCD}\epsilon_{CD}$. We introduce four complex 3-algebra valued scalar fields Z_a^A as well as their complex conjugates $\bar{Z}_{A\bar{a}}$. Similarly, we denote the fermions by Ψ_{Aa} and their complex conjugates by $\bar{\Psi}_{\bar{a}A}$. A raised A index indicates that the field is in the $\mathbf{4}$ of $SU(4)$; a lowered index transforms in the $\bar{\mathbf{4}}$. We assign Z_a^A and Ψ_{Aa} a $U(1)$ charge of 1. Complex conjugation raises or lowers the A index, flips the sign of the $U(1)$ charge, and interchanges $a \leftrightarrow \bar{a}$. We postulate the following supersymmetry transformations

$$(24) \quad \begin{aligned} \delta Z_a^A &= i\bar{\epsilon}^{AB}\Psi_{Ba}, \\ \delta\Psi_{Bd} &= \gamma^m \mathcal{D}_m Z_d^A \epsilon_{AB} + f_1^{abc} Z_a^C \bar{Z}_{C\bar{b}} Z_c^A \epsilon_{AB} + f_2^{abc} Z_a^C Z_b^D \bar{Z}_{B\bar{c}} \epsilon_{CD}, \\ \delta\bar{\Psi}_{\bar{a}c} &= i\bar{\epsilon}_{AB}\gamma_m Z_a^A \bar{\Psi}_{\bar{b}}^B f_3^{abc} + i\bar{\epsilon}^{AB}\gamma_m \bar{Z}_{A\bar{a}} \Psi_{Bb} f_4^{\bar{a}bc}, \end{aligned}$$

where f_1^{abc} , f_2^{abc} , f_3^{abc} and $f_4^{\bar{a}bc}$ are tensors of the 3-algebra. The covariant derivative is defined by $\mathcal{D}_m Z_d^A = \partial_m Z_d^A - \bar{A}_{m\bar{c}}^c Z_c^A$. Next, we consider the closure of (24) on the scalars. Then we find that $[\delta_1, \delta_2]Z_d^A$ only closes on to translations and a gauge symmetry if $f_1^{abc} = f_2^{ac\bar{b}}$. Next, we examine the closure of the algebra on the fermions, that gives $f_4^{\bar{a}bc} = -f_3^{b\bar{a}c}$ and $f_3^{abc} = f_2^{ac\bar{b}}$. Finally the closure of (24) onto translations and gauge transformations for the gauge field is succeed when f_2^{abc} satisfies the fundamental identity (22) and that $\mathcal{D}_m(f_2^{abc}) = 0$. This is just the statement that f_2^{abc} is an invariant tensor of the gauge algebra.

With these results, it is not difficult to show that an invariant Lagrangian is of the Chern–Simons form with interacting scalars, fermions and vectors that take values in a 3-algebra. As with the $\mathcal{N} = 8$ model, the Lagrangian is entirely determined by specifying of a triple product on a 3-algebra that satisfies the fundamental identity and can be written as:

$$(25) \quad \begin{aligned} \mathcal{L} &= -\text{Tr}(\mathcal{D}^m \bar{Z}_A, \mathcal{D}_m Z^A) - i\text{Tr}(\bar{\Psi}^A, \gamma^m \mathcal{D}_m \Psi_A) - V + \mathcal{L}_{CS} \\ &\quad - i\text{Tr}(\bar{\Psi}^A, [\Psi_A, Z^B; \bar{Z}_B]) + 2i\text{Tr}(\bar{\Psi}^A, [\Psi_B, Z^B; \bar{Z}_A]) \\ &\quad + \frac{i}{2}\epsilon_{ABCD}\text{Tr}(\bar{\Psi}^A, [Z^C, Z^D; \Psi^B]) - \frac{i}{2}\epsilon^{ABCD}\text{Tr}(\bar{Z}_D, [\bar{\Psi}_A, \Psi_B; \bar{Z}_C]), \end{aligned}$$

where the scalar potential

$$(26) \quad V = \frac{2}{3}\text{Tr}(\Upsilon_B^{CD}, \bar{\Upsilon}_{CD}^B),$$

$$\Upsilon_B^{CD} = [Z^C, Z^D; \bar{Z}_B] - \frac{1}{2}\delta_B^C[Z^E, Z^D; \bar{Z}_E] + \frac{1}{2}\delta_B^D[Z^E, Z^C; \bar{Z}_E],$$

and \mathcal{L}_{CS} is given by

$$(27) \quad \mathcal{L}_{CS} = \frac{1}{2}\varepsilon^{mnl}(f^{ab\bar{c}\bar{d}}A_{m\bar{c}b}\partial_n A_{l\bar{d}a} + \frac{2}{3}f^{ac\bar{d}}f^{ge\bar{f}\bar{b}}A_{m\bar{b}a}A_{n\bar{d}c}A_{l\bar{f}e}).$$

Note that the Lagrangian (25) is automatically gauge invariant since it is supersymmetric and supersymmetries close into gauge transformations

$$(28) \quad [\delta_1, \delta_2]Z_d^A = v^m \mathcal{D}_m Z_d^A + \Lambda_{\bar{c}b} f^{ab\bar{c}} Z_b^A,$$

where

$$(29) \quad v^m = \frac{i}{2}\varepsilon_2^{CD}\gamma^m \epsilon_{1CD}, \quad \Lambda_{\bar{c}b} = i(\bar{\epsilon}^{DE2}\epsilon_{1CE} - \bar{\epsilon}_1^{DE}\epsilon_{2CE})\bar{Z}_{D\bar{c}}Z_b^A.$$

The second term in (28) is a gauge transformation: $\delta_\Lambda Z_d^A = \tilde{\Lambda}_d^a Z_a^A$. On the field $\bar{Z}_{A\bar{d}}$ we find $\delta_\Lambda \bar{Z}_{A\bar{d}} = \Lambda_{\bar{c}b}^* f^{*\bar{a}bc} \bar{Z}_{A\bar{a}}$. If we assume the existence of a gauge invariant metric, namely $\delta_\Lambda(h^{\bar{a}b}\bar{Z}_{A\bar{a}}Z_b^A) = 0$, we must require

$$(30) \quad f^{ab\bar{c}\bar{d}} = f^{ab\bar{c}} h^{\bar{d}e} = f^{*\bar{c}dab}.$$

This implies that $(\tilde{\Lambda}^{c\bar{d}})^* = -\tilde{\Lambda}^{d\bar{c}}$, therefore the transformation parameters $\tilde{\Lambda}_b^a$ are elements of $u(N)$. In this example we see that the general form of three-dimensional Lagrangians with $\mathcal{N} = 6$ supersymmetry, $SU(4)$ R-symmetry and a $U(1)$ global symmetry with gauge group $U(N) \times U(N)$ is entirely determined by specifying a triple product on a 3-algebra that satisfies the fundamental identity. It would certainly be interesting to see if there are other examples and hence other models with different gauge groups. A matrix realization of the Hermitian 3-algebra [20] is proved by $[X, Y; Z] = XZ^\dagger Y - YZ^\dagger X$, $\langle X, Y \rangle = \text{tr}(XY^\dagger)$. The matrix-value fields X, Y, Z are expanded as $X = X_a T^a$ etc., where T^a is a basis of $(M \times N)$ matrices and T_a are their Hermitian conjugates. The 3-bracket is then a map from $M \times N$ matrices to itself as the first requirement of an algebra. Moreover, the bracket satisfies the fundamental identity (22). Hence, it is a realization of the Hermitian 3-algebra. An explicit solution of the fundamental identity can also be realized in terms of the generators t^α of the associated semi-simple Lie algebra as

$$(31) \quad f^{ab}_{cd} = (t^\alpha)_d^a (t_\alpha)_c^b,$$

where $(t^\alpha)_b^a$ are the generators in the bi-fundamental representation. The index α is lowered by the inverse of Killing form $\kappa^{\alpha\beta}$ of the Lie algebra. This realization does not in general satisfy antisymmetry with respect to a, b or c, d indices. Imposing this property restricts possible choices of the Lie algebras and hence the Lie group. With the Lie group $\mathcal{G} = G_L \otimes G_R$, a, b, c, d ranges over $1, \dots, \text{rank}(G_L)\text{rank}(G_R)$ and α ranges over $1, \dots, \text{rank}(G_L) + \text{rank}(G_R)$. As shown in works [21] after analysis of all possible compact Lie groups and their representations, only allowed gauge groups leading to the manifest $\mathcal{N} = 6$ supersymmetry are, up to discrete quotients, $SU(N) \times U(1)$, $Sp(N) \times U(1)$, $SU(N) \times SU(N)$, and $SU(N) \times SU(M) \times U(1)$ with possibly additional $U(1)$'s. Matter representations are restricted to be the bi-fundamentals. But we have to emphasize the role of triple products and 3-algebras even though the resulting Lagrangians can be viewed as relatively familiar Chern-Simons-matter gauge theories based on Lie algebras. From the point of view declared here, the dynamical fields have interactions that are most naturally defined in terms of a triple product.

After the proposal of BLG, a lot of attempts have been done to extract and understand various aspects of this theory. One of the important articles in this direction is the paper of [22] in which it was shown that if one of the scalars, for example X^8 , has a nonzero expectation value, one can reduce the membrane action to D2 brane action which shows an important notion of reliability of the BLG theory.

In [23] it has been constructed the classical action of the ABJM model in the $\mathcal{N} = 3$, $d=3$ harmonic superspace². Our motivation comes in part from corresponding studies in AdS_5/CFT_4 where the $\mathcal{N} = 2$ formulation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) has been an extremely efficient tool for studies of anomalous dimensions, non-renormalization properties and integrability. In such a formulation three out of six supersymmetries are realized off shell while the other three mix the superfields and close on shell. The superfield action involves two hypermultiplet superfields in the bifundamental representation of the gauge group and two Chern-Simons gauge superfields corresponding to the left and right gauge groups. The $\mathcal{N} = 3$ superconformal invariance allows only a minimal gauge interaction of the hypermultiplets. One may wonder how the sextic scalar potential of the ABJM model can appear in the absence of an original superpotential. We show that, upon reducing the superfield action to the component form, the scalar potential naturally arises as a result of eliminating some auxiliary fields from the gauge multiplet and from the harmonic expansion of the off-shell q^+ hypermultiplets. This is a striking new feature of the $\mathcal{N} = 3$ superfield formulation as compared to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ ones. Besides the original $U(N) \times U(N)$ ABJM model, we also constructed $\mathcal{N} = 3$ superfield formulations of some generalizations. For the $SU(2) \times SU(2)$ case we give a simple superfield proof of its enhanced $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R-symmetry. To clarify the significance of the $\mathcal{N} = 3$ superfield formulation presented in [23], let us resort to the analogy between the ABJM theory and the $\mathcal{N} = 4$, $d=4$ super Yang-Mills (SYM_4^4) theory, which describe the low-energy dynamics of multiple M2 and D3 branes, respectively. It is well known, the SYM_4^4 model is the maximally supersymmetric and superconformal gauge theory in four dimensions, a fact is crucial for the string theory / field theory correspondence (see e.g. [5]). The $\mathcal{N} = 2$, $d=4$ harmonic superspace [25] provides the appropriate off-shell $\mathcal{N} = 2$ superfield description of SYM_4^4 as SYM_4^2 plus an $\mathcal{N} = 2$ hypermultiplet in the adjoint representation minimally coupled to the $\mathcal{N} = 2$ gauge superfield. Such a formulation was successfully used to study the low-energy quantum effective action and the correlation functions of composite operators in $\mathcal{N} = 2$ superspace. Analogously to SYM_4^4 , the ABJM model is the maximally supersymmetric and superconformal Chern-Simons-matter theory in three dimensions. The ABJM construction opened up ways for studying the AdS_4/CFT_3 correspondence between three-dimensional field models and four-dimensional supergravity in AdS space [19]. We believe that the $\mathcal{N} = 3$ superfield description of the ABJM model and its generalizations developed in the paper [23] will be useful for studying their algebraic and quantum structure as the $\mathcal{N} = 2$ harmonic superspace approach has proved to be for SYM_4^4 . In particular, we expect that it will be very efficient for investigating the low-energy quantum effective action in three-dimensional $\mathcal{N} = 6$ supersymmetric field models superspace, because the manifest off-shell $\mathcal{N} = 3$ supersymmetry is respected at each step of the computation. One of

²For the $\mathcal{N} = 1, 2$ superspace formulations of the BLG and ABJM model see [24]

the most interesting features of the BLG model is the subtle interplay between the gauge algebra and supersymmetry, and we hope that our manifestly supersymmetric formulations will shed more light on this issue.

3. M-BRANE BOUND STATES AND THE SUPERSYMMETRY OF BPS SOLUTIONS IN THE BLG THEORY

If the BLG theory is provided an authentic description of multiple M2-branes, it must be able to incorporate the various M-branes which are known to exist. They are supersymmetric objects of the 11 dimensional quantum supergravity, and will appear as classical BPS solutions in the dual field theory, the BLG Lagrangian. They imply the existence of M-branes in addition to the 'background M2-branes' whose dynamics is describe by the BLG theory in question. Since in \mathcal{M} -theory for an M2 brane there is an M5 brane which is electric-magnetic dual of each other, one natural task is to find the relation between the M2 and M5 brane dynamics in the context of the BLG theory.

The original motivation of Bagger and Lambert was to write down a theory capable of reproducing the Basu–Harvey equation [7] that describes an M2-brane ending on an M5-brane as a BPS equation. This generalized Nahm’s equation [26], usually called ADHM construction, for the moduli space of monopoles in the gauge theory which describes a D1-brane ending on a D3-brane. Nontrivial BPS solutions would have less supersymmetries and they disintegrate into three categories: the vortices, the domain walls and the spacetime-filling configurations. Simple 1/2-BPS equations can be readily written and also the solutions have been studied, see [10]. The energy bound corresponding to this particular BPS configuration should appear in the superalgebra of the theory as a central charge term.

When we compute the anticommutator of two supercharges, we obtain the following result:

$$(32) \quad \{Q^\alpha, Q^\beta\} = -2P_m(\Gamma^m\Gamma^0)^{\alpha\beta} + Z_{IJ}(\Gamma^{IJ}\Gamma^0)^{\alpha\beta} + Z_{IJKL}(\Gamma^{IJKL}\Gamma^i\Gamma^0)^{\alpha\beta} + Z_{IJKL}(\Gamma^{IJKL})^{\alpha\beta},$$

where α, β are the 11 dimensional spinor index, and $i = x, y$. In the above we have, in addition to the usual energy momentum vector P_m defined as $P^m = \int d^2x T^{0m}$, three types of central charges:

$$(33) \quad \begin{aligned} Z_{IJ} &= - \int d^2x \text{tr}(\mathcal{D}_i X^I \mathcal{D}_j X^J \varepsilon^{ij} - \mathcal{D}_0 X^K F^{KIJ}), \\ Z_{IJKL} &= \frac{1}{3} \int d^2x \text{tr}(\mathcal{D}_j X^{[I} F^{JKL]} \varepsilon^{ij}), \\ Z_{IJKL} &= \frac{1}{4} \int d^2x \text{tr}(F^{M[IJ} F_M^{KL]}), \end{aligned}$$

where we also introduced a short-hand notation for 3-products: $F^{IJK} = [X^I, X^J, X^K]$. The stress-energy tensor T_{mn} can be computed in the usual way. In case where the fermions are set to zero, it results in

$$(34) \quad T_{mn} = \mathcal{D}_m X_a^I \mathcal{D}_n X^{aI} - \eta_{mn}(\frac{1}{2} \mathcal{D}_\rho X^{aI} \mathcal{D}^\rho X_a^I + V).$$

We note that the Chern–Simons like term does not contribute to the stress-energy tensor because this term is topological one and does not depend on the worldvolume metric. The first two classes are actually topological terms, since they can be

expressed as surface integrals. They are boundary terms and they are equal to zero for field configurations that are non-singular and topologically trivial. The last one, $Z_{IJKL} = Z_{[IJKL]}$ can be actually shown to vanish as well, but for a different reason: one should make use of the invariance, the fundamental identity and skew-symmetry.

Now we are almost ready to consider simple BPS equations and identify the central charge terms as different combinations of M-branes. We consider vortex configurations that describe two stacks of membranes intersecting along the time direction where only the scalars $X^3 = \Phi + \bar{\Phi}$, $X^4 = i(\Phi - \bar{\Phi})$ and the gauge vector \tilde{A}_{na}^b are excited. Thus, considering a configuration such that $\mathcal{D}_0\Phi = \mathcal{D}_0\bar{\Phi}$, the BPS conditions that follow from supersymmetry variations (18) are reduced to ³

$$(35) \quad \mathcal{D}_z\Phi\Gamma^z\Gamma^\Phi\epsilon + \mathcal{D}_{\bar{z}}\Phi\Gamma^{\bar{z}}\Gamma^\Phi\epsilon + \mathcal{D}_z\bar{\Phi}\Gamma^z\Gamma^{\bar{\Phi}}\epsilon + \mathcal{D}_{\bar{z}}\bar{\Phi}\Gamma^{\bar{z}}\Gamma^{\bar{\Phi}}\epsilon = 0.$$

For this configuration, the energy density is given by

$$\mathcal{H} = 4\text{tr}(\mathcal{D}_z\Phi, \mathcal{D}_{\bar{z}}\bar{\Phi}) + 4\text{tr}(\mathcal{D}_{\bar{z}}\Phi, \mathcal{D}_z\bar{\Phi}) = \frac{1}{2}\mathcal{Z}^0 + 8\text{tr}(\mathcal{D}_{\bar{z}}\Phi, \mathcal{D}_z\bar{\Phi}),$$

where \mathcal{Z}^0 is the density of the 0-form central charge Z_{IJ} evaluated for this field configuration. Thus $\mathcal{H} \geq \frac{1}{2}\mathcal{Z}^0$ and the bound is saturated when BPS configuration is given by (anti)holomorphic curves

$$\mathcal{D}_{\bar{z}}\Phi = \mathcal{D}_z\bar{\Phi} = 0.$$

If this last condition is satisfied, it follows from the BPS equation (35) that the solution preserve half of the supersymmetries (18) satisfying $\Gamma^z\Gamma^\Phi\epsilon = 0$. Thus, for the case when the gauge field is equal to zero, the vortex configuration is given by

$$\Phi = c_a T^a \frac{1}{z},$$

where c_a are arbitrary constants. For the case of the configuration when also the gauge vector $A_m^{\tilde{a}} T^{\tilde{a}}$ exists, we find that half-BPS exist if

$$[\Phi, A_{\bar{z}}] = [\bar{\Phi}, A_z] = 0,$$

where $[\cdot, \cdot]$ is the usual Lie commutator. In this model, the 3-algebra indices a are split into $a = (+, -, \tilde{a})$ and the structure constants are given by

$$f^{+\tilde{a}\tilde{b}\tilde{c}} = f^{-\tilde{a}\tilde{b}\tilde{c}} = C^{\tilde{a}\tilde{b}\tilde{c}}, \quad f^{+-\tilde{a}\tilde{b}} = f^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = 0,$$

where $C^{\tilde{a}\tilde{b}\tilde{c}}$ are the structure constants of a compact semi-simple Lie algebra satisfying the usual Jacobi identity. This implies that with respect to the single M2-brane theory, the vortex solutions of the BLG theory include extra degrees of freedom, given by the components of the gauge vector that commute with the scalar fields.

To describe a stack of M2-branes ending on an M5-brane it is necessary to switch on the X^3, X^4, X^5, X^6 scalar fields [7]. Given that these fields depend only on the worldvolume coordinate σ^2 , the BPS condition is [10]

$$(36) \quad \frac{dX^A}{d\sigma^2}\Gamma^A\Gamma^2\epsilon - \frac{1}{6}\varepsilon^{BCDA}\Gamma^A[X^B, X^C, X^D]\Gamma^{3456}\epsilon = 0,$$

where $A, \dots = 3, 4, 5, 6$. For this field configuration the energy density is given by

$$\mathcal{H} = \frac{1}{2}\text{tr}(\partial_2 X^A, \partial_2 X^A) + \frac{1}{12}\text{tr}([X^A, X^B, X^C], [X^A, X^B, X^C]).$$

³ $\Gamma^z = \Gamma^1 + i\Gamma^2, \quad \Gamma^{\bar{z}} = \Gamma^1 - i\Gamma^2, \quad \Gamma^\Phi = \Gamma^3 + i\Gamma^4, \quad \Gamma^{\bar{\Phi}} = \Gamma^3 - i\Gamma^4$

As usual, we can write the potential as $V(X) = \frac{1}{2}\text{tr}(\frac{\partial W}{\partial X^A}, \frac{\partial W}{\partial X^A})$, where

$$(37) \quad W = \frac{1}{2}m\text{tr}(X^A, X^A) + \frac{1}{24}\varepsilon^{ABCD}\text{tr}(X^A, [X^B, X^C, X^D]).$$

Here we add an $SO(4)$ symmetric mass deformation term. Thus

$$\mathcal{H} = \frac{1}{2}\text{tr}(\partial_2 X^A + \frac{\partial W}{\partial X^A}, \partial_2 X^A + \frac{\partial W}{\partial X^A}) - \text{tr}(\partial_2 X^A, \frac{\partial W}{\partial X^A}),$$

where $\mathcal{Z}_1 = -2\text{tr}(\partial_2 X^A, \frac{\partial W}{\partial X^A})$ is the density of $Z_i^{\alpha\beta}$ the 1-form central charge. For this field configuration $\mathcal{H} \geq \frac{1}{2}\mathcal{Z}_1$ and the bound is saturated when

$$(38) \quad \frac{dX^A}{d\sigma^2} - \frac{1}{6}\varepsilon^{BCDA}[X^B, X^C, X^D] = mX^A.$$

When the (38) with $m = 0$ are satisfied, then it follows from (36) that the field configuration proposed by Basu and Harvey as the M2-brane worldvolume solution describing the M2-M5 system is half-BPS and the preserved supersymmetries satisfy $\Gamma^2\epsilon = \Gamma^{3456}\epsilon$.

Vacuum solutions require $\partial_A W = 0$, or $mX^A = -\frac{1}{6}\varepsilon^{BCDA}[X^B, X^C, X^D]$, where the T^A satisfy $[T^A, T^B, T^C] = \varepsilon^{ABCD}T^D$. In addition to the trivial solution $X^A = 0$, this Eq. has a fuzzy S^3 solution in which the M2' puff up into a fuzzy three-sphere with radius proportional to \sqrt{m} . This implies that $\text{tr}(T^A, T^A) \sim n^2$ at large integers n . Unfortunately, we do not know enough about the representations of three-algebras to confirm this prediction. The two solutions describe two zero-energy vacuum states of the M2-brane in the four-flux background.

The fuzzy funnel solution of Eq. (38) can be found by taking

$$(39) \quad X^A = f(\sigma^2)T^A.$$

The equation for f is $f' = mf - f^3$; the solution is

$$f = \frac{\sqrt{m}}{\sqrt{1 - ce^{-2m\sigma^2}}}.$$

If $c = +1$ and $m > 0$ the solution behaves as $f = \frac{1}{\sqrt{\sigma^2}}$ for small but positive σ^2 . These solutions describe fuzzy funnels in which an infinite radius fuzzy three-sphere at $\sigma^2 = 0$ relaxes into the fuzzy sphere or the trivial vacuum, respectively, as $\sigma^2 \rightarrow \infty$. The spacetime interpretation of these solutions is that they correspond to M2-branes that end on a single M5-brane, located at $\sigma^2 = 0$ and infinitely extended along the (x^0, \dots, x^5) directions. On the other hand, if $c = -1$ and $m > 0$, the function f is bounded. It vanishes exponentially as $\sigma^2 \rightarrow -\infty$ and approaches $f \rightarrow \sqrt{m}$ as $\sigma^2 \rightarrow \infty$. Here there is no divergent fuzzy funnel, i.e. no M5-brane. This solution smoothly interpolates between the trivial and fuzzy sphere vacua. In other words, it is a traditional domain wall that interpolates between two degenerate vacuum solutions of the worldvolume effective action.

Here we have only considered the simplest solutions and it would be interesting to systematically work out more solutions and study their properties. While very little is known about explicit nontrivial examples of the n -algebra, its correspondence with Nambu brackets is very helpful.

4. DYNAMICAL SYMMETRY AND NAMBU MECHANICS

In this section we review some standard facts about the Nambu brackets and n -algebras (Filippov–Nambu algebras). More than three decades ago, Nambu [9] proposed a generalization the classical Hamiltonian mechanics. Dynamics according to Nambu consists in replacing Poisson bracket by a ternary (n -ary) operation on algebra of observables A and requires two $(n-1)$ 'Hamiltonians' H_1, H_2 (H_1, \dots, H_{n-1}) to describe the evolution. This dynamical picture is consistent if and only if the evolution operator is an isomorphism of algebra of observables.

This mechanics is remarkable in several respect. First, it treats all conserved quantities of a mechanical system on the same footing. It is clearly a most attractive feature from a quantum perspective. In his formalism, Nambu replaced the usual pair of canonical variables found in Hamiltonian mechanics with a triplet of coordinates in an odd dimensional 'phase space' possessing as fundamental symmetries the volume preserving diffeomorphisms group in the place of symplectic diffeomorphisms. It has been shown that several Hamiltonian systems possessing dynamical or hidden symmetries can be realized within the framework of Nambu's generalized mechanics. Among such systems are the Euler equations for the angular momentum of a rigid body in three dimensions, the $SU(n)$ -isotropic harmonic oscillator, the $SO(4)$ -Kepler problem and others somewhat exotic examples. As required by the formulation of the Nambu dynamics, the integrals of motion needed for complete integrability of these systems necessarily become the so-called generalized 'Hamiltonians'. Corresponding phase flow preserves the phase volume so that the analog of the Liouville theorem is still valid, which is fundamental for the formulation of statistical mechanics with two temperature-like intensive parameters. Yet, the fundamental principles of a canonical formalism of Nambu's generalized mechanics and on the principle of least action, similar to the invariant geometrical form of the Hamiltonian mechanics [27], has only recently been given an elegant geometric formulation by Takhtajan [11]. For further applications of the classical and quantum Nambu brackets the reader may consult [28].

The basic properties of the associative algebra $\mathcal{A} = C^\infty(\mathbb{R}^n)$, what equipped with a Nambu bracket of order n $\{\cdot, \dots, \cdot\}$ the same as in n -Lie algebra Filippov is: Linearity, Antisymmetry, Leibnitz rule and the Fundamental Identity

$$(40) \quad \{\{f_1, \dots, f_n\}, f_{n+1}, \dots, f_{2n-1}\} = \{\{f_1, f_{n+1} \dots, f_{2n-1}\}, f_2, \dots, f_n\} + \{f_1, \{f_2, f_{n+1} \dots, f_{2n-1}\}, f_3, \dots, f_n\} + \dots + \{f_1, \dots, f_{n-1}, \{f_n, f_{n+1}, \dots, f_{2n-1}\}\}.$$

On the vector space V the linear Nambu brackets are related to the notion of Filippov–Nambu n -algebra. If we introduce the basis T^a of V then n -bracket can be defined through the structure constants (3) and fundamental identity (4).

The identity (40) guarantees the fact that if each of $(f_i)_{i=1, \dots, n}$ is the conserved quantity, then the observable $\{f_1, \dots, f_n\}$ is also conserved. The new equations of motion in the phase space $M \equiv \mathbb{R}^n$ are analogous to the Hamilton–Poisson equations:

$$(41) \quad \frac{dx^i}{dt} = \{x^i, H_1, \dots, H_{n-1}\},$$

where the n -bracket on an oriented n -dimensional manifold is defined as Jacobian (volume form):

$$(42) \quad \{f_1, \dots, f_n\} = \varepsilon^{i_1 \dots i_n} \partial_{i_1} f_1 \partial_{i_2} f_2 \dots \partial_{i_n} f_n,$$

for any functions $f_1, \dots, f_n \in C^\infty(\mathbb{R}^n)$ and $i = 1, \dots, n$. The $n - 1$ Hamiltonians H_1, \dots, H_{n-1} determine the phase-space trajectory in a geometrical way. These are also a corresponding Liouville equation for any observable $f \in C^\infty(\mathbb{R}^n)$

$$(43) \quad \frac{df}{dt} = \partial_i f \cdot \dot{x}^i = \{f, H_1, \dots, H_{n-1}\}.$$

The $n - 1$ Hamiltonians are conserved in time. Given the initial position in the phase-space $x_0^i = x^i(t = 0)$ they take the values $h_i = H_i(x_0)$. The intersection of hypersurfaces $h_i; i = 1, \dots, n - 1$ gives the geometrical shape of the trajectory passing through the point $x_0 \in \mathbb{R}^n$. This is the reason why the Nambu 3-d dynamical system is regarded as a toy model for completely integrable systems. To make a case for the physical relevance of this new formalism, Nambu pointed out a specific realization for the $n = 3$; namely the asymmetric Euler top. Here the triplet \vec{x} is naturally identified with the angular momentum \vec{l} in the body fixed frame. There are two guaranteed conserved quantities: the total kinetic energy $H_1 = \frac{1}{2}(\frac{l_1^2}{I_1} + \frac{l_2^2}{I_2} + \frac{l_3^2}{I_3})$ and $H_2 = \frac{1}{2}(\vec{l})^2$ the Casimir invariant. The corresponding phase space is S^2 which provides a spherical foliation of \mathbb{R}^3 with varying radius $\sqrt{2H_2}$ for various conditions l_0^i with Poisson algebra $SO(3)$: $\{l^i, l^j\} = \varepsilon^{ijk}l^k$. The classical Nambu Eqs. of motion are $\dot{l}^i = \varepsilon^{ijk}\partial_j H_1 \partial_k H_2$ or

$$(44) \quad \dot{l}_1 = (\frac{1}{I_2} - \frac{1}{I_3})l_2 l_3, \quad \dot{l}_2 = (\frac{1}{I_3} - \frac{1}{I_1})l_3 l_1, \quad \dot{l}_3 = (\frac{1}{I_1} - \frac{1}{I_2})l_1 l_2,$$

which are just the Euler force-free rigid body equations. In this example the time evolution of the Euler top in Nambu mechanics is described by two Hamiltonian functions. These two Hamiltonians lie in the same $SO(3)$ Lie algebra⁴ and are interpreted, the first one as the one who defines the 2-d phase space geometry, embedded in the 3-d phase space, while the second one gives the dynamics of the trajectories on the 2-d phase space. Specifically the Euler equations for the asymmetric top naturally describe geodesic flows on a triaxial ellipsoid and can be solved in terms of Jacobi elliptic function [29]. It should be noted that these equations have reincarnated during recent decades in the celebrated Nahm equations [26] for the $SU(2)$ self-dual Yang–Mills field relevant to theories of extended objects such as monopoles and membranes.

In addition it is important that we can lay the basis for a generalized quantum mechanics based on the classical Nambu dynamics [30] of Euler's asymmetric top. This Nambu quantum mechanics naturally possesses, besides Planck constant, new deformation parameters. One of its defining experimental signatures is a nonlinear time evolution generated by Jacobi elliptic functions, as compared to the standard exponential time evolution of standard quantum mechanics. The new deformation parameters are given by the moduli of the elliptic functions. In the limit when these are set to zero, the usual geometric formulation of quantum mechanics, based on the Kähler structure of the space of rays in a complex Hilbert space, is recovered. This motivates the general expression for what we call the Nambu wave function $\Psi^a = \sum_i l_i^a(t)e_i$ ($a = 1, \dots, N$) where e_i are the usual quaternion imaginary units such that $e_i e_j = -\delta_{ij} + \varepsilon_{ijk}e_k$. The quaternion conjugate Nambu wave function is $\bar{\Psi}^a = -\sum_i l_i^a \bar{e}_i$. The inner product reads $\bar{\Psi}\Phi = \delta_{ij}\Psi_i \Phi_j - \varepsilon_{ijk}e_k(\bar{\Psi} \times \vec{\Phi})$. The

⁴This situation contrasts with Dirac's mechanics where the constraints appear as subsidiary conditions

second term in the above equation is the quaternionic counterpart of the symplectic 2-form. It is at the basis of the 3-form, characteristic of Nambu's original mechanics [9]. Due to the non-linear nature of the Nambu–Schrödinger equation that describes a collection of N (which could be infinite) free abstract Euler tops the superposition principle apparently no longer holds for Ψ .

Next the fundamental analog of the symplectic 2-form of usual Hamiltonian dynamics is a closed non-degenerate 3-form [11]: $\omega^{(3)} = dl_1 \wedge dl_2 \wedge dl_3$, and the action given as an integral of the corresponding Poincaré–Cartan 2-form $S = \int l_1 dl_2 \wedge dl_3 - H_1 dH_2 \wedge dt$. This form of the action shows that initial and final states in this type of the Nambu dynamics are described by loops rather than points, because the integrand of the action is a two form, rather than a one form, as in the usual Hamiltonian dynamics. Now, we can say that, just as the simple harmonic oscillator is the prototype classical and quantum system of the standard Hamiltonian mechanics, the Euler asymmetric top is the prototypical representative of Nambu's ternary mechanics.

It is straightforward to generalize presented results for the case of the Nambu bracket of order n . The analog of the Poincaré–Cartan integral invariant is defined as the following $n - 1$ form

$$(45) \quad \omega^{(n-1)} = x_1 dx_2 \wedge \dots \wedge dx_n - H_1 dH_2 \wedge \dots \wedge dH_{n-1} \wedge dt.$$

The action functional is given by

$$(46) \quad S(C_{n-1}) = \int_{C_{n-1}} \omega^{(n-1)}$$

and is defined on the $n - 1$ -chains in the extended phase space. In its formulation admissible variations are those which do not change projections of the boundary ∂C_{n-1} on the $x_2 x_3 \dots x_n$ -hyperplanes; in this case the "share" of "configuration space" in a phase space is $1 - \frac{1}{n}$.

Next we consider the all-familiar classical Coulomb problem $\frac{dz^i}{dt} = \{z^i, H\}$, with z^i standing for the phase-space 6-vector (\vec{r}, \vec{p}) , and $H = \frac{p^2}{2} - \frac{1}{r}$. Because H possesses rotational symmetry, the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is an integral of motion. This rotational symmetry implies that the orbit lies in some two dimensional plane, though it is not enough to ensure that the orbit is closed. An extra dynamical symmetry must exist for closed orbit. Such an integral was first discovered by Laplace (but is called the Runge–Lenz vector in classical mechanics or the Lenz–Pauli vector in quantum mechanics) and is given by $\vec{A} = \vec{p} \times \vec{L} - \frac{\vec{r}}{r}$. Multiplying it by $\vec{n} = \frac{\vec{r}}{r}$ instantly yields Kepler's elliptical orbits, $\vec{n} \cdot \vec{A} + 1 = \vec{L}^2 / r$. Since $\vec{A} \cdot \vec{L} = 0$, it follows that $H = \frac{A^2 - 1}{2L^2}$. One can easily check that $\{A_i, L_j\} = \varepsilon_{ijk} A_k$ and $\{A_i, A_j\} = -2HL_k$. For bound state problems ($E < 0$), one can define a new conserved vector $\vec{D} = \frac{\vec{A}}{\sqrt{-2E}}$ and further $\mathcal{R} \equiv \vec{L} + \vec{D}$, $\mathcal{L} \equiv \vec{L} - \vec{D}$. These six simplified invariants obey the standard $SU(2) \times SU(2) \sim SO(4)$ symmetry algebra (Note that for scattering problems where $E > 0$, one instead find the Lorentzian Lie algebra $SO(3, 1)$),

$$\{\mathcal{R}_i, \mathcal{R}_j\} = \varepsilon_{ijk} \mathcal{R}_k, \quad \{\mathcal{R}_i, \mathcal{L}_j\} = 0, \quad \{\mathcal{L}_i, \mathcal{L}_j\} = \varepsilon_{ijk} \mathcal{L}_k$$

and depend on each other and the Hamiltonian through $H = \frac{-1}{2\mathcal{R}^2} = \frac{-1}{2\mathcal{L}^2}$, so only five of the invariants are algebraically independent. Equivalently to the Hamiltonian law of motion $\frac{dz^i}{dt} = \{z^i, H\}$ the same classical evolution may also be specified by

Nambu's equation of motion that is the case for all superintegrable systems with five of the above six L_i, A_i (or products thereof) as the generalized Hamiltonians

$$\frac{dz^i}{dt} = \frac{1}{L_1 \cdot (L_1^2 + L_2^2 + L_3^2)} \frac{\partial(z^i, A_2, A_3, L_1, L_2, L_3)}{\partial(p_1, p_2, p_3, r_1, r_2, r_3)},$$

or

$$\frac{dz^i}{dt} = H^2 \{z^i, \ln(\mathcal{R}_3 + \mathcal{L}_3), \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{L}_2\},$$

etc.

So we feel that these new examples may help in further understanding of the elements of the Nambu's theory such as its algebraic structure and its possible quantization. Then we need to understand in every specifically case which properties are essential from the physical viewpoint. As noted by Pauli, extension to operators requires a hermitian version of \vec{A} : $\vec{A}' = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - \frac{\vec{r}}{r}$ so that $(\vec{A}')^2 = 2H(\vec{L}^2 + \hbar^2) + 1$, leading to $\mathbf{D}' = \frac{\mathbf{A}'}{\sqrt{-2H}}$ and further to the respective chiral reduction \mathcal{R}' and \mathcal{L}' , which obey

$$[\mathcal{R}'_i, \mathcal{R}'_j] = 2i\hbar\varepsilon_{ijk}\mathcal{R}'_k, \quad [\mathcal{R}'_i, \mathcal{L}'_j] = 0, \quad [\mathcal{L}'_i, \mathcal{L}'_j] = 2i\hbar\varepsilon_{ijk}\mathcal{L}'_k$$

and hence

$$(47) \quad H = \frac{-1}{2(\mathcal{R}'^2 + \hbar^2)} = \frac{-1}{2(\mathcal{L}'^2 + \hbar^2)}.$$

We can now recall the eigenvalues of the quadratic Casimir invariants of $SU(2)$ for $s = 0, \frac{1}{2}, 1, \dots$ leading to the Balmer spectrum for the Hamiltonian $E = \frac{-1}{2\hbar^2(2s+1)^2} = \frac{-1}{2\hbar^2 n^2}$. The size of these $SU(2) \times SU(2)$ multiplets $(2s+1)^2 = n^2$ is the corresponding degeneracy.

Now we will present the Lie algebra of volume preserving diffeomorphisms $SDiff(R^3)$ in the Clebsch–Monge gauge, their relation with the Filippov–Nambu 3-algebras on \mathbb{R}^3 as well as on \mathbb{T}^3 and the Nambu mechanics, which can be represented as flow equations of incompressible fluids [31]. Since the famous paper by V. Arnold [32] where he proved that the solution of the Euler Eqs. for perfect (incompressible and inviscid) fluids are the geodesics of the infinite dimensional volume preserving diffeomorphisms group, there have been many developments. Here we will focus in the description of $SDiff(R^3)$, in a particular gauge, the Clebsch–Monge gauge, thus establishing the connection with the Nambu dynamics (flows) in \mathbb{R}^3 . This discussion easily extends to three dimensional manifolds with a metric and a smooth Nambu tensor field. Let $\mathcal{A} = C^\infty(R^3)$ be the space of smooth functions on \mathbb{R}^3 and $\mathcal{G} = SDiff(R^3)$ be the set of smooth maps of $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ with the determinant of the Jacobian at each point of \mathbb{R}^3 equals to one, i.e. $J(f)(x) = \det[\partial_i f^i(x)] = 1$. This set forms a group under composition of functions. The elements $X(v) = -v^i \partial_i$ with the Lie algebra $[X(u), X(v)] = X(w)$ for $f^i(x) = x^i + v^i(x)$ with $\partial_i v^i = 0$ have composition law: $w = (u \cdot \partial)v - (v \cdot \partial)u = \partial \times (u \times v)$. We will impose conditions at infinity for $v^i(x)$: $v^i(x) \stackrel{|x| \rightarrow \infty}{\rightarrow} 0$ such that the total kinetic energy is finite $E = \frac{1}{2} \int d^3x v^2(x) < \infty$. For any infinitesimal element we define the flow $\frac{dx^i}{dt} = v^i(x)$ with initial conditions $x_0^i = x^i(t=0)$. This Eq. describes the motion of a particle which is immersed in a fluid of given stationary velocity field at the point x_0^i , at $t=0$. For every divergenceless vector field $v^i(x) \in \mathbb{R}^3$, with above boundary conditions we can find a vector potential $A^i(x)$ such that $v^i = \varepsilon^{ijk} \partial_j A_k$. For such $A^i(x)$ Clebsch and Monge introduced three scalar potentials $\alpha, \beta, \gamma \in C^\infty(R^3)$ such

that: $A_i = \partial_i \alpha + \beta \partial_i \gamma$. So finally we get $v^i(x) = \varepsilon^{ijk} \partial_j \beta \partial_k \gamma$. The scalar function $\alpha(x)$ becomes the gauge degree of freedom of $A^i(x)$. From the last relation we see that the intersection of the surfaces $\beta = \text{const}, \gamma = \text{const}$ define locally the flow lines. The existence of the scalar potentials β, γ (Clebsch–Monge potentials) is guaranteed locally if $v^i(x)$ is an analytic function in the region of a point say $x^i = 0$. Then there exists two integrals of motion of the flow equation: $\frac{dx^i}{v^i(x)} = dt$ through which we can determine β and γ . The generators of the flow, in terms of the Clebsch–Monge potentials, become $X(\beta, \gamma) = -\varepsilon^{ijk} \partial_j \beta \partial_k \gamma \partial_i$ and the action of $X(\beta, \gamma)$ on a smooth function $\alpha \in C^\infty(\mathbb{R}^3)$ is: $X(\beta, \gamma)\alpha = -\{\alpha, \beta, \gamma\}$, the Nambu bracket of α, β, γ . The volume-preserving property is nothing but the identity $\partial_i X^i(\beta, \gamma) = \partial_k (\varepsilon^{ijk} \partial_i \beta \partial_j \gamma) = 0$. The flow becomes $\dot{x}^i = \{x^i, \beta, \gamma\}$; and so the Clebsch–Monge potentials of the flow are just the two Hamiltonians $H_1 = \beta, H_2 = \gamma$ of the Nambu dynamics. We conclude that the flow equations of incompressible fluids can be described by the Nambu dynamics and vice versa. By considering now the commutation relations of the algebra in the Clebsch–Monge gauge we obtain: $[X(\beta_1, \gamma_1), X(\beta_2, \gamma_2)] = X(\{\beta_1, \gamma_1, \beta_2\}, \gamma_2) + X(\beta_2, \{\beta_1, \gamma_1, \gamma_2\})$. Acting both sides of this relations on functions α we get the fundamental identity:

$$\{\beta_1, \gamma_1 \{\beta_2, \gamma_2, \alpha\}\} - \{\beta_2, \gamma_2 \{\beta_1, \gamma_1, \alpha\}\} = \{\{\beta_1, \gamma_1, \beta_2\}, \gamma_2, \alpha\} + \{\beta_2, \{\beta_1, \gamma_1, \gamma_2\}, \alpha\}.$$

Thus we observe that all the information of the commutation relations of $SDiff(\mathbb{R}^3)$ is contained in the Filippov–Nambu 3-algebra for a basis of functions in \mathbb{R}^3 . So, if both Hamiltonians are linear, $H_1 = a \cdot x, H_2 = b \cdot x$, then the flows $X(a, b) = (a \times b)^i \partial_i$ represent translations along the direction $a \times b$ (constant laminar flow). The next interesting case is of the linear Nambu flow with an axis of symmetry \vec{a} , which can be derived from a pair of Hamiltonians, $H_2 = \vec{a} \cdot \vec{x}$ and $H_1 = 1/2(\vec{x}, B\vec{x})$, where $a, x \in \mathbb{R}^3$ and B is a real, symmetric, 3×3 matrix. The corresponding trajectory of the linear Nambu flow is given by $\frac{dx^i}{dt} = \varepsilon^{ijk} a^j B^{kl} x^l = x^l M^{li}$. The solutions, given an initial condition $x^i(0)$, lie on the intersection of the plane with the normal vector \vec{a} and the quadratic surface given by $H_1 = 1/2(\vec{x}(0), B\vec{x}(0))$. We can integrate the equation of motion explicitly and find $\vec{x}(t) = \vec{x}(0)e^{tM}$. Since the matrix M is traceless, $A = e^M$ is an element of the group $SL(3, R)$. It is possible to compactify the linear Nambu flow on \mathbb{T}^3 , if we consider the linear evolution equation, modulo the size of the torus, i.e. we take x^i to belong to the elementary cell, $x^i = x^i + L^i$, where L^i is the length of the torus along direction x^i . If we choose these units so that $L^i = 2\pi$ then the action of the matrix A on every point of \mathbb{T}^3 is then taken modulo 2π . These flows are known [32] as toral automorphisms. The motion in this case, even though the equation is linear, can be chaotic, depending on the matrix elements of A . We can check that, for linear Nambu flows in \mathbb{R}^3 , we have, essentially, a reduction to a two-dimensional phase space problem on the plane orthogonal to the vector \vec{a} . In the case of \mathbb{T}^3 , if the vector has rational components, then we have a finite number of different images of the plane; if, however, the components are irrationals, then we have a truly three-dimensional evolution for the system.

The other illustrative example for this construction is an electric charge in a homogeneous magnetic field. At this point we review the relationship between the noncommutativity and incompressibility of the quantum Hall state and the effective theory of the incompressible fluid for the quantum Hall effect that is one of the most remarkable phenomena in condensed matter physics.

The classical phase space is defined by the H_2 function: $H_2 = \frac{e}{m^2 c} \vec{v} \cdot \vec{B}$ and so the Nambu algebra of the phase-space coordinates v^i is according to Eq. $\{X^i, X^j\}_{H_2} = \epsilon^{ijk} \partial_k H_2$,

$$(48) \quad \{v^i, v^j\} = \frac{e}{m^2 c} \epsilon^{ijk} B_k.$$

The phase space is a plane transverse to B embedded in R^3 . The dynamics is defined through $H_1 = \frac{1}{2} m v^2$ and the Hambu Eqs

$$\dot{v}^i = \frac{e}{m c} \epsilon^{ijk} v^j B^k$$

produce the correct physical Eqs. of motion for the Landau problem. The density of states in the lowest Landau level (LLL) is uniform and in proportion to the strength of the magnetic field, $\rho_0 = \frac{1}{2\pi l_0^2}$, where $l_0 = 1/\sqrt{B}$ is the magnetic length characterizing the scale of the wave function, and thus almost all electrons fall into the LLL in strong magnetic limit. Since the density is spatially constant, occupied area is exactly determined by fixing the number of particles. While the area is preserved, positions of particles can be changed by gauge transformation. Therefore, the electron state in the strong magnetic field behaves as incompressible fluid. Although any dynamical degrees of freedom do not exist because we neglect excitations to higher Landau levels, we should consider residual degrees of freedom for the fluid, geometrical configurations of particles, related to area preserving transformation. Thus Chern-Simons theory which is also non-dynamical theory captures the feature of the incompressible fluid. An important property of the incompressible fluid is that it possesses no dynamical degree of freedom and the residual degree of freedom comes from geometry of the fluid, which is related to area preserving diffeomorphism. Indeed one can derive Chern-Simons action that is the effective theory of the LLL state integrating over fermion modes [33]. We firstly introduce integration constants of the cyclotron motion describing the residual degrees of freedom called a guiding center: $X = x + l_0^2 \Pi_y, Y = y - l_0^2 \Pi_x$, where $\vec{\Pi} = \vec{p} + \vec{A}$ is the magnetic momentum. These operators satisfy the following commutation relations $[X, Y] = i l_0^2, [\Pi_x, \Pi_y] = -\frac{i}{l_0^2}$. When the magnetic field becomes very strong, contributions of the magnetic momentum to the guiding center and the canonical momentum can be neglected so that $\vec{X} \sim x$ and $\vec{p} \sim -\vec{A}$. Thus the Lagrangian can be written in terms of the guiding center coordinates $\mathcal{L} = \vec{p} \cdot \dot{\vec{x}} - \mathcal{H} = \frac{B}{2} (X\dot{Y} - \dot{X}Y), \quad \mathcal{H} = \frac{1}{2m} \vec{\Pi}^2$ and for the n -body state action we have

$$(49) \quad S = \frac{B}{2} \int dt \sum_{i=1}^n \epsilon_{ab} X_i^a \dot{X}_i^b.$$

In the large n limit, a fluid dynamical description becomes available $\sum_{i=1}^n \rightarrow \int d^2 x \rho(x), \vec{X}_i(t) \rightarrow \vec{X}(x, t), \vec{X}(x, 0) = \vec{x}$. The initial state is a reference configuration of the fluid. We will consider fluctuation modes from the reference state as the residual degree of freedom. The constraint for the incompressibility is the constant density condition, $\rho(x) = \rho_e$. Since the density of particles is the Jacobian of the fluid dynamical field, the constraint can be written with the Poisson bracket form

$$(50) \quad \rho_e = \rho(x) = \rho_e |\partial \vec{X} / \partial \vec{x}| = \frac{1}{2} \rho_e \epsilon_{ab} \{X^a, X^b\}.$$

Adding this Jacobian preservation constraint to action with temporal gauge field A_0 as the Lagrange multiplier, the action is modified as

$$(51) \quad S = \frac{B}{2} \rho_e \int dt d^2x [\varepsilon_{ab} X^a (\dot{X}^b - \theta \{X^b, A_0\}) + 2\theta A_0],$$

where $\theta = 1/2\pi\rho_e$ will become the noncommutative parameter. The Lagrangian has an exact gauge invariance under area preserving diffeomorphisms of the X plane. Then, satisfying the constraint, we can decompose X^a as $X^a = x^a + \theta^{ab} A_b$, $\theta^{ab} = \theta\varepsilon^{ab}$. Here we can regard gauge fields as the fluctuation mode from the reference state, and the gauge transformation corresponds to area preserving transformation of the fluid. Writing the action (51) in terms of the gauge fields, we obtain

$$(52) \quad S = \frac{1}{4\pi\nu} \int d^3x \varepsilon^{mnl} (\partial_m A_n A_l + \frac{\theta}{3} \{A_m, A_n\} A_l).$$

The constant $1/\nu = 1/(B\theta) = n$ is an integer, which is the level of the Chern–Simons theory, and $\nu = \rho_e/\rho_0$ is a filling fraction for the LLL states. Both the odd and even integer cases describe quantum Hall states, the odd cases corresponding to fermions and the even to bosons. Furthermore, this action can be regarded as a leading contribution of the noncommutative Chern-Simons action

$$(53) \quad S = \frac{1}{4\pi\nu} \int d^3x \varepsilon^{mnl} (\partial_m A_n \star A_l - \frac{2i}{3} A_m \star A_n \star A_l),$$

where \star -product is the Moyal product defined as $f(x) \star g(x) = f(x) \exp(\frac{i}{2} \overleftarrow{\partial}_m \theta^{mn} \overrightarrow{\partial}_n) g(x)$ that exactly reproduces the quantitative connection between filling fraction (level in the Chern–Simons description) and statistics required by Laughlin’s theory [33]. In this example, we have discussed the incompressible fluid as the LLL state and its effective theory.

In order to construct non-trivial examples of n -algebras, the crucial observation that of [11] where it was noted that the Nambu n -brackets (42) in \mathbb{R}^n create a tower of lower dimensional brackets of order $n - 1, n - 2, \dots$ including the family of Poisson structures on submanifolds which are embedded in \mathbb{R}^n . Namely, for a fixed H we can define a new bracket $\{f_1, \dots, f_{n-1}\}_H = \{H, f_1, \dots, f_{n-1}\}$, which turns out to be the Nambu bracket of order $n - 1$. Let us consider a smooth 3-manifold \mathcal{M}_3 embedded in \mathbb{R}^4 through a level-set Morse function $h(x^1, \dots, x^4) = c$ with $c \in \mathbb{R}$ fixed. Then by using the fundamental identity (40) in \mathbb{R}^4 we can check that the 3-bracket on \mathbb{R}^4

$$(54) \quad \{f_1, f_2, f_3\} = \omega^{ijk}(x) \partial_i f_1 \partial_j f_2 \partial_k f_3, \quad \omega^{ijk} = \varepsilon^{ijkl} \partial_l h,$$

satisfies the relation

$$(55) \quad \omega^{plm} \partial_p \omega^{ijk} = \omega^{pj k} \partial_p \omega^{ilm} + \omega^{ip k} \partial_p \omega^{jlm} + \omega^{ij p} \partial_p \omega^{klm}.$$

For example if h is a linear function $h(x^1, \dots, x^4) = a_i x^i$ then we obtain the constant Nambu 3-algebra $\{x^i, x^j, x^k\} = \varepsilon^{ijkl} a_l$. If h is a quadratic function, representing the sphere $S^3 \subset \mathbb{R}^4$: $h = \frac{1}{2} x^i x_i$ then we have the linear Nambu 3-algebra $\{x^i, x^j, x^k\}_{S^3} = \varepsilon^{ijkl} x^l$. If we use polar coordinates to project on the surface $e^4 = \cos \vartheta_3$, $e^3 = \cos \vartheta_2 \sin \vartheta_3$, $e^2 = \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3$, $e^1 = \cos \vartheta_1 \sin \vartheta_2 \sin \vartheta_3$ then the 3-sphere algebra is

$$\{e^i, e^j, e^k\}_{S^3} = \frac{1}{\sin^2 \vartheta_3 \sin \vartheta_2} \varepsilon^{pqr} \partial_{\vartheta_p} e^i \partial_{\vartheta_q} e^j \partial_{\vartheta_r} e^k = \varepsilon^{ijkl} e^l.$$

By using the Leibniz property, it is possible to write down the 3-algebra on S^3 explicitly for a basis of hyperspherical harmonics the corresponding Nambu S^3 3-algebra $Y_a = Y_{nlm}(\vartheta_3, \vartheta_2, \vartheta_1)$, $m = -l, \dots, l$, $l = 0, 1, \dots, n-1$,

$$(56) \quad \{Y_a, Y_b, Y_c\} = f_{abc}^d Y_d,$$

where f_{abc}^d can be expressed in terms of 6j symbols of $SU(2)$ ($SO(4) \sim SU(2) \times SU(2)$).

We observe that, the most general Nambu 3-algebra (54) $\{x^i, x^j, x^k\}_h = \epsilon^{ijkl} \partial_l h$ has h as Casimir. Such restriction of this algebra on the surface $h = c$ gives a non-degenerate 3-form ω^{ijk} which satisfies the fundamental identity. Let us now present two examples of 3-algebras such as \mathbb{R}^3 and \mathbb{T}^3 . Obvious that the 3-algebra \mathbb{R}^3 of coordinates is $\{x^i, x^j, x^k\} = \epsilon^{ijk}$. For the 3-torus \mathbb{T}^3 the algebra for the periodic function basis: $e^n = e^{in \cdot x}$, with $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$ and $x = (x_1, x_2, x_3) \in (0, 2\pi)^3$ is given by $\{e^n, e^m, e^l\} = -in \cdot (m \times l) e^{n+m+l}$.

In the long history of the study of the Nambu bracket there have been many attempts to quantize the Nambu mechanics, based on the deformation theory, path integral formulation and on Nambu-Heisenberg relation. However, the issue of the quantization is still a difficult child and does not seem to be unique. An explicit realization of the quantum Nambu bracket in terms of matrices, as posed in the original paper by Nambu, still seems to be lacking. As an interesting approach authors [34] introduced many-index objects (as particular case three-index objects called 'cubic matrices') to realize the quantum version of the Nambu bracket. The most mathematically complete quantization scheme for the Nambu 3-bracket up to now is given in ref. [35] where an algebraic topological quantization, the Zariski \star quantization which is based on factorization of polynomials in several real variables and variations thereof, has been proposed, but the algebraic complexity of the scheme seems to hide important physical and geometrical aspects of the problem. All the other present proposals are violate, in general, the basic properties of the 3-bracket such as Leibnitz and the Fundamental Identity [11]. One of different approach to quantization is a canonical formalism. It is based on the Heisenberg commutation relations, which for the phase space $X = \mathbb{R}^2$ with canonical Poisson bracket look like the following $[a, a^\dagger] = I$, where operators a^\dagger, a act in a linear space of quantum states. Being one of a fundamental principles of quantum mechanics, the Heisenberg commutation relations have remarkable mathematical properties. In particular, one has celebrated Stone-von Neumann theorem that all irreducible representations of the Heisenberg commutation relations are unitary equivalent. In [9] proposed the following generalization of the Heisenberg commutation relation

$$(57) \quad [\hat{A}_1, \hat{A}_2, \hat{A}_3] = \hat{A}_1 \hat{A}_2 \hat{A}_3 - \hat{A}_1 \hat{A}_3 \hat{A}_2 + \hat{A}_3 \hat{A}_1 \hat{A}_2 - \hat{A}_3 \hat{A}_2 \hat{A}_1 + \hat{A}_2 \hat{A}_3 \hat{A}_1 - \hat{A}_2 \hat{A}_1 \hat{A}_3 = i\hbar_N I,$$

where $\hat{A}_1, \hat{A}_2, \hat{A}_3$ are linear operators, I is a unit and \hbar_N is a constant. Nambu-Heisenberg relation with $\hbar_n = \sqrt{3}$ admits the following representation in the Hilbert space \mathcal{H}_3

$$(58) \quad \hat{A}_1 |\omega \rangle = (\omega + 1 + \rho) |\omega + 1 \rangle, \quad \hat{A}_2 |\omega \rangle = (\omega + \rho) |\omega + \rho \rangle, \\ \hat{A}_3 |\omega \rangle = (\omega + \rho^2) |\omega + \rho^2 \rangle.$$

A direct calculation proves the following result : $[\hat{A}_1, \hat{A}_2, \hat{A}_3] |\omega \rangle = \rho^2 (1 - \rho^2) |1 + \rho + \rho^2 + \omega \rangle$ with $1 + \rho + \rho^2 = 0$, i.e. $\rho = \frac{-1 + \sqrt{-3}}{2}$. Then the Nambu-Heisenberg

commutator can have both finite and infinite dimension linear Hilbert space realization \mathcal{H}_3 with the basis $\{|\omega \rangle\}$ parametrized by a ring of algebraic integers $\mathbb{Z}[\rho]$, i.e.

$$(59) \quad \omega = m_1 + m_2\rho \in \mathbb{Z}[\rho], \quad m_1, m_2 \in \mathbb{Z}.$$

It is interesting to note that the Nambu–Heisenberg relation suggests the cubic form of the uncertainty principle $\Delta P \Delta Q \Delta R \sim \hbar_N$. In [11] it was also mentioned that the Nambu–Heisenberg commutation relations for general n admit a natural representation in the vector space \mathcal{H}_n . In [36] has been presented the representation the Nambu–Heisenberg commutation relation for $n = 5$ and $n = 7$ which can be proved directly with the use of a symbolic calculations package Wolfram Research Mathematica. However, although we are absolutely certain that there exists a natural representation for any n , we are unable to construct it explicitly.

In this section we presented a geometrical perspective for the classical and quantum Nambu dynamics in three dimensional phase space manifolds. The two Hamiltonians are interpreted in following way: one of the two sets defines the 2- dim phase space geometry embedded in the 3-dim phase space, while the second one gives the dynamics of the trajectories on the 2-dim phase space. This view persists in all higher n -dimensions of phase space where exists $n - 1$ Hamiltonians. Then we choose $n - 2$ of them to define a 2-dim phase space embedded in n -dimensions with the $(n - 1)$ th Hamiltonians to define the trajectories. This perspective stressed, in fact, the importance of the $SDiff(M_3)$ group as the all embracing framework of possible Nambu 3-dim Hamiltonian systems which, after all, are the flow equations for stationary incompressible fluids in the manifold. We presented an explicit constructions, in the Clebsch–Monge gauge, of the structure constants of the Nambu 3-algebras for the cases of \mathbb{R}^3 , the torus \mathbb{T}^3 and the sphere S^3 as well as of quadratic 3-dim manifolds embedded in \mathbb{R}^4 . The foliation of the three dimensional phase space by arbitrary two dimensional symplectic manifolds, whose quantization is well known either by operator methods or \star -quantization techniques, motivates the definition of the quantum 3-bracket (or 3-geometry) as a foliation of quantum 2-brackets. For this purpose the authors [31] define an associative quantization of the algebra $\{x^i, x^j\}_{H_2} = \varepsilon^{ijk} \partial_k H_2$ promoting the phase space coordinates x^i at $t = 0$ to hermitian operators \hat{X}^i with commutation relations: $[\hat{X}^i, \hat{X}^j] = i\hbar \varepsilon^{ijk} P^k(\hat{X})$ having as a classical limit $\frac{1}{i\hbar} [\hat{X}^i, \hat{X}^j]_{\hbar \rightarrow 0} = \{x^i, x^j\}_{H_2}$. If H_2 is a quadratic function of the canonical phase coordinates there is no ordering problem. For H_2 cubic or higher (non-linear Lie algebra) there is no unique way to quantize. The quantum 3-commutator should be viewed as the corresponding quantum volume density element. It is associated, in our case, with the intersection of quantum (fuzzy) surfaces. It is hoped that quantum 3-algebras is a new interesting area of mathematics in itself, with importance as well for the quantization of fluid dynamics and more generally for the geometry of 3-d manifolds (branes) such as our physical space (quantum gravity).

5. M5 FROM M2

When a neutral material put in an external electric field the electric charges separate from each other and form an electric dipole. Similar phenomena exists in string theory where there are extended objects such as D_p branes and higher rank antisymmetric form fields as charges of these objects. In particular, when a set of

neutral D branes put in an external antisymmetric background field one observes the "polarized" D branes which are expanded into a higher dimensional world volume theory [38]. One may naturally expect such phenomena for membranes in \mathcal{M} -theory.

The eleven dimensional supergravity contains an antisymmetric three-form field C_3 and its magnetic dual six-form field C_6 . It is well known that any p -brane can covariantly couple to a $p+1$ form field and also to $p-1, \dots$ form fields due to existence of world volume antisymmetric gauge fields or Kalb–Ramond fields. Meyers in [38] showed that p -brane can also couple to $p+3, \dots$ form fields via the fact that the commutators of transverse scalar fields in non-Abelian theories are non zero. Then interesting subject in this crossing pattern is the coupling of supergravity form fields as background fields with world volume of M2 and M5 branes. By placing a system of N D_p branes in an external background form field causes that the system would has a vacuum in which is noncommutative an stable against of the commutative one. Similarly, in \mathcal{M} -theory, if a collections of membranes would be in an external C_6 form field the system has a vacuum in which membranes are polarized due to field strength effect and are formed into a fuzzy S^3 sphere. This can be interpreted as the formation of spherical branes ended on five-brane. Thus, a single M5-brane may contain multiple M2-branes and is therefore a promising starting point for a construction of the BLG model. Another indication of this is that the Nambu-bracket realization of the BLG theory introduces some 'internal' Riemannian 3-manifold \mathcal{N}_3 , so that the total space dimension is $2+3=5$. In this realization, the BLG model is essentially an exotic gauge theory for the group $SDiff(S^3)$ of volume-preserving diffeomorphisms of the 3-sphere.

This is not the first occasion on which exotic gauge theories based on volume-preserving diffeomorphisms have appeared [42]. One can use various methods to study M5 brane theory by using M2 theory but the authors of [39] described an interesting approach to achieve this goal using the Nambu 3-brackets. In fact, by considering an 3 dimensional internal space in the world volume of M2 brane, they were able to find a six dimensional theory which has some desired properties of an M5 brane. For example, they found the action of a self dual two-form gauge field living on the world volume of M5 brane.

For the construction of M5-brane, the authors [39] introduce an "internal" three-manifold \mathcal{N} and use the Nambu 3-bracket

$$\{f, g, h\} = P^{\mu\nu\lambda}(y)\partial_\mu f \partial_\nu g \partial_\lambda h$$

on \mathcal{N} as a realization of three-algebra. Here y^μ ($\mu = 1, 2, 3$) is the local coordinates on \mathcal{N} . One of the most important properties of the Nambu 3-bracket is that it satisfies the analog of the fundamental identity for arbitrary functions f_i on \mathcal{N} ,

$$(60) \quad \{f_1, f_2\{f_3, f_4, f_5\}\} = \\ \{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} + \{f_3, f_4\{f_1, f_2, f_3\}\}.$$

This gives a very severe constraint on the coefficient $P^{\mu\nu\lambda}(y)$. Actually it is known that by the suitable choice of the local coordinates, it can be reduced to the Jacobian

$$(61) \quad \{f, g, h\} = \varepsilon^{\mu\nu\lambda} \frac{\partial f}{\partial y^\mu} \frac{\partial g}{\partial y^\nu} \frac{\partial h}{\partial y^\lambda}.$$

This property is referred to as the 'decomposability' in the literature. If we choose the basis of functions on \mathcal{N} as $\chi(y)$ ($a = 1, 2, \dots$) and write the Nambu-Poisson

bracket as a Lie 3-algebra,

$$(62) \quad \{\chi^a, \chi^b, \chi^c\} = \varepsilon^{\mu\nu\lambda} \partial_\mu \chi^a \partial_\nu \chi^b \partial_\lambda \chi^c = f^{abc}{}_d \chi^d(y).$$

Eq. (60) implies that the structure constant $f^{abc}{}_d$ here satisfies the fundamental identity. The integration $\langle f, g \rangle = \int_{\mathcal{N}} d^3y f(y)g(y)$ over the y -space can be used to define the invariant metric $h^{ab} = (\chi^a, \chi^b)$. Except for the trivial case ($\mathcal{N} = \mathbb{R}^3$), we have to cover \mathcal{N} by local patches and the coordinates y^μ are the local coordinates on each patch. If we need to go to the different patch where the local coordinates are y'^μ , the coordinate transformation between y and y' (say $y'^\mu = f^\mu(y)$) should keep the Nambu 3-bracket (61). It implies that $\{f^1, f^2, f^3\} = 1$. Namely $f^\mu(y)$ should be the volume-preserving diffeomorphisms. As we will see, the gauge symmetry of the BLG model for this choice of the Filippov-Nambu 3-algebra is the volume-preserving diffeomorphisms of \mathcal{N} which is very natural in this set-up.

Now, we will show that the BLG model with a Filippov-Nambu structure on a 3-dimensional manifold contains the low energy degrees of freedom on an M5-brane. Before going on, let us count the number of degrees of freedom in the bosonic and fermionic sectors in our model. The fermion is a Majorana spinor in 10+1 dimensions with a chirality condition, and thus it has 16 real fermionic components, equivalent to 8 bosonic degrees of freedom. For a 5-brane there are 5 transverse directions corresponding to 5 scalars X^i . For an ordinary 2-form gauge field in 6D, there are 6 propagating modes. But since we do not have the usual kinetic term for A_m , but rather a Chern–Simons term, there are only 3 propagating modes. The low energy effective theory of an M5-brane contains the same number of bosonic and fermionic degrees of freedom. But a salient feature of the M5-brane is that the 2-form gauge field is self-dual. Hence our major challenge is to show that the gauge field of the BLG model is equivalent to a self-dual 2-form gauge field.

A comment on the notation: we will use I, J, K to label the transverse directions to the membrane worldvolume \mathcal{M} . We decompose this eight dimensional space as a direct product of \mathcal{N} and remaining 5 dimensional space. We use μ, ν, λ to label \mathcal{N} and i, j, k to label the transverse directions of the M5-brane. By combining the basis of $C(\mathcal{N})$, we can treat $X_a^I(x)$ and $\Psi_a(x)$ as six-dimensional local fields

$$(63) \quad X^I(x, y) = \sum_a X_a^I(x) \chi^a(y), \quad \Psi(x, y) = \sum_a \Psi_a(x) \chi^a(y).$$

Similarly, the gauge field A_m^{ab} can be regarded as a bi-local field:

$$(64) \quad A_m(x, y, y') = A_m^{ab}(x) \chi^a(y) \chi^b(y').$$

The existence of such a bi-local field does not mean that the theory is non-local. Let us expand it with respect to $\Delta y^\mu = y'^\mu - y^\mu$ as

$$A_m(x, y, y') = a_m(x, y) + b_{m\mu}(x, y) \Delta y^\mu + \frac{1}{2} c_{m\mu\nu}(x, y) \Delta y^\mu \Delta y^\nu + \dots$$

Because A_m^{ab} always appears in the action in the form $f^{bcd}{}_a A_m{}_{bc} X_a^I$, the field $A_m(y, y')$ is highly redundant, and only the component $b_{m\mu}(x, y) = \frac{\partial}{\partial y'^\mu} A_m(x, y, y')|_{y'=y}$ contributes to the action. For example, the covariant derivative of BLG model is rewritten for our case as,

$$(65) \quad \begin{aligned} \mathcal{D}_m X^I(x, y) &= (\partial_m X_a^I(x) - g f^{bcd}{}_a A_m{}_{bc} X_a^I(x)) \chi^a(y) \\ &= \partial_m X^I(x, y) - g \varepsilon^{\mu\nu\rho} \frac{\partial^2 A_m(x, y, y')}{\partial y^\mu \partial y'^\nu} \Big|_{y=y'} \frac{\partial X^I(x, y)}{\partial y^\rho} \end{aligned}$$

$$= \partial_m X^I(x, y) - g \varepsilon^{\mu\nu\rho} \partial_\mu b_{m\nu}(x, y) \partial_\rho X^I(x, y) = \partial_m X^I - g \{b_{m\nu}, y^\nu, X^I\}.$$

The covariant derivative for the fermion field is similarly,

$$(66) \quad \mathcal{D}_m \Psi(x, y) = \partial_m \Psi(x, y) - g \{b_{m\nu}, y^\nu, \Psi\}.$$

In [39], this theory written in terms of fields on six dimensions is identified with the theory describing a single M5-brane. However we still have SO(8) global symmetry, which is different from the SO(5) symmetry expected in the M5-brane theory. Then to interpret the six-dimensional theory, we must take a partial static gauge for three among six world-volume coordinates. As we mentioned above, however, we do not have full diffeomorphisms in the y^μ space. The action is invariant only under volume-preserving diffeomorphisms. This implies that we cannot completely fix the fields X^μ , and there are remaining physical degrees of freedom. For this reason, we should loosen the static gauge condition as

$$(67) \quad X^\mu(x, y) = \frac{1}{g} y^\mu + b^\mu(x, y), \quad b_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho} b^\rho.$$

As was shown in [39], the tensor field $b_{\mu\nu}$ is identified with a part of the 2-form gauge field on a M5-brane. The gauge transformations of the scalar fields X^I and fermion fields are given by

$$(68) \quad \delta_\Lambda X^I(x, y) = g \Lambda_{ab}(x) f^{abc}{}_d X_c^I(x) \chi^d(y) = g \Lambda_{ab}(x) \{\chi^a, \chi^b, X^I\} \\ = g(\delta_\Lambda y^\mu) \partial_\mu X^I(x, y), \quad \delta_\Lambda \Psi(x, y) = g(\delta_\Lambda y^\mu) \partial_\mu \Psi(x, y),$$

where we used $f^{abc}{}_d = (\{\chi^a, \chi^b, \chi^c\}, \chi_d)$, $\sum_a \chi^a(y) \chi_a(y') = \delta(y - y')$ and $\delta_\Lambda y^\rho$ is defined as

$$\delta_\Lambda y^\rho = \varepsilon^{\rho\mu\nu} \partial_\mu \Lambda_\nu(x, y), \quad \Lambda_\mu(x, y) = \partial'_\mu \tilde{\Lambda}(x, y, y')|_{y=y'}, \quad \tilde{\Lambda}(x, y, y') = \Lambda_{ab}(x) \chi^a(y) \chi^b(y').$$

The transformation (68) may be regarded as the infinitesimal reparametrization $y'^\mu = y^\mu - g \delta y^\mu$. Since $\partial_\mu \delta y^\mu = 0$, it represents the volume-preserving diffeomorphisms. As the symmetry is local on \mathcal{M} , the gauge parameter is an arbitrary function of x . So what we have obtained is a gauge theory on \mathcal{M} whose gauge group is the volume-preserving diffeomorphisms of \mathcal{N} which preserves the volume form $\omega = dy^1 \wedge dy^2 \wedge dy^3$.

The following combination defines the 'covariant' derivative along the fiber direction:

$$(69) \quad \mathcal{D}_\mu \Phi \equiv \frac{1}{2} g^2 \varepsilon_{\mu\nu\rho} \{X^\nu, X^\rho, \Phi\} \\ = \partial_\mu \Phi + g(\partial_\lambda b^\lambda \partial_\mu - \partial_\mu b^\lambda \partial_\lambda) \Phi + \frac{1}{2} g^2 \varepsilon_{\mu\nu\rho} \{b^\mu, b^\nu, \Phi\}.$$

Together with (65) we have a set of covariant derivatives on M5 world-volume. Just like the case of ordinary gauge theories, the field strength of the tensor field \mathcal{H} arises in the commutator of the covariant derivatives defined above:

$$(70) \quad [\mathcal{D}_\mu, \mathcal{D}_\nu] \Phi = g^2 \varepsilon_{\mu\nu\rho} \{\mathcal{H}_{123}, X^\rho, \Phi\}, \quad [\mathcal{D}_m, \mathcal{D}_\mu] \Phi = g^2 \{\mathcal{H}_{m\rho\mu}, X^\rho, \Phi\}, \\ [\mathcal{D}_m, \mathcal{D}_n] \Phi = -\frac{g}{V} \varepsilon_{mnl} \mathcal{D}_p \tilde{\mathcal{H}}^{p\ l\ \mu} \mathcal{D}_\mu \Phi,$$

where V is the 'induced volume' $V = g^3 \{X^1, X^2, X^3\}$, and $\tilde{\mathcal{H}}$ is a dual field strength. Equation (70), in which Φ is taken to be X^μ is nothing but the Bianchi identity $\mathcal{D}_l \tilde{\mathcal{H}}^{lmn} + \mathcal{D}_\mu \tilde{\mathcal{H}}^{\mu mn} \equiv 0$, where $\tilde{\mathcal{H}}^{lmn}$ and $\tilde{\mathcal{H}}^{\mu mn}$ are Hodge dual of $\mathcal{H}_{\mu\nu\lambda}$ and $\mathcal{H}_{\mu\nu m}$.

Now we rewrite the various parts of the BLG action in terms of the six dimensional fields and their covariant derivatives

$$(71) \quad S_X + S_{pot} = \int d^3x < -\frac{1}{2}(\mathcal{D}_m X^i)^2 - \frac{1}{2}(\mathcal{D}_\mu X^i)^2 - \frac{1}{4}\mathcal{H}_{m\mu\nu}^2 - \frac{1}{12}\mathcal{H}_{\mu\nu\rho}^2 - \frac{1}{2g^2} \\ - \frac{g^4}{4}\{X^\mu, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 >$$

$$(72) \quad S_\Psi + S_{int} = \int d^3x < \frac{i}{2}\bar{\Psi}\Gamma^m\mathcal{D}_m\Psi + \frac{i}{2}\bar{\Psi}\Gamma^\rho\Gamma_{123}\mathcal{D}_\rho\Psi \\ + i\frac{g^2}{2}\bar{\Psi}\Gamma_{\mu i}\{X^\mu, X^i, \Psi\} + i\frac{g^2}{4}\bar{\Psi}\Gamma_{ij}\{X^i, X^j, \Psi\} >,$$

where $< f, g > = \int d^3y f g$. The Chern–Simons term cannot be rewritten in manifestly gauge-covariant form

$$(73) \quad S_{CS} = \int d^3x \varepsilon^{mnp} < -\frac{1}{2}\varepsilon^{\mu\nu\lambda}\partial_\mu b_{m\nu}\partial_n b_{p\lambda} + \frac{g}{6}\varepsilon^{\mu\nu\lambda}\partial_\mu b_{\nu\sigma}\varepsilon^{\rho\sigma\tau}\partial_\sigma b_{\rho\rho}(\partial_\lambda b_{m\tau} - \partial_\tau b_{m\lambda}) > .$$

However, the equation of motion which is derived from these actions turns out to be manifestly gauge-covariant. This was confirmed recently in [40] where it was shown that solving the field equations associated with $b_{m\mu}$ and $b_{\mu\nu}$ is tantamount to imposing the Hodge self-duality condition on the non-linear field strength. This allowed the authors to rewrite the gauge field Lagrangian in a gauge covariant form as

$$(74) \quad S = - \int d^3x d^3y \left\{ \frac{1}{8}\mathcal{H}_{m\mu\nu}\mathcal{H}^{m\mu\nu} + \frac{1}{12}\mathcal{H}_{\mu\nu\rho}\mathcal{H}^{\mu\nu\rho} - \frac{1}{144}\varepsilon^{mnl}\varepsilon^{\mu\nu\rho}\mathcal{H}_{mnl}\mathcal{H}_{\mu\nu\rho} \right. \\ \left. - \frac{1}{12g}\varepsilon^{mnl}\mathcal{H}_{mnl} \right\}.$$

The last term in this expression can be interpreted as a coupling of the M5-brane to the constant background C_3 field which has non-zero components $C_{mnl} = \frac{1}{g}\varepsilon_{mnl}$. Following [40], it is possible to rewrite this as $\frac{1}{2}\int \mathcal{H}_3 \wedge C_3$. This action possesses full volume preserving diffeomorphism invariance. However the Lorentz symmetry is broken by the presence of the three-form field.

Next we can rewrite the supersymmetry transformations in terms of the six-dimensional covariant derivatives and field strength:

$$\delta X^i = i\bar{\epsilon}\Gamma^i\Psi, \quad \delta b_{\mu\nu} = -i\bar{\epsilon}\Gamma_{\mu\nu}\Psi, \quad \delta b_{m\nu} = -iV(\bar{\epsilon}\Gamma_m\Gamma_\nu\Psi) + ig(\bar{\epsilon}\Gamma_m\Gamma_i\Gamma_{123}\Psi)\partial_\nu X^i, \\ \delta\Psi = \mathcal{D}_m X^i\Gamma^m\Gamma^i\epsilon + \mathcal{D}_\mu X^i\Gamma^\mu\Gamma^i\epsilon - \frac{1}{2}\mathcal{H}_{m\nu\rho}\Gamma^m\Gamma^{\nu\rho}\epsilon - \left(\frac{1}{g} + \mathcal{H}_{123}\right)\Gamma_{123}\epsilon \\ - \frac{g^2}{2}\{X^\mu, X^i, X^j\}\Gamma^\mu\Gamma^{ij}\epsilon + \frac{g^2}{6}\{X^i, X^j, X^k\}\Gamma^{ijk}\Gamma^{123}\epsilon.$$

A peculiar property of this supersymmetric transformation is that the perturbative vacuum (the configuration with all fields vanishing) is not invariant under this transformation due to the term in $\delta\Psi$ proportional to $1/g$. We can naturally interpret this term as a contribution of the background C-field. In the M5- brane action coupled to background fields, the self-dual field strength is defined by $H = db + C$. The inclusion of C-field in the field strength is required by the invariance of the action under C-field gauge transformations. The shift of the field strength \mathcal{H}_{123} by $1/g$ in the action as well as in the supersymmetric transformation suggests that the relation $C \sim g^{-1}$ between the Nambu structure and the C-field background. In

fact, M5-brane in a constant C -field background is still 1/2 BPS. The effect of the C -field is changing which half of 32 supersymmetry remain unbroken. We can find this phenomenon in our six-dimensional theory. In addition to 16 supersymmetries we described above, the theory has 16 non-linear fermionic symmetries δ^{nl} , which shift the fermion by a constant spinor $\delta^{nl}\Psi = \xi$. The action is invariant under this transformation because constant functions in y^μ space are in the center of the 3-algebra. The perturbative vacuum is invariant under the combination of two fermionic symmetries $\delta_\epsilon - \frac{1}{g}\delta^{nl}$. In the weak coupling limit $g \rightarrow 0$, the transformation laws for this combined symmetry agree with those of an $N = (2, 0)$ tensor multiplet:

$$\delta X^i = i\bar{\epsilon}\Gamma^i\Psi, \quad \delta\Psi = \partial_\mu X^i\Gamma^\mu\Gamma^i\epsilon - \frac{1}{12}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\epsilon \quad \delta b_{\underline{\mu\nu}} = -i\bar{\epsilon}\Gamma_{\underline{\mu\nu}}\Psi.$$

The gauge symmetry of the M5 world-volume theory is the volume-preserving diffeomorphisms on \mathcal{N} . The transformation law for both X^i and Ψ are given in the same form (68) where the volume-preserving coordinate transformation is parametrized by three arbitrary functions Λ_μ . While $b_{m\mu}$ and $b_{\mu\nu}$ are viewed as the gauge potentials for the gauge symmetry of coordinate transformations preserving a given Nambu structure, B_m^μ and b^μ should be viewed as two types of deformation parameters of the Nambu structure of the M5-brane world-volume. We have b^μ specifying the change of the Nambu structure due to a change of coordinates δy^μ in \mathcal{N} (so that the volume form is changed), and B_m^μ specifying the change due to a mixing of the two classes of coordinates x^m and y^μ . The gauge symmetry corresponds to redundant descriptions of deformations of the Nambu structure. Then the M5-brane theory with a self-dual gauge field can thus be interpreted as a dynamical theory of the Nambu structure.

6. REFORMULATION OF DIRAC–NAMBU–GOTO ACTION BY NAMBU BRACKET

In the late 80's in [3] has been remarked that "Eleven dimensional supergravity remains enigma". In a papers [42] (and references therein) it was suggested that, just as 10-dimensional supergravity is related to superstring theory, so 11-dimensional supergravity may be related to supermembrane theory. In support of this connection it was built an 11-dimensional supermembrane action and shown that the preservation of local symmetries of this action in an 11-dimensional background requires that the background satisfy certain constraints, which are equivalent to the equations of motion of 11-dimensional supergravity. Furthermore, it was argued that the spectrum of the 11-dimensional supermembrane contains the massless states of 11d supergravity. Then one can hope that a supermembrane theory will provide a quantum consistent extension of 11-dimensional supergravity just as superstring theories are thought to provide a quantum consistent extension of 10-dimensional supergravity theories. Already classically the possibilities for super p -brane actions are severely limited. It is well-known that the Green–Schwarz superstring action exists for $d = 3, 4, 6, 10$ and one can similarly show that the supermembrane action exists for $d = 4, 5, 7, 11$. Then we might expect quantum considerations to impose yet further restrictions. Indeed we know that only the 10d superstring action is quantum consistent (i.e. free from anomalies). This might lead one to suspect that the only quantum consistent super p -brane is 11d supermembrane.

With an embedding of $(p + 1)$ -dimensional worldvolume coordinates into D -dimensional target spacetime, $X(\xi) : \xi^m \rightarrow X^M$, where $m = 0, 1, \dots, p$ and $M =$

$0, 1, \dots, D - 1$, the Dirac–Nambu–Goto (DNG) action for a p -brane reads [4]

$$(75) \quad S_{DNG} = -T \int d^{p+1}\xi \sqrt{-\det(\partial_m X^M \partial_n X_M)},$$

where T is the membrane tension. Let us decompose, formally, the p -brane world-volume coordinates into two parts: $\{\xi^m\} = \{\sigma^\mu, \zeta^i\}$, where $\mu = 0, 1, \dots, d - 1$ and $i = 1, \dots, \hat{d}$. The decomposition is a priori arbitrary for any non-negative integers d, \hat{d} . One natural application of the splitting will be the case where p -brane is extended over two topologically different spaces, e.g. compact and non-compact spaces. With the decomposition above, a square root free reformulation of the DNG action was achieved in:

$$(76) \quad S = \int d^d\sigma \text{Tr}(\sqrt{-h}\mathcal{L}), \quad \text{Tr} := \int d^{\hat{d}}\zeta, \quad \mathcal{L} = -h^{\mu\nu} \mathcal{D}_\mu X^m \mathcal{D}_\nu X_M \\ - \frac{1}{4\hat{d}!} e^{d-1} \{X^{M_1}, X^{M_2}, \dots, X^{M_{\hat{d}}}\} \{X_{M_1}, X_{M_2}, \dots, X_{M_{\hat{d}}}\} + (d-1)e,$$

where the action contains three kinds of auxiliary fields: scalar e , d -dimensional metric $h_{\mu\nu}$ and a gauge connection A_μ^i which defines the ‘covariant derivative’: $\mathcal{D}_\mu X^M := \partial_\mu X^M - A_\mu^i \partial_i X^M$. The classical equation of motion that follows from (75) may equivalently be obtained from the action (76). Integrating out all the auxiliary fields, using their on-shell values, the action reduces to the DNG action, $S \equiv S_{DNG}$, and hence the classical equivalence. The novelty of the above reformulation was the appearance of the gauge interaction and the Nambu bracket squared potential. The latter basically stems from an identity rewriting the determinant as the Nambu bracket squared:

$$(77) \quad \det(\partial_i X^M \partial_j X_M) = \frac{1}{\hat{d}!} e^{d-1} \{X^{M_1}, X^{M_2}, \dots, X^{M_{\hat{d}}}\} \{X_{M_1}, X_{M_2}, \dots, X_{M_{\hat{d}}}\}.$$

A physical picture after the reformulation can be described as a single brane as a condensation of multiple lower-dimensional branes, i.e. a p -brane by $(d-1)$ -branes. Obviously, the choice of $\hat{d} = 0$ and $d = p+1$ corresponds to the well-known ‘Polyakov’ action, which was actually first conceived by the authors [37]. On the other hand, with a gauge fixing for e to be constant, the other extreme choice of $d = 0$, $\hat{d} = p+1$ leads to the Schild action. Furthermore, the association of the digits, 2 and 3 to string and \mathcal{M} -theory becomes manifest within the reformulation: For example, the fact that the codimension of D -branes is 2 suggests to choose $\hat{d} = 2$, which leads to the two-algebra as in the Yang–Mills theory. Likely the choice of $p = 5$, $d = 3$, $\hat{d} = 3$ suggests that the BLG model with an infinite dimensional gauge group describes a M5-brane as a condensation of multiple M2-branes. Here we presented a generalization of the Polyakov method, a novel scheme to take off the square root of DNG action for a p -brane. While the square root free Polyakov action is a $(p+1)$ -dim field theory, the resulting action (76) lives in an arbitrary lower dimensional d which is smaller than $p+1$. Such a reformulation shows a the general phenomenon that non-Abelian structure of lower dimensional gauge theories can capture the description of higher dimensional objects. It suggests that a single p -brane can be described via different but equivalent actions, either $(p+1)$ -dimensional Polyakov action or various lower dimensional gauge theories with the Nambu bracket interactions of different degrees. This implies the existence of a web of duality relations among large classes of gauge theories. In particular, a theory

with the Yang–Mills interaction i.e. Poisson bracket of $\hat{d} = 2$, is equivalent to lower dimensional theories based on Nambu bracket structure.

The reformulation of the DNG action (76) is purely bosonic. In order to establish a firm connection to string/ \mathcal{M} -theory one needs to supersymmetrize them. The requirement of supersymmetry may give rise to a constraint on the a priori arbitrary decomposition, $p + 1 = d + \hat{d}$. Our main interest is to supersymmetrize the action (76). For $d = 1$ case, supersymmetric actions are ready to be read-off from an earlier work [41]. These authors listed light-cone gauge fixed supersymmetric actions for various p -branes in diverse spacetime dimensions. As usual, the Fierz identity required for the supersymmetry invariance, restricts the possible values of p and the spacetime dimension D :

$$(78) \quad \begin{aligned} p = 1, \quad D = 3, 4, 6, 10; \quad p = 2, \quad D = 4, 5, 7, 11; \\ p = 3, \quad D = 6, 8; \quad p = 4, \quad D = 9; \quad p = 5, \quad D = 10. \end{aligned}$$

In the string theory, the lightcone gauge $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{D-1}) = \tau$ is convenient for quantization because it allows the elimination of all unphysical degrees of freedom and unitarity is guaranteed. Of course, one loses manifest Lorentz invariance and one must be careful to check that it is not destroyed by quantization. In membrane theory, however, the lightcone gauge does not eliminate all unphysical degrees of freedom. For membranes, however, only $(D - d)$ variables are physical. Thus the lightcone gauge must leave a residual gauge invariance [42]. Utilizing the identity (77), in terms of the Nambu p -bracket, their light-cone gauge fixed supersymmetric p -brane actions can be reexpressed in a compact form:

$$(79) \quad \begin{aligned} \mathcal{L}_{L.C.} = & \frac{1}{2}(\mathcal{D}_\tau X^I)^2 - \frac{1}{2p!}\{X^{I_1}, \dots, X^{I_p}\}^2 + \frac{i}{2}\bar{\Psi}\mathcal{D}_\tau\Psi \\ & + \frac{1}{2(p-1)!}\bar{\Psi}\Gamma^{I_1\dots I_{p-1}}\{X_{I_1}, \dots, X_{I_{p-1}}\}\Psi, \end{aligned}$$

where $\mathcal{D}_\tau = \partial_\tau + u^a(\sigma, \tau)\partial_a$ is a 'covariant time derivative' with 'gauge field' u^a satisfying $\partial_a u^a = 0$. For a membrane ($p = 2$) of spherical topology, the solution of this constraint is $u^a = \varepsilon^{ab}\partial_b\omega$. Remarkably that for correspondence $X^I \rightarrow A^I, \omega \rightarrow A_0$, this looks like a $(D - 1)$ dimensional supersymmetric Yang–Mills theory dimensionally reduced to one time dimension with infinite dimensional gauge group. This group is, in fact, the subgroup of the worldvolume diffeomorphisms group that preserves the Nambu bracket $\{f, g\} = \varepsilon^{ab}\partial_a f \partial_b g$ and is known as the group of area-preserving diffeomorphisms. The nature of this infinite-dimensional gauge group depends critically on the topology of the membrane. As example, for spherical topology it was shown to be $SU(\infty)$ by Hoppe [44]. This has an important application in regularization of membrane theories by replace the gauge theory of $S\text{Diff}(S^2)$ by gauge theory of $SU(N)$. Hoppe has studied the canonical quantization of a relativistic spherical membrane in the light cone gauge. He find that the classical $SU(N)$ Yang–Mills theories, in the large N limit

$$(80) \quad \lim_{N \rightarrow \infty} N[A_\mu, A_\nu] = \{A_\mu, A_\nu\},$$

can be described as a new type of gauge principle. The gauge potentials become c-number functions

$$(81) \quad A_\mu(x, \vartheta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l A_\mu^{lm}(x) Y_{lm}(\vartheta, \varphi),$$

of two additional coordinates, which parametrize the surface of an internal sphere at every space-time point. The new gauge transformations

$$(82) \quad \delta A_\mu(x, \vartheta, \varphi) = \partial_\mu v(x, \vartheta, \varphi) + \{A_\mu, v\},$$

where the Poisson bracket of two functions is defined as $\{f, g\} = \frac{\partial f}{\partial \varphi} \frac{\partial g}{\partial \cos \vartheta} - \frac{\partial f}{\partial \varphi} \frac{\partial g}{\partial \cos \vartheta}$, is isomorphic to the infinite dimensional Lie algebra of area preserving (or symplectic) diffeomorphisms of the sphere $SDiff(S^2)$ which is the symmetry of the membrane after gauge fixing. In the case p -brane u^a can be written in terms of functions A_k , ($k = 1, \dots, p-1$) as

$$(83) \quad u^a = \varepsilon^{a_1 \dots a_{p-1} a} \frac{\partial A_1}{\partial \sigma^{a_1}} \dots \frac{\partial A_{p-1}}{\partial \sigma^{a_{p-1}}},$$

and then the covariant time-derivative can be written in the form

$$(84) \quad \mathcal{D}_\tau X^i = \frac{\partial X^i}{\partial \tau} + \{A_1, \dots, A_{p-1}, X^i\}.$$

As a result, the action (79) is invariant under the p -dimensional volume preserving diffeomorphisms:

$$(85) \quad \delta X^i = \lambda^a \partial_a X^i, \quad \delta u^a = -\partial_\tau \lambda^a - u^b \partial_b \lambda^a + \lambda^b \partial_b u^a.$$

In this case λ^a is written in terms of functions Λ_k , ($k = 1, \dots, p-1$) in the same way as for u^a and the transformation laws of p -dimensional volume preserving diffeomorphisms are rewritten as

$$(86) \quad \delta X^i = \{\Lambda_1, \dots, \Lambda_{p-1}, X^i\},$$

$$\delta u^a = -\partial_\tau \lambda^a - \{A_1, \dots, A_{p-1}, \lambda^a\} + \{\Lambda_1, \dots, \Lambda_{p-1}, u^a\}.$$

In the paper [43], authors consider an alternative choice of $d = 0$. In particular, they focus on a supermembrane propagating in eleven-dimensional flat spacetime and proposed to following action for the three-algebra description of a supermembrane in eleven dimensions:

$$(87) \quad S_{M2} = \int d^3 \xi (\mathcal{L}_\omega + \mathcal{L}_{WZ}),$$

$$\mathcal{L}_\omega = \frac{1}{12} \omega^{-1} \langle E^M, E^N, E^P \rangle \langle E_M, E_N, E_P \rangle - \frac{1}{2} \omega,$$

$$\mathcal{L}_{WZ} = -\frac{i}{2} \varepsilon^{ijk} \bar{\theta} \Gamma_{MN} \partial_i \theta (E_j^M \partial_k X^N - \frac{1}{3} \bar{\theta} \Gamma^M \partial_j \theta \bar{\theta} \Gamma^N \partial_k \theta),$$

which contains eleven-dimensional target spacetime coordinates X^M , a Majorana spinor θ and a scalar density field ω . The former two are dynamical while the last one is auxiliary. With the supersymmetry invariant pull-back

$$E_i^M = \partial_i X^M - i \bar{\theta} \Gamma^M \partial_i \theta,$$

we set

$$\langle E^M, E^N, E^P \rangle = \varepsilon^{ijk} E_i^M E_j^N E_k^P,$$

which has the following expansion in terms of the Nambu-bracket

$$\langle E^L, E^M, E^N \rangle = \{X^L, X^M, X^N\} - 3i \bar{\theta} \Gamma^{[L} \{X^M, X^N\}, \theta\}$$

$$+ 3\bar{\theta} \{\Gamma^{[L} \theta, X^M, \bar{\theta} \Gamma^{N]}\} \theta - i \bar{\theta}_\alpha \bar{\theta}_\beta \bar{\theta}_\gamma \{(\Gamma^{[L} \theta)^\alpha, (\Gamma^M \theta)^\beta, (\Gamma^N] \theta)^\gamma\}.$$

Similarly, the Wess–Zumino part of the action can be also reexpressed in terms of the Nambu-bracket:

$$(88) \quad \mathcal{L}_{WZ} = -\frac{i}{2}\bar{\theta}\Gamma_{MN}\{X^M, X^N, \theta\} + \frac{1}{2}\bar{\theta}_\alpha\bar{\theta}_\beta\{(\Gamma_{MN}\theta)^\alpha, (\Gamma^M\theta)^\beta, X^N\} \\ -\frac{i}{6}\bar{\theta}_\alpha\bar{\theta}_\beta\bar{\theta}_\gamma\{(\Gamma_{MN}\theta)^\alpha, (\Gamma^M\theta)^\beta, (\Gamma^N\theta)^\gamma\}.$$

Thus, all the derivatives appear only through the Nambu three-brackets. Let us now introduce a shorthand notation for the induced metric: $g_{ij} = E_i^M E_{Mj}$ and denote its determinant by $g = \det(g_{ij})$ as usual. All the equations of motion are then summarized by:

$$(89) \quad \omega - \sqrt{-g} = 0, \quad g^{ij}E_i^M \Gamma_M(1-\Gamma)\partial_j \theta = 0, \quad \partial_i(\sqrt{-g}g^{ij}E_j^M) - i\varepsilon^{ijk}\partial_i\bar{\theta}\Gamma^M_N\partial_j\theta\Pi_k^N = 0.$$

From an identity $\frac{1}{6}\langle E^M, E^N, E^P \rangle \langle E_M, E_N, E_P \rangle = \det(E_i^M E_{Mj})$ integrating over the auxiliary scalar assuming the on-shell value $\omega = \sqrt{-g}$, our proposed action (87) reduces to the well-known supersymmetric DNG action for M2-brane [42]:

$$(90) \quad S_{M2} = \int d^3\xi \left[-\sqrt{-\det(E_i^M E_{Mj})} - \frac{i}{2}\varepsilon^{ijk}\bar{\theta}\Gamma_{MN}\partial_i\theta(E_j^M\partial_k X^N - \frac{1}{3}\bar{\theta}\Gamma^M\partial_j\theta\bar{\theta}\Gamma^N\partial_k\theta) \right].$$

The action (87) is invariant under the following transformations:

a) Target-spacetime supersymmetry:

$$(91) \quad \delta_\epsilon\theta = \epsilon, \quad \delta_\epsilon X^M = -i\bar{\theta}\Gamma^M\epsilon, \quad \delta_\epsilon\omega = 0;$$

b) Local 32-component fermionic symmetry:

$$(92) \quad \delta_\zeta\theta = (1 + (\omega/\sqrt{-g})\Gamma)\zeta, \quad \delta_\zeta X^M = i\bar{\theta}\Gamma^M\delta_\zeta\theta, \quad \delta_\zeta\omega = 4i\omega(g^{-1})^{ij}E_i^M\partial_j\bar{\theta}\Gamma_M\zeta,$$

where ζ is an arbitrary local 32-component spinorial parameter and Γ is as in [42]: $\Gamma = \frac{1}{6\sqrt{-g}}\Gamma_{LMN}\langle E^L, E^M, E^N \rangle$ satisfying $\Gamma^2 = 1$. In particular, taking the choice $\zeta = (1 + (\omega/\sqrt{-g})^{-1}(1 + \Gamma)\kappa)$ leads to a symmetry:

$$(93) \quad \delta_\kappa\theta = (1 + \Gamma)\kappa, \quad \delta_\kappa X^M = i\bar{\theta}\Gamma^M\delta_\kappa\theta, \quad \delta_\kappa\omega = 4i\frac{\omega\sqrt{-g}}{\omega + \sqrt{-g}}(g^{-1})^{ij}E_i^M\partial_j\bar{\theta}\Gamma_M\delta_\kappa\theta,$$

where κ is an arbitrary local fermionic parameter so that the transformations of θ and X^M coincide with the kappa-symmetry in [42];

c) Worldvolume diffeomorphisms:

$$(94) \quad \delta_v X^M = v^i\partial_i X^M, \quad \delta_v\theta = v^i\partial_i\theta, \quad \delta_v\omega = \partial_i(\omega v^i),$$

where $v^i = \delta\xi^i$ is an arbitrary local bosonic parameter, and the Lagrangian transforms to a total derivative as $\delta_v\mathcal{L} = \partial_i(v^i\mathcal{L})$.

Assume that the target space of the super- p -brane is a curved supermanifold with $E_M^A(z)$ as its corresponding supervielbeins. The $A = a, \alpha$ are the tangent space indices. Then the super- p -brane action is given by

$$(95) \quad S = -T_p \int d^{p+1}\sigma \left(\sqrt{-\det(E_i^a E_j^b \eta_{ab})} + \frac{2}{(p+1)!} \varepsilon^{i_1 \dots i_{p+1}} E_{i_1}^{A_1} \dots E_{i_{p+1}}^{A_{p+1}} B_{A_{p+1} \dots A_1} \right),$$

where $E_i^A = \partial_i Z^M E_M^A$ is the pull-back of the supervielbeins E_M^A . The field $B_{A_{p+1} \dots A_1}(z)$ is the superspace $p+1$ -form potential. In fact, due to the κ -symmetry of the action,

only special values of p and D are allowable [42]. In this action the $p + 1$ -algebra also can be introduced. Since

$$(96) \quad \det(E_i^a E_j^b \eta_{ab}) = \frac{1}{(p + 1)!} \langle E^{a_1}, \dots, E^{a_{p+1}} \rangle \langle E_{a_1}, \dots, E_{a_{p+1}} \rangle, \\ \langle E^{a_1}, \dots, E^{a_{p+1}} \rangle = \varepsilon^{i_1 \dots i_{p+1}} E_{i_1}^{a_1} \dots E_{i_{p+1}}^{a_{p+1}},$$

the action (95) can be reformulated in terms of the Nambu $p + 1$ -brackets. The novelty of this reformulation is the appearance of the Filippov–Nambu $p+1$ -algebra.

As shown in [43], double dimensional reduction of supermembrane action (87), putting $\xi^2 = X^{10}$, $\Gamma^{(11)} = \Gamma^{10}$, straightforwardly leads to the well-known formulation of the type IIA superstring action by Green and Schwarz [3]. In a similar fashion to type IIA superstring action, the Schild version of type IIB superstring covariant action in ten dimensions also appears. All the derivatives therein appear through the Nambu brackets such that the two-algebra structure of superstring theory and the three-algebra structure of \mathcal{M} -theory become manifest. The Nambu two- and three-brackets naturally arise since the dimensions of the string worldsheet and the membrane worldvolume are two and three respectively. One advantage to employ the Nambu brackets is the simplicity of the double dimensional reduction: The three-bracket clearly reduces to the two-bracket. Hence the Filippov–Nambu $p + 1$ -algebra structure becomes apparent for the super- p -brane theory.

The most intriguing question is: "What is \mathcal{M} -theory?" It is instructive to analyze the situation from the perspective of spectrum of elementary excitations. Superstrings describe massless modes of lower spins $s \leq 2$ like graviton ($s = 2$), gravitino ($s = 3/2$), vector bosons ($s = 1$) and matter fields with spins 1 and $1/2$, as well as certain antisymmetric tensors. On the top of that there is an infinite tower of massive excitations of all spins. Since the corresponding massive parameter is supposed to be large, massive higher spin excitations are not directly observed at low energies. They are important however for the consistency of the theory. Assuming that \mathcal{M} -theory is some relativistic theory admitting a covariant perturbative interpretation, we conclude that it should necessarily contain higher spin modes to describe superstring models as its particular vacua. There are two basic alternatives: (i) $m \neq 0$: higher spin modes in \mathcal{M} -theory are massive or (ii) $m = 0$: higher spin modes in \mathcal{M} -theory are massless. Each of these alternatives is not straightforward. In the massive case it is generally believed that no consistent superstring theory exists beyond ten dimensions and therefore there is no good guiding principle towards \mathcal{M} -theory from that side. For the massless option the situation is a sort of opposite: there is a very good guiding principle but it looks like it might be too strong. Indeed, massless fields of high spins are gauge fields. Therefore this type of theories should be based on some higher spin gauge symmetry principle with the symmetry generators corresponding to various representations of the Lorentz group. It is very well known however that it is a hard problem to build a nontrivial theory with higher spin gauge symmetries. One argument is due to the Coleman–Mandula theorem and its generalizations [46] which claim that symmetries of S-matrix in a non-trivial (i.e., interacting) field theory in a flat space can only have sufficiently low spins. These arguments convinced most of experts that no consistent nontrivial higher spin gauge theory can exist at all.

However, it was realized (see [47] and references therein) that the situation changes drastically once, instead of the flat space, the problem is analyzed in the AdS space with nonzero curvature Λ . This generalization led to the solution of

the problem of consistent higher spin gravitational interactions in all orders in interactions at the level of equations of motion. An important general conclusion is that Λ should necessarily be nonzero in the phase with unbroken higher spin gauge symmetries since it cancels the Coleman–Mandula argument which is hard to implement in the *AdS* background. However, up to date a fully consistent action describing interactions of propagating higher spin fields is not known. The nonlinear higher spin theory in four dimensions was shown to be consistent up to cubic order at the action level and, later, at all orders at the level of equations of motion. Concerning the problem of finding a consistent higher spin action, it should be noted that one example does exist: the Chern–Simons action in $d=3$ constructed by Blencowe based on a higher spin algebra [48] (see also [49], [50] in a related context) as the algebra of volume-preserving diffeomorphisms $\partial_a(\sqrt{g}\Omega^{ab}) = 0$ of a manifold \mathcal{M} . This algebra is a subalgebra of the general diffeomorphisms algebra of manifold \mathcal{M} and corresponds to the residual symmetry of an extended object in the light-cone gauge. The symplectic diffeomorphisms on \mathcal{M}^p are generated by $L_\lambda = \Omega^{ab}\partial_b\lambda\partial_a$, where λ is an arbitrary function of σ and $\Omega^{ac}\Omega_{cb} = \delta_b^a$. The generators L_λ obey the algebra $[L_{\lambda_1}, L_{\lambda_2}] = L_{\lambda_3}$ where $\lambda_3 = \Omega^{ab}\partial_b\lambda_1\partial_a\lambda_2 = \{\lambda_1, \lambda_2\}$. Expanding the parameter $\lambda(\sigma)$ (whenever possible) in terms of a complete infinite set of basis functions, one obtains an infinite dimensional algebra from this composition law. This algebra can be gauged by making the parameter λ local, $\lambda = \lambda(x, \sigma)$ in a spacetime, e.g. in a $2+1$. then one can introduce a gauge field $\Gamma_m(x, \sigma)$ defined on $\mathcal{M}^3 \otimes \mathcal{M}^p$: $\delta\Gamma_m = \partial_m\lambda + \{\Gamma_m, \lambda\}$. Next one can write down an action for this field, in particular, a Chern-Simons term

$$(97) \quad S_{CS} = \int d^3x \int d^p\sigma \sqrt{g}\varepsilon^{mnl}(\Gamma_m\partial_n\Gamma_l + \frac{1}{3}\{\Gamma_m, \Gamma_n\}\Gamma_l) .$$

It is important to realize that this action is invariant under the gauge transformations only if the volume-preservation condition is satisfied. This action was considered in [49] for the case of a 2-sphere S^2 and 2-hyperboloid H^2 . For example, for the case of a 2-sphere it describes infinitely many spin-1 gauge fields, while for a 2-hyperboloid it describes infinitely many higher spin gauge fields including the gravitation field in *AdS* space. In order to reveal the infinite dimensional algebraic structure of the algebra $SDiff(H^2)$, one needs an expansion on the 2-hyperboloid H^2 , such that the Lorentz transformation properties of the generators will be manifest [50]. This can be done by using a harmonic parametrization of H^2 [25] defined as follows. Consider a set of variables $u^{\pm\alpha}$ parametrizing the group $SL(2, R) \sim SO(2, 1)$. The index α of $u^{\pm\alpha}$ is an $SL(2, R)$ one, and the index $+$ or $-$ refers to a charge of the $SO(2)$ subgroup of $SL(2, R)$ and $u^{+\alpha}u_{\alpha}^- = 1$. The coset $SL(2, R)/SO(2)$ will be realized on functions $f^{(q)}(u)$ of $u^{\pm\alpha}$ having a definite $SO(2)$ charge: $\partial^0 f^{(q)} = (u^{+\alpha}\partial_{u^{+\alpha}} - u^{-\alpha}\partial_{u^{-\alpha}})f^{(q)} = qf^{(q)}$. In other words, those functions are given by the harmonic expansion (for $q \geq 0$):

$$(98) \quad f^{(q)} = \sum_{n=0}^{\infty} f^{(\alpha_1 \dots \alpha_{n+q} \beta_1 \dots \beta_n)} u_{\alpha_1}^+ \dots u_{\alpha_{n+q}}^+ u_{\beta_1}^- \dots u_{\beta_n}^- ,$$

and similarly for $q < 0$. Note that the coefficients in this expansion are now irreducible spin-tensors of $SL(2, R)$. The two derivatives $\partial/\partial\sigma^a$ on the coset are now represented by the operators:

$$(99) \quad \partial^{++} = u^{+\alpha} \frac{\partial}{\partial u^{-\alpha}}, \quad \partial^{--} = u^{-\alpha} \frac{\partial}{\partial u^{+\alpha}}, \quad [\partial^{++}, \partial^{--}] = \partial^0 .$$

Integration on H^2 is defined as follows: $\int du \cdot 1 = 1$, $\int duu_{(\alpha_1}^+ \dots u_{\alpha_r}^+ u_{\beta_1}^- \dots u_{\beta_s}^-) = 0$ for $r + s > 0$. This formal definition of the integral has all the desired properties, in particular, it allows integration by parts. Using the parametrization of H^2 introduced above we can get an insight into the structure of the infinite dimensional algebra of $SDiff(H^2)$. Now the composition law can be rewritten in the following form $\lambda_3 = \partial^{++} \lambda_1(u) \partial^{--} \lambda_2(u) - (1 \leftrightarrow 2)$. Expanding the harmonic functions we find

$$\lambda_{12} = \sum_{r,s,k} C_k^{2r,2s} \lambda_1^{\alpha_1 \dots \alpha_k \beta_{k+1} \dots \beta_{2r}} \lambda_2^{\alpha_{k+1} \dots \alpha_{2s}}_{\beta_{k+1} \dots \beta_{2r}} u_{(\alpha_1}^+ \dots u_{\alpha_k}^+ u_{\alpha_{k+1}}^- \dots u_{\alpha_{2s})}^- - (1 \leftrightarrow 2),$$

where $C_k^{2r,2s}$ are $SL(2, R)$ Clebsh–Gordon coefficients. Note that this composition law for area-preserving diffeomorphisms of H^2 had be already given in [49]. In order to make contact with familiar Chern–Simons theories based on finite dimensional spacetime algebras, one must examine the finite dimensional truncation of $SDiff(H^2)$. Now the generalization the concepts discussed above to the case of a supermanifold is straightforward [50]. If we take $\Gamma_m = (\omega_m^{\alpha\beta} + \theta^4 e_m^{\alpha\beta}) u_{(\alpha}^+ u_{\beta)}^-$, the Chern–Simons action is just the action for the Poincare gravity in $2 + 1$ dimensions

$$(100) \quad S = \int d^3x \varepsilon^{mnl} (e_m^{\alpha\beta} \partial_n \omega_{l\alpha\beta} - e_{m,\beta}^\alpha \omega_{n\rho}^\beta \omega_{l\alpha}^\rho).$$

Since the Chern-Simons theory is a true gauge theory, the resulting higher spin theory (how shown in the above toy example H^2) is consistent by construction and naturally extends the Einstein-Hilbert action (which in $d3$ also has an interpretation as a Chern-Simons action). It is, however, only of limited use since it is topological theory that does not give rise to propagating degrees of freedom. It is possible to describe a super AdS (also superconformal) field theory for an infinite tower of integer and half-integer higher spin field in a similar fashion.

7. CONCLUSIONS AND OUTLOOK

A (1+2)-dimensional relativistic gauge theory based on the Filippov 3-algebra [6] rather than on the Lie algebra, was proposed recently by BLG [10], as a model of multiple M2-branes. The model has an $OSp(8|4)$ conformal symmetry as expected for the infra-red fixed point of the Yang–Mills type gauge theory on coincident D2-branes. The construction requires a metric on the 3-algebra and if this metric is positive definite then the structure constants of the 3-algebra define a totally-antisymmetric fourth-rank tensor satisfying a fundamental identity. When the structure constants vanish one has a 'trivial' 3-algebra and the model reduces to a free theory for the $\mathcal{N} = 8$ scalar multiplet, as expected for the conformal limit of a single planar M2-brane. A non-trivial realization based on the Lie algebra $so(4)$ was given by BLG, and it appears to describe two coincident M2-branes on an orbifold $\mathbb{R}^8/\mathbb{Z}_2$. Other possibilities emerge when one allows for Lorentzian metrics on the 3-algebra [15], [16] but these models have ghosts. Various other facets of BLG models have been addressed in other papers; an incomplete list can be found in [18]-[21]. In the context of the original BLG model, with positive definite metric, there remains one other possibility: there is an infinite-dimensional realization of the 3-algebra in terms of the Nambu bracket on a three-dimensional space. In this realization, the BLG model is essentially an exotic gauge theory for the group of volume-preserving diffeomorphisms of this space, where by 'exotic' we mean that the gauge theory is not of the Yang–Mills type. This is not the first occasion on which exotic gauge theories based on volume preserving diffeomorphisms have appeared. They also arise

from light-cone gauge fixing of relativistic p -brane actions for $p > 2$; these are exotic gauge theories with a group of p -volume-preserving diffeomorphisms, $SDiff_p$, as the gauge group [45]. As is well-known, the flux of the 2-form potential on the M5-brane may be interpreted as M2-branes 'dissolved' in the M5-brane. Thus, a single M5-brane may contain multiple M2-branes and is therefore a promising starting point for a construction of the BLG model for multiple M2-branes. Another indication of this is that the Nambu-bracket realization of the BLG theory introduces some 'internal' Riemannian 3-manifold M_3 , so that the 'total' space dimension is $2+3 = 5$. In fact, it has been proposed in recent papers that the Nambu-bracket realization of the BLG model is equivalent to the M5-brane action [39]. One of the many obvious questions is whether analogous results might emerge by considering M5-branes of other topologies, for example $S^1 \times M_4$ for some closed 4-manifold M_4 . One might imagine that this could be related to some exotic $(1+1)$ -dimensional gauge theory based on the Filippov 4-algebra. A natural question is whether there exist gauge theories for which the gauge group is the group $SDiff(M_n)$ of volume-preserving diffeomorphisms of some n -dimensional manifold M_n for $n \geq 3$; we assume that M_n is closed and compact with respect to some volume n -form. Another outstanding problem is the nature of the $6d$ conformal field theory governing the low energy dynamics of N coincident M5-branes. In light of what we now know about multiple coincident M2-branes, it seems likely that this problem will simplify in the $N \rightarrow \infty$ limit. Given that a condensate of M2-branes may be viewed, in some sense, as an M5-brane, then is there a similar sense in which an M5 condensate could be viewed as a yet higher-dimensional M-brane? Recalling that the recent advances in the M2 case were prompted by the Basu–Harvey proposal that the boundary of multiple M2-branes on an M5-brane might be understood in terms of fuzzy 3-spheres, it is natural to reconsider the implications of the recent demonstration [51] that an M5-brane can have a boundary on an M9-brane, which is a boundary of the 11-dimensional bulk spacetime of \mathcal{M} -theory. In this context we should mention that higher-dimensional generalizations of the Basu–Harvey equation have been considered in [52]. In general, could say that \mathcal{M} -theory must be a new kind of theory, which should perhaps be formulated in terms of completely new degrees of freedom, and requires new physical principles. Now it seems the main aim of this development programme is to put the Nambu-bracket realization of the BLG theory into a larger context by developing further the general principles of $SDiff$ gauge theory.

Though this a brief survey of some important trends in recent investigations poses more questions than provides the answers, we feel that the subject of the Filippov–Nambu higher algebraic operations might be relevant for future development of mathematical structure related to a great many physical problems and then certainly deserve further studies.

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