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A NOTE ON $\sigma(*)$-RINGS AND THEIR EXTENSIONS

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Abstract. Let R be an associative ring with identity $1 \neq 0$, and σ an endomorphism of R. We recall $\sigma(*)$ property on R (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R). Also recall that a ring R is said to be 2-primal if and only if the prime radical $P(R)$ and nil radical are same, i.e. if the prime radical is a completely semiprime ideal. It can be seen that a $\sigma(*)$ ring is a 2-primal ring.

Let R be a ring and σ an automorphism of R. Then we know that σ can be extended to an automorphism of the skew polynomial ring $R[x;\sigma]$. In this paper we show that if R is a Noetherian ring and σ is an automorphism of R such that R is a $\sigma(*)$-ring, then $R[x;\sigma]$ is also a $\sigma(*)$-ring.

Keywords: minimal prime, prime radical, automorphism, $\sigma(*)$-ring.

1. Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The set of prime ideals of R is denoted by $Spec(R)$. The sets of minimal prime ideals of R is denoted by $Min.Spec(R)$. The prime radical and the nil radical of R are denoted by $P(R)$ and $N(R)$ respectively. The set of positive integers is denoted by $\mathbb{N}$, the ring of integers is denoted by $\mathbb{Z}$, the field of rational numbers is denoted by $\mathbb{Q}$ and the field of complex numbers is denoted by $\mathbb{C}$. Let R be a ring and σ an automorphism of R. Let I be an ideal of R such that $\sigma^m(I) = I$ for some $m \in \mathbb{N}$. We denote $\cap_{i=1}^{m}\sigma^i(I)$ by I^0.

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This article concerns the study of skew polynomial ring (Ore extension) over a $\sigma(*)$-ring R, where σ is an automorphism of R.

Definition 1.1. (Krempa [3]) Let R be a ring and σ an endomorphism of R. Then σ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$. The ring R is said to be a σ-rigid ring if there exists a σ-rigid endomorphism R.

Example 1.2. *Let $R = \mathbb{C}$, and $\sigma : \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a+ib) = a-ib$, $a, b \in \mathbb{R}$. Then it can be seen that σ is a rigid endomorphism of R.*

Definition 1.3. (Kwak [4]) Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1.4. *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ Let $\sigma : R \to R$ be defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.*

We note that the above ring is not σ-rigid. For let $0 \neq a \in F$. Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Recall that a ring R is 2-primal if and only if $N(R) = P(R)$ if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a commutative ring is 2-primal and so is a reduced ring.

Recall that $R[x;\sigma]$ is the usual polynomial ring with coefficients in R, in which multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x;\sigma]$ by $S(R)$. If an ideal I of a ring R is σ-stable (i.e. $\sigma(I) = I$), then we denote as usual $I[x;\sigma]$ by $S(I)$.

We also note that if σ is an automorphism of R, then it can be extended to an automorphism of $R[x;\sigma]$ such that $\sigma(x) = x$; i.e. $\sigma(\Sigma_{i=0}^{n} x^i a_i) = \Sigma_{i=0}^{n} x^i \sigma(a_i)$. The study of skew polynomial rings has been of interest to many authors. For example [1, 2, 4].

Definition 1.5. (Goodearl and Warfield [2]) A ring R is said to be right(left) Noetherian ring if it satisfies the ascending chain condition on right(left) ideals. Explicitly this means: given an increasing sequence of right(left) ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$$

there exists an $n \in N$ for which

$$I_n = I_{n+1} = I_{n+2} = ...$$

Equivalently R is right(left) Noetherian if every right(left) ideal of R is finitely generated. R is said to be Noetherian if it is both right and left Noetherian.

Some examples: The ring of integers $\mathbb{Z}$, any field, any principal ideal domain, the ring of polynomials in finitely-many variables over the integers or a field.

The ring of polynomials in infinitely-many variables, $x_1, x_2, x_3, ...$ is not Noetherian as the ascending sequence of ideals $(x_1) \subseteq (x_1, x_2) \subseteq (x_1, x_2, x_3) \subseteq ...$ does not terminate.

We note that a right Noetherian ring need not be left Noetherian. For example consider $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}\}$. Then R is right Noetherian but not left Noetherian.

We now state the main result of this paper in the form of the following Theorem [Theorem (2.6]:

Theorem: *If R is a Noetherian ring and σ is an automorphism of R such that R is a $\sigma(*)$-ring, then $R[x;\sigma]$ is also a $\sigma(*)$-ring.*

2. Skew polynomial rings over $\sigma(*)$-rings

We begin this section with the following Proposition:

Proposition 2.1. *Let R be a ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies R is 2-primal.*

Proof. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

The following example shows that there exists an endomorphism σ of a ring R such that the converse of the above Proposition does not hold.

Example 2.2. *Let $R = F[x]$, F a field. Then R is a commutative domain, and therefore is 2-primal with $P(R) = 0$. Let $\sigma : R \to R$ be defined by $\sigma(f(x)) = f(0)$. Let $f(x) = xa$, $0 \neq a \in F$. Then $f(x)\sigma(f(x)) \in P(R)$, but $f(x) \notin P(R)$. Therefore R is not a $\sigma(*)$-ring.*

Recall that an ideal J of R is called completely prime if $ab \in J$ implies $a \in J$ or $b \in J$ for $a, b \in R$.

We know that for any ring R, the prime radical $P(R)$ is the intersection of prime ideals of R. Now Proposition (3.3) of Goodearl and Warfield [2] implies that every prime ideal of R contains a minimal prime ideal. Therefore, the prime radical $P(R)$ is the intersection of minimal prime ideals of R (Proposition (3.10) of Goodearl and Warfield [2]). With this, we have the following:

Theorem 2.3. *Let R be a Noetherian ring, and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. Let R be a Noetherian ring such that for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. Now for each i, $a \in U_i$ or $\sigma(a) \in U_i$ as U_i are completely prime. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring and let $U = U_1$ be a minimal prime ideal of R. Now by Proposition (2.1), $P(R)$ is completely semiprime. Now $Min.Spec(R)$ is finite by Theorem (3.4) of Goodearl and Warfield [2]. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Suppose that $\sigma(U) \neq U$. Then $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$.

Now suppose that $U = U_1$ is not completely prime. Then there exist $a, b \in R \setminus U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$. So $c \in P(R)$ and, thus $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \notin U$, $U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime.

Note that in above Theorem the condition of completely primeness of minimal prime ideals can not be deleted. Towards this we have the following:

Remark 2.4. Let R be a Noetherian ring and σ an automorphism of R such that $\sigma(U) = U$ for each minimal prime ideal U of R. Then $R[x; \sigma]$ need not be a $\sigma(*)$-ring. (Example (2.2))

We note that if R is a Noetherian ring, then as mentioned above, $Min.Spec(R)$ is finite. Now if σ is an automorphism of R, then $\sigma^j(U) \in Min.Spec(R)$ for any $U \in Min.Spec(R)$ for all $j \in \mathbb{N}$. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(U) = U$ for all $U \in Min.Spec(R)$. We denote $\cap_{i=1}^{m} \sigma^i(U)$ by U^0. We now have the following:

Theorem 2.5. *Let R be a Noetherian ring and σ an automorphism of R. Then : $P \in Min.Spec(S(R))$ if and only if there exists $Q \in Min.Spec(R)$ Such that $S(P \cap R) = P$ and $P \cap R = Q^0$.*

Proof. See Theorem (2.4) of Bhat [1].

Theorem 2.6. *Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. Then $R[x; \sigma]$ is also a $\sigma(*)$-ring.*

Proof. First of all we show that $\sigma(P) = P$, for all $P \in Min.Spec(S(R))$. Let $P \in Min.Spec(S(R))$. Then by Theorem (2.5) there exists $U \in Min.Spec(R)$ such that $P = U^0[x; \sigma]$. Now R is a $\sigma(*)$-ring implies that $\sigma(U) = U$ by Theorem (2.3), and therefore $U^0 = U$. So $P = U[x; \sigma]$ and thus $\sigma(P) = P$.

We now show that P is completely prime. Let

$$f(x) = x^n a_n + x^{n-1} a_{n-1} + ... + a_0,$$

and

$$g(x) = x^m b_m + x^{m-1} b_{m-1} + ... + b_0 \text{ in } R[x; \sigma] \text{ be such that}$$
$$f(x)g(x) \in P = U[x; \sigma], \text{ and } g(x) \notin U[x; \sigma].$$

This implies that

$$x^{n+m}\sigma^m(a_n)b_m + x^{n+m-1}\sigma^m(a_{n-1})b_m + x^{n+m-1}\sigma^{m-1}(a_n)b_{m-1} + ... + a_0 b_0 \in U[x; \sigma].$$

Now $g(x) \notin U[x; \sigma]$ (say $b_m \notin U$). Now $\sigma^m(a_n)b_m \in U$. Also U is completely prime by Theorem(2.3), therefore, $\sigma^m(a_n) \in U$; i.e. $a_n \in U$.

Now $\sigma^m(a_{n-1})b_m + \sigma^{m-1}(a_n)b_{m-1} \in U$ implies that $\sigma^m(a_{n-1})b_m \in U$. Now $b_m \notin U$ implies that $\sigma^m(a_{n-1}) \in U$; i.e. $a_{n-1} \in U$.

With the same process in a finite number of steps it can be seen that $a_i \in U$ for all i, $0 \leq i \leq n-2$ also.

Therefore $a_i \in U$ for all i, $0 \leq i \leq n$; i.e. $f(x) \in P = U[x; \sigma]$.

Thus $\sigma(P) = P$ and P is completely prime for all $P \in Min.Spec(S(R))$. Moreover $S(R)$ is Noetherian by Theorem (1.14) of Goodearl and Warfield [2]. Hence by Theorem (2.3), we get that $R[x;\sigma]$ is also a $\sigma(*)$-ring.

Remark 2.7. (1) Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. Then $R[x;\sigma]$ is also a $\sigma(*)$-ring. Therefore Proposition (2.1) implies that $R[x;\sigma]$ is 2-primal.

(2) If R is 2-primal Noetherian ring, then $R[x;\sigma]$ need not be 2-primal. For example consider $\mathbb{Z}_2$ and let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring with $P(R) = 0$, and therefore, R is 2-primal. Define $\sigma : R \to R$ by $\sigma(a,b) = (b,a)$. Then it can be seen that $P(R[x;\sigma]) = 0$, but $P(R[x;\sigma])$ is not completely semiprime as $((1,0)x)^2 = 0 = P(R[x;\sigma])$, but $(1,0)x \notin P(R[x;\sigma])$. Thus $R[x;\sigma]$ is not 2-primal.

References

[1] V. K. Bhat, *Associated prime ideals of skew polynomial rings.* Beitrдge Algebra Geom. **49**: 1 (2008), 277–283. MR2410584 (2009e:16046).

[2] K. R. Goodearl and R. B. Warfield Jr., *An introduction to non-commutative Noetherian rings.* Second Edition, Cambridge Uni. Press, 2004. MR2080008 (2005b:16001).

[3] J. Krempa, *Some examples of reduced rings.* Algebra Colloq. **3**: 4 (1996), 289–300. MR1422968 (98e:16027).

[4] T. K. Kwak, *Prime radicals of skew-polynomial rings.* Int. J. Math. Sci. **2**: 2 (2003), 219–227. MR2061508. (2006a:16035).

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