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TORSION AND CURVATURE FOR THE 6-COMMUTATOR

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Abstract. Torsion and curvature for 6-commutator on $Vect(2)$ are calculated. It is proved that 6-curvature (6-torsion) can be presented as a exterior product of usual curvature (usual torsion) by some 4-form.

Keywords: N-commutator, torsion, curvature, Bianchi identities

Let s_k be a standard skew-symmetric associative polynomial

$$s_k(t_1,\ldots,t_k)=\sum_{\sigma\in Sym_k} sign\,\sigma\, t_{\sigma(1)}\cdots t_{\sigma(k)}.$$

Let $M=M^n$ be a n-dimensional manifold, U a space of smooth functions on U and L a space of vector fields on M^n. Let $Diff_{[k]}$ be a space of differential operators of differential order k. Recall that a homogeneous differential operator $u_\alpha\partial^\alpha$ has differential order k if $\partial^\alpha=\prod_{i=1}^n\partial_i^{\alpha_i}$, $\partial_i=\frac{\partial}{\partial x_i}$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbf{Z}_+^n, \sum_{i=1}^n\alpha_i=k$. One can make identifications $U=Diff_{[0]}$ and $L=Diff_{[1]}$. Let

$$Diff_{(k)}=\oplus_l Diff_{[k]}$$

be a space of differential operators of differential order $\le k$ and $Diff$ a space of all differential operators on M^n. Endow $Diff$ by a structure of associative algebra under composition. Then L became U-module under natural action $(u,X)\mapsto u\cdot X$.

One can consider s_k as a k-ary operation on a space $Diff$. In general,

$$X_1\in Diff_{(l_1)},\ldots,X_k\in Diff_{(l_k)}\Rightarrow X_1\cdots X_k\in Diff_{(l_1+\cdots+l_k)},$$

and $s_k(X_1,\ldots,X_k)$ is a differential operator of order $l_1+\cdots+l_k-1$. In case of $k=2$ it is a well-defined operation on $Vect(n)$,

$$s_2(X_1,X_2)=[X_1,X_2]=X_1X_2-X_2X_1$$

(Lie commutator).

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One asks whether s_N might be well-defined on $Vect(n)$ for $N > 2$. Call s_N as *N-commutator* if it is a such case,

$$\forall X_1,\dots,X_N \in Vect(n) \Rightarrow s_N(X_1,\dots,X_N) \in Vect(n).$$

In [1] it was established that for any n there exist some N depending on n such that N-commutators are well-defined. Namely, N- commutator on $Vect(n)$ is well-defined if $N = n^2 + 2n - 2$. For example, $Vect(2)$ has non-trivial 6-commutator.

Denote by $End\,L$ be a space of endomorphisms of L, i.e., space of $\mathbf{R}$-linear maps. Let $End_U L$ be space of U-linear maps on L.

Let $X_i = u_i^l \partial_l$ vector fields, $i = 0, 1, \dots, 6$. Here we use Einshtein notation. Summations are made over repeated indexes. Let $\nabla_X \in End\,L$ be covariant derivation. Recall that Riemann curvature $R : \wedge^2 L \to End_U L$ for a covariant derivation ∇_X is defined by

$$R(X_1, X_2) = [\nabla_{X_1}, \nabla_{X_2}] - \nabla_{[X_1,X_2]}.$$

Our aim is to study curvature and torsion for 6-commutator on $Vect(2)$. From now let L be space of vector fields on 2-dimensional space. Let us follow usual defintions of curvature and torsion (see for example, [2]) to generalize them for N-commutators. Define 6 - torsion form and 6- curvature form for 6-commutator as a polylinear skew-symmetric maps

$$T_6 : \wedge^6 L \to L$$

$$R_6 : \wedge^6 L \to End_U L$$

by

$$T_6(X_1,\dots,X_6) = \sum_{i=1}^{6} (-1)^{i+1} s_5(\nabla_{X_1},\dots,\widehat{X_i},\dots,\nabla_{X_6})(X_i) - s_6(X_1,\dots,X_6),$$

$$R_6(X_1,\dots,X_6) = s_6(\nabla_{X_1},\dots,\nabla_{X_6}) - \nabla_{s_6(X_1,\dots,X_6)}.$$

Here notation $\hat{X}_i$ means that corresponding element is omitted. Let $\eta : \wedge^4 L \to U$ be 4-form defined by

$$\eta(X_1,X_2,X_3,X_4) = -3 \begin{vmatrix} \partial_1 X_1(x_1) & \partial_1 X_2(x_1) & \partial_1 X_3(x_1) & \partial_1 X_4(x_1) \\ \partial_2 X_1(x_1) & \partial_2 X_2(x_1) & \partial_2 X_3(x_1) & \partial_2 X_4(x_1) \\ \partial_1 X_1(x_2) & \partial_1 X_2(x_2) & \partial_1 X_3(x_2) & \partial_1 X_4(x_2) \\ \partial_2 X_1(x_2) & \partial_2 X_2(x_2) & \partial_2 X_3(x_2) & \partial_2 X_4(x_2) \end{vmatrix}$$

Our main result is

Theorem 1. *Torsion and curvature for* 6*-commutators can be given by the following formulas*

$$T_6 = \eta \wedge T,$$

$$R_6 = \eta \wedge R.$$

In other words,

$$T_6(u_1^{l_1}\partial_{l_1},\dots,u_6^{l_6}\partial_{l_6}) =$$

$$-3 \sum_{\sigma \in Sym_{2,4}} sign\,\sigma\, u_{\sigma(1)}^{l_1} u_{\sigma(2)}^{l_2}\, T_{l_1 l_2}^{s} \begin{vmatrix} \partial_1 u_{\sigma(3)}^1 & \partial_1 u_{\sigma(4)}^1 & \partial_1 u_{\sigma(5)}^1 & \partial_1 u_{\sigma(6)}^1 \\ \partial_2 u_{\sigma(3)}^1 & \partial_2 u_{\sigma(4)}^1 & \partial_2 u_{\sigma(5)}^1 & \partial_2 u_{\sigma(6)}^1 \\ \partial_1 u_{\sigma(3)}^2 & \partial_1 u_{\sigma(4)}^2 & \partial_1 u_{\sigma(5)}^2 & \partial_1 u_{\sigma(6)}^2 \\ \partial_2 u_{\sigma(3)}^2 & \partial_2 u_{\sigma(4)}^2 & \partial_2 u_{\sigma(5)}^2 & \partial_2 u_{\sigma(6)}^2 \end{vmatrix} \partial_s,$$

and

$$R_6(u_1^{l_1}\partial_{l_1},\dots,u_6^{l_6}\partial_{l_6})(u_0^{l_0}\partial_{l_0}) =$$

$$-3\sum_{\sigma\in Sym_{2,4}} sign\,\sigma\, u_0^{l_0}u_{\sigma(1)}^{l_1}u_{\sigma(2)}^{l_2}\,R^s_{l_0l_1l_2}\begin{vmatrix} \partial_1 u^1_{\sigma(3)} & \partial_1 u^1_{\sigma(4)} & \partial_1 u^1_{\sigma(5)} & \partial_1 u^1_{\sigma(6)} \\ \partial_2 u^1_{\sigma(3)} & \partial_2 u^1_{\sigma(4)} & \partial_2 u^1_{\sigma(5)} & \partial_2 u^1_{\sigma(6)} \\ \partial_1 u^2_{\sigma(3)} & \partial_1 u^2_{\sigma(4)} & \partial_1 u^2_{\sigma(5)} & \partial_1 u^2_{\sigma(6)} \\ \partial_2 u^2_{\sigma(3)} & \partial_2 u^2_{\sigma(4)} & \partial_2 u^2_{\sigma(5)} & \partial_2 u^2_{\sigma(6)} \end{vmatrix}\partial_s.$$

Here

$$T^l_{ij} = \Gamma^l_{ij} - \Gamma^l_{ji},$$

is torsion tensor and

$$R^l_{ijk} = -\partial_k\Gamma^l_{ij} + \partial_j\Gamma^l_{ik} + \Gamma^l_{mj}\Gamma^m_{ik} - \Gamma^l_{mk}\Gamma^m_{ij}$$

is curvature tensor and summations are made by shuffle permutations

$$Sym_{2,4} = \{\sigma\in Sym_6|\sigma(1)<\sigma(2),\quad \sigma(3)<\sigma(4)<\sigma(5)<\sigma(6)\}.$$

Theorem 2. *Take place the following analoges of Bianchi identities*

$$\sum_{i=1}^{7}(-1)^i R_6(X_1,\dots,\widehat{X_i},\dots,X_7) = 0,$$

$$\sum_{i=1}^{7}(-1)^i \nabla_{X_i} R_6(X_1,\dots,\widehat{X_i},\dots,X_7) = 0.$$

If Cristoffel symbols are compatible with Riemann metric $g_{i,j}$, and $\langle\ ,\ \rangle$ is a scalar product, then

$$\langle R_6(X_1,\dots,X_6)(X_7),X_8\rangle = -\langle R_6(X_1,\dots,X_6)(X_8),X_7\rangle.$$

Here $X_1,\dots,X_8$ are any vector fields on M^2.

So, 6-torsion and 6-curvatures are defined by 1-jets of vector fields. Recall that in classical cases torsion and curvature forms are U-linear maps. In case of 6-commutators U-linearity fall and remains only linearity over scalars.
Forms $T_6(X_1,\dots,X_6)$ and $R_6(X_1,\dots,X_6)(X_0)$ are polylinear over $\mathbf{R}$ and skew-symmetric under arguments $X_1,\dots,X_6$ but 6-curvature is U-linear under X_0.

Since 6-form η is not identically 0, notions of flatness in terms of 6-curvature and usual curvature are coincide. Similar things happen for torsions.

Corollary 3. *$R_6 = 0$ is an identity if and only if $R = 0$ is an identity.*
$T_6 = 0$ is an identity if and only if $T = 0$ is an identity.

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